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A

TREATISE

ON

ALGEBRA.

BY

GEORGE PEACOCK, M.A. F.R.S. F.G.S.
F. Ast. S. and F.C.P.S.

FELLOW AND TUTOR OF TRINITY COLLEGE,
CAMBRIDGE.

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MDCCCXXX.

TO THE

REVEREND JAMES TATE, M.A.

MASTER OF RICHMOND SCHOOL, YORKSHIRE,

AND

FORMERLY FELLOW OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

I BEG leave to inscribe this Work to you, in the first place, as a tribute of affection to one of my best and dearest friends: in the second place, as an expression of my gratitude for your instructions when I was your pupil, and for your kindness and encouragement at a period of my life when they were invaluable to me: and lastly, as the most public testimony which I can give of the respect which I feel for your learning and for those various happy arts of communicating knowledge to others for which you are so greatly and so justly celebrated.

I trust that you will be pleased to find some topics of discussion in the following Work, which will recall to your mind the subject of many interesting conversations with you, both when I was your pupil, and on subsequent occasions; and I feel gratified in being able to refer some portion of my own fondness for such speculations to the same person to whom I am under so many other obligations.

DEDICATION.

That you may long live to enjoy health and piness, and to receive the well-merited homage o affection and respect of your friends and of the ration of your numerous pupils, is the prayer of sincere friend and grateful pupil

THE AUTH

PREFACE.

THE Work which I have now the honour of presenting to the public, was written with a view of conferring upon Algebra the character of a demonstrative science, by making its first principles co-extensive with the conclusions which were founded upon them: and it was in consequence of the very particular examination of those principles to which I was led in the course of this enquiry, that I have felt myself compelled to depart so very widely from the form under which they have been commonly exhibited. The object which I proposed to effect is undoubtedly one of great importance, and of no small difficulty, inasmuch as it brought me into immediate contact with the discussion of many subjects of dispute and controversy, which have not hitherto been settled upon satisfactory grounds: and though I am very sensible of the great responsibility which I incur by an attempt of this nature, accompanied as it is by the proposal of so many innovations, yet I shall be perfectly satisfied if I may be considered as having succeeded in removing any difficulties or imperfections from the elements of this beautiful and most comprehensive science.

If the first principles of Algebra had been consistent with themselves, or had led to no difficulties either in the reasoning immediately connected with them, or in their remoter consequences, which did not admit of a simple and uniform explanation, we should very properly hesitate before we acceded to any innovations in those principles or in their exposition: for under such circumstances, the perfect union and attachment of the parts of the fabric would furnish the best evidence of the sufficiency of the foundations: but it is the admitted existence of difficulties in the consequences of the principles of Algebra, as they are commonly stated, both immediate and remote, which naturally, and indeed necessarily, induces us to suspect the existence likewise of imperfections or inaccuracies in the principles themselves: a suspicion which becomes confirmed when it appears, after the most careful examination of them, that the difficulties in question are not referable to their imperfect developement.

Algebra has always been considered as merely such a modification of Arithmetic as arose from the use of symbolical language, and the operations of one science have been transferred to the other without any statement of an extension of their meaning and application: thus symbols are assumed to be the general and unlimited representatives of every species of quantity: the operations of Addition and Subtraction in their simple arithmetical sense, are assumed to be denoted by the signs + and −, and to be used in connecting such symbols with each other: Multiplication and Division, two inverse operations in Arithmetic, are supposed to be equally applicable to all quantities which symbols may denote, without any necessary modification of their meaning: but at the same time that the primitive assumption of such signs and operations is thus carefully limited in the extent of their signification, there is no such limitation imposed upon the extent of their application: thus it is not considered necessary that the operations of Addition and Subtraction should be confined to quantities of the same kind, or that the quantities subtracted should be less than the quantities from which they are subtracted: and when the violation of this restriction, which would appear to be rendered necessary by the primitive meaning of those operations, has led to the independent existence of the signs + and −, as an assumption which is also necessary in order to preserve the assumed universality of the values of the symbols and of the possibility of the operations which they designate, it is not considered that by this additional usage of them, we have altogether abandoned the definitions of those operations in practice, though we have retained them in name: for the consequences of those operations, and of the assumptions connected with them, must be determined by the fundamental rules for performing them, which are independent of each other, or whose necessary connection is dependent upon their assumed universality only: and the imposition of the names of Addition and Subtraction upon such operations, and even their immediate derivation from a science in which their meaning and applications are perfectly understood and strictly limited, can exercise no influence upon the results of a science, which regards the combinations of signs and symbols *only*, according to determinate laws, which are altogether independent of the specific values of the symbols themselves.

It is this immediate derivation of Algebra from Arithmetic, and the close connection which it has been attempted to preserve between those sciences, which has led to the formation of the opinion, that one is really *founded* upon the other: there is one sense, which we shall afterwards examine, in which this opinion is true: but in the strict and proper sense in which we speak of the principles of a demonstrative science, which constitute the foundation of its propositions, it would appear from what we have already stated, that such an opinion would cease to be maintainable: in order however to establish this conclusion more completely, it may be proper to exhibit at some length the successive transitions which are made from the principles and operations of Arithmetic to those of Algebra, in order to shew that their connection is not necessary but conventional, and that Arithmetic can only be considered as a Science of Suggestion, to which the principles and operations of Algebra are adapted, but by which they are neither limited nor determined.

In our first transition from Arithmetic to Algebra, we consider symbols as the general representatives of numbers, and the signs of operation and other modes of combining them as designating operations with arithmetical names and arithmetical meanings: but in the very first applications of such operations, the mere use of general symbols renders the proper limitation of their values, which is necessary in order to prevent the exhibition or performance of impossible operations or of such as have no prototypes in Arithmetic, extremely difficult and embarrassing, inasmuch as such limitations can very rarely be conveyed to the eye or to the mind by the symbols themselves: thus $a-(a+b)$ would obviously express an impossible operation in such a system of Algebra; but if $a+b$ was replaced by a single symbol c, the expression $a-c$, though equally impossible with $a-(a+b)$, would cease to express it. The assumption however of the independent existence of the signs $+$

and – removes this limitation, and renders the performance of the operation denoted by – equally possible in all cases: and it is this assumption which effects the separation of arithmetical and symbolical Algebra, and which renders it necessary to establish the principles of this science upon a basis of their own: for the assumption in question can result from no process of reasoning from the principles or operations of Arithmetic, and if considered as a generalization of them, it is not the last result of a series of propositions connected with them: it must be considered therefore as an independent principle, which is suggested as a means of evading a difficulty which results from the application of arithmetical operations to general symbols.

It is the admission of this principle, in whatever manner we are led to it, which makes it necessary to consider symbols not merely as the general representatives of numbers, but of every species of quantity, and likewise to give a form to the definitions of the operations of Algebra, which must render them independent of any subordinate science: for in the first place the symbols, whatever they denote, must be unlimited in value, and it is only by their ceasing to be abstract numbers that we shall be enabled to interpret the affections which the signs + or – (or any other signs) essentially attached to them may be supposed to express: and in the second place, in framing the definitions of algebraical operations, to which symbols thus affected are subjected, we must necessarily omit every condition which is n any way connected with their specific value or representation: in other words, the definitions of those operations ust regard the laws of their combination *only*: thus the erations denoted by + and – must regard the *affection* of symbols (with their proper signs + and –, whether accompanied or not by any other signs of affection which they are capable of receiving) by them, according to an *assumed* law for the *concurrence* of those signs: and the operations denoted by × and ÷, or equivalent modes of denoting them, must regard in the first place the result of the combinations of the symbols, and in the second place the result of the combination of the peculiar signs which belong to them. Again, in order that such operations may possess an invariable meaning and character, when the symbols with their proper signs, which are submitted to them, are the same, we shall suppose them independent of any mere accident of position, or order of succession; or, in other words, when any number of such operations are to be performed and of symbols to be combined by means of them, we shall suppose the results to be the same, in whatever order those operations succeed each other.

If we should rest satisfied with such assumed rules for the combinations of symbols and of signs by such operations, which are perfectly independent of any interpretation of their meaning, or of their relation to each other, we should retain in the results obtained all the symbols which were incorporated, without possessing the power of any further simplification: it is as a first step to effect such further reduction of the results, and in order to define the symbolical relation of pairs of those operations to each other, that we assume the operation denoted by + to be the *inverse* of that which is denoted by −, and conversely; and the operation denoted by × to be the *inverse* of that which is denoted by ÷, or conversely: or, in other words, we consider $a+b-b$ and $a-b+b$, $a\times b\div b$ or $a\div b\times b$ or $\frac{ab}{b}$ to be identical in signification with the simple symbol a.

Under such a form, the fundamental operations of Algebra are altogether symbolical, and we might proceed to deduce symbolical results and equivalent forms by means of them without any regard to the principles of any other science; and it would merely require the introduction of some such sign as = in the place of the words *algebraical result of*, or, *algebraically equivalent to*, to connect the results obtained with the symbolical representation of the operations which produce them, in order to supersede altogether the use of ordinary language.

It is at this point that the essential connexion of Algebra and Arithmetic may properly be said to commence: for a science of mere signs and symbols must terminate in the consequences of their laws of combination, unless they can be associated by interpretation, with real operations upon real magnitudes with specific representations: and it is with a view to such applications of this science that we have considered, even in the assumptions which we have hitherto made, Arithmetic or Arithmetical Algebra as the *science* of *suggestion:* that is, as the science, whose operations and the general consequences of them should serve as the guides to the assumptions which become the foundation of symbolical Algebra: thus granting or assuming in the first instance the universality of the values and of the representations of the symbols and also the independent existence of the signs + and −, assumptions to which there is nothing corresponding in Arithmetic, we assume likewise the existence of operations denoted by + and −, × and ÷ in other cases, which in anticipation of their subsequent and ordinary interpretation, we call Addition, Subtraction, Multiplication and Division: in rithmetical Algebra where signs do not exist independently, the operations of Addition and Subtraction lead to a rule for the change of signs in the quantities to be subtracted, which suggests the assumed rule for their *concurrence* in symbolical Algebra, and which to that extent defines the corresponding operations: in a similar manner, the operation of multiplication in arithmetical Algebra leads to a rule for the incorporation of the signs + and −, which suggests likewise the assumption of a corresponding rule in symbolical Algebra: it appears also in Arithmetic and arithmetical Algebra, that the result of any number of operations is the same in whatever order they are taken, and when symbols are used, in whatever order they are written: it is therefore assumed in order to preserve this accordance between the two sciences, and to render the operations themselves independent of any accident of position, that the same rule should prevail in symbolical Algebra likewise; lastly, the pairs of operations of Addition and Subtraction, Multiplication and Division, are respectively the inverse of each other in Arithmetic and arithmetical Algebra, and the relation of the corresponding operations in symbolical Algebra is determined by being defined likewise to be respectively the inverse of each other.

Having thus established the necessary identity of the results of arithmetical and symbolical Algebra, as far as this agreement can extend without violating the necessary limitations of the first of these sciences, we may legitimately assume the operations denoted by + and −, × and ÷ in symbolical Algebra, to coincide strictly in meaning with the operations of Addition and Subtraction, Multiplication and Division in Arithmetic, when the quantities which are submitted to them are arithmetical; and in their ordinary meaning are applicable to all quantities of the *same* kind, whether arithmetical or not; and inasmuch as Multiplication and Division are operations, when the multiplier or divisor is a number or numerical fraction, which are equally applicable to all quantities whatever be their nature, we may legitimately extend this interpretation of the meaning of such operations to all such cases likewise: it is only when the quantities which are subjected to such operations are of a different nature, whether that difference arises from their possessing different algebraical signs, or different specific properties independently of such signs, that we are obliged to seek for an interpretation, when possible, of the meaning of such operations and of their results, which may be in strict accordance with the different assumptions which have been made concerning them.

Thus $a+a$ will mean the double of a and $-a+(-a)$ or $-a-a$ will mean the double of $-a$, for a and a in one case, and $-a$ and $-a$ in the other, are quantities of the same kind, whatever they may denote: also $2a$ will denote $a+a$ or the double of a, and $2(-a)$ or $-2a$, will denote $-a-a$ or the double of $-a$, in conformity with the interpretation of simple numerical factors in multiplication which has been given above: in a similar manner $a+a+a+a+a$, where the same symbol is repeated five times, will mean five times a and will be denoted by $5a$ and $-a+(-a)+(-a)+(-a)+(-a)$ or $-a-a-a-a-a$, where $-a$, subjected to the sign $+$, is repeated five times, will mean 5 times $-a$, and will be denoted by $5(-a)$ or $-5a$: in a similar manner we may derive the rule for the collection of *like* terms into one: thus $5a+3a$ is equivalent to $8a$; for

$$\begin{aligned} &a+a+a+a+a+(a+a+a) \\ &=a+a+a+a+a+a+a+a=8a, \end{aligned}$$

where 8 is the arithmetical sum of the coefficients of $5a$ and $3a$; and $5a-3a=2a$; for $5a-3a$

$$\begin{aligned} &=a+a+a+a+a-(a+a+a) \\ &=a+a+a+a+a-a-a-a \\ &=a+a+a-a+a-a+a-a \\ &=a+a=2a, \end{aligned}$$

obliterating $a-a$ whenever it appears; it is not necessary to proceed further with such reductions which may be easily generalized so as to lead to the common rule which is given for effecting them.

Again, in considering the interpretation of the meaning of the result of any operation or operations, it is to the final result only that we are required to look, and not to its form or state at any intermediate part of the process which leads to it; for the connection of such successive forms is algebraically necessary, and is governed by laws which must likewise govern, and to a certain extent determine, their interpretation, and not conversely: thus the expression $-b+a$ is algebraically equivalent to $a-b$: if a and b be quantities of the same kind, and if a be greater than b, then $a-b$ admits of an immediate and simple interpretation: but it is only by a reference to this second and equivalent form that we are properly enabled to interpret the first: in a similar manner $-5a$ is algebraically equivalent to $5(-a)$, and inasmuch as the second form means *five* times $-a$, the first must or may admit likewise of the same interpretation.

If however the general laws of the combination of symbols or signs should lead to identical results from different sources, we should no longer be enabled to infer the identity of those sources from the identity of the results:—such cases present themselves continually in arithmetic, where the incorporation of the digits leads to results which bear no visible or discoverable traces of their origin; thus the number 24 may be the product of 12×2, or 8×3, or 6×4, or $2 \times 3 \times 4$, and in passing from the result to its factors there is no indication which should direct us to the selection of one of them in preference to any other: in Algebra however symbols which are not subjected to inverse operations, or which do not appear in consequence of them, are indestructible and must equally present themselves in every possible equivalent form: but the same observation does not apply to the signs, which admit of incorporation according to determinate laws, when the same sign may result from different combinations: thus $+-$ and $-+$ are replaced in all cases by the single sign $-$, and $++$ and $--$ by the single sign $+$, and similarly for other combinations of these and other signs whose existence is recognized in Algebra: it is for this reason that direct and inverse processes in Algebra are not coextensive, and ambiguities which have no existence in the first will necessarily present themselves in the second: thus $a-b$ may arise from $a+(-b)$ or $a-(+b)$: ab may arise from $+a \times +b$, or from $-a \times -b$: $-ab$ may arise from $+a \times -b$, or $-a \times +b$, and similarly in other instances which are of perpetual occurrence in Algebra: the same ambiguities must present themselves also in the corresponding interpretations, when they are to be determined from the result alone, and not from its primitive elements.

It is only when the signs of operation of Al
present themselves under circumstances which have
totypes in Arithmetic, that simple and direct intep
tions of the results can be deduced by their identity
arithmetical operations: in all other cases interpretat
when possible, are merely limited by their accordanc
non-accordance with the general laws of Algebra, and
the possibility of the symbols degenerating into arith
tical values: it is upon this principle that we inter
ab or $-ab$ when a and b are lines, or when one of t
is a line and the other an area, or when one of ther
time, and the other velocity, and so on in all other cas
it is upon this principle likewise that we shall be enab
to interpret

$$a+b\sqrt{-1} \text{ and } a-b\sqrt{-1}, \quad a(\cos\theta+\sqrt{-1}\sin$$

and similar quantities, when the symbolical laws of the si
$\sqrt{-1}$ have once been determined.

The repetition of the operation of addition has be
shewn, from the primary association of Algebra with Arit
metic, to be denoted by multiplying the symbol affect
with the sign $+$ by a number or coefficient which is equ
to the number of repetitions of the symbol itself: in
similar manner, the repetition of the operation of Multipl
cation with the same symbol may be denoted by writin
this symbol with an index equal to the number of rep
titions of the symbol in the expression written at fu
length: the simplifications of expressions which are thu
affected are in both cases equally independent of the specifi
nature of the quantity which the symbol denotes: and a
in one case, if m and n be whole numbers, then

$$ma+na=(m+n)a,$$

so likewise in the other, under the same circumstances

$$a^m \times a^n = a^{m+n}:$$

and inasmuch as in one case, the *principle of the permanence of equivalent forms* would shew that

$$ma + na = (m + n)\,a,$$

when m and n are general symbols affected with any signs whatever, so likewise in the other, the same principle, under the same circumstances, would equally shew that

$$a^m \times a^n = a^{m+n}:$$

the interpretation of the meaning of particular values of the index, whether fractional or negative, is involved in this conclusion, which becomes the principle of indices; and it becomes therefore the general principle which must not only determine the interpretation of indices when they are assumed, but must guide us conversely to the determination of the indices, which must be assumed to suit a specific interpretation.

The principle of the permanence of equivalent forms, which appears to me so important in generalizing the results of algebraical operations, must derive its authority from the view which I have taken of the principles of Algebra and of their connection with Arithmetic, considered as a science of suggestion: for in the first place, this principle assumes the operations of Algebra and their results as altogether independent of the specific values of the symbols, and equivalent forms as existing therefore whatever values such symbols may be supposed to possess, so long as they are general in form: and in the second place, it enables us to consider the equivalent forms obtained in arithmetical Algebra, where the symbols are

general in form, though specific in value, as neces
the same likewise in symbolical Algebra, if any such
exist as the result of algebraical operations, wheth
finable or not: the necessity and use of this principle
been shewn in the generalization of the principle of in
which we have noticed in the last paragraph, in pr
$(a^m)^n = a^{mn}$, in the developement of the Binomial The
when the index is a general symbol, and in establi
the general existence of equivalent series in many
cases where the operations which produce them ar
definable in words: so many occasions however have
sented themselves in the course of this Work of discu
this most comprehensive principle, both on its first
blishment and in many of its applications, that I do
feel it necessary to resume the general consideration
in this place: there is one subject, however, whic
essentially connected with it, which is of such great
portance and of such peculiar delicacy and difficulty,
I shall proceed to notice it somewhat in detail.

When an equivalent form results from the perform
of definable operations, its existence is *necessary*, as a
sequence of them: but if an equivalent form exists,
supposed to exist, when the operations which produc
are not definable, its existence is no longer *necessary*,
the sense in which mathematical necessity is commonly
derstood, in whatever manner such existence may be
sumed: thus if n be a whole number, the existence of
equivalent series for $(1 + x)^n$ is *necessary*, inasmuch as
operation which produces it may be completely define
but if n be a general symbol, we are unable to define
operation by which we pass from $(1 + x)^n$ to its equival
series, which exists therefore under such circumstances, o

in virtue of the principle of the permanence of equivalent forms: the connection between one and the other therefore only becomes necessary, when its existence is assumed: in other words, if such an equivalent series *does* exist, it *must* be the series in question, and no other; and if such a series does exist, it will be an equivalent series, in the most extended sense of the word: so far, therefore, as the consequences of the existence of such a series are concerned, it becomes a matter of indifference whether it exists by mathematical necessity or not.

Again, if we suppose u to represent any expression involving the symbol x, whose form is neither defined or exhibited, and if we suppose u' to denote the same expression, when x becomes $x+h$, then the existence of the series which is equivalent to u', or

$$u + Du \, . \, h + D^2u \, . \, \frac{h^2}{1.2} + D^3u \, . \, \frac{h^3}{1.2.3} + \text{\&c.}$$

will be necessary, when the connection of u and Du, Du and D^2u, D^2u and D^3u, &c. is necessary: but if the operations which connect these successive derivative expressions be not definable in the general form which they are supposed to possess, and the law of their connection be determinable from the application of the principle of equivalent forms only, then the existence of the series ceases to be necessary, though possible, and it is the equivalence of u' and the series which thus results, which alone can be considered as determined by a mathematical necessity; the only remaining case would be that in which the operations which connected u and Du were definable under no circumstances which could furnish a foundation for the application of the law of the permanence of equivalent forms,

when the existence of an equivalent series may or ma
be possible, and when a necessary mathematical conne
can only exist between u' and this hypothetical se
under such circumstances, the assumption of its exist
unless discoverable from other sources, must termina
general reasoning concerning it, and lead to no conclu
which do not equally belong to all series of the same

The preceding observations are important, inasn
as we are frequently obliged to reason concerning hyp
tical equivalent forms, as if they were really existing,
a view to discover some property in the forms themse
or in others derived from them, by which their possibl
necessary existence may be ascertained: the discover
such properties for such purposes constitutes a great
of the substance of the investigations and artifice
Algebra, and it becomes a question of the greatest co
quence, not merely to ascertain whether such forms e
by mathematical necessity or not, but likewise to shew
the nature of that connection will exercise no influence u
the conclusions which are founded upon them.

One of the most important consequences of this vie
the principles and operations of Algebra, is the compl
separation which it effects of the laws for the combinat
of symbols from the principles of their interpretation :
common systems of Algebra, the previous interpretati
assumed or understood, of the operations of Algebra,
termines, or is supposed to determine, the results which
obtained, and the laws of symbolical combinations:
the case is reversed in the system which I have ventur
to propose, where the laws of symbolical combinations
assumed, not arbitrarily, but with a general referen

to their anticipated interpretation in the subordinate science of arithmetic, whilst the interpretations of the results obtained are entirely determined in accordance with those laws by a reference to the specific values of the symbols: thus the interpretations of the signs + and − will be different for all different values of the symbols affected by them, and the determination of them when possible, in this and other cases, will constitute a distinct and most important subject of investigation: it being kept in mind however, that such interpretations are never mathematically necessary in any single case, though the connection of the interpretations of a series of dependent results may become so: or, in other words, the admitted interpretation of any one of them upon which the others depend may impose a mathematical necessity upon all the others in the series, but not conversely.

It is by means of such interpretations, and the principles which limit and govern them, that Algebra becomes accommodated to the form and peculiar character of every subordinate science: to Arithmetic in the first instance, as the *general science of suggestion:* to Geometry, as defining the relations of lines to each other, with respect to magnitude and position: to Mechanics and Dynamics, as defining forces, their directions and effects, whether to produce rest or motion: and similarly to every other branch of natural philosophy, which can be made to depend, by approximation, at least, upon fixed and invariable principles.

I have thought it proper to give the preceding general exposition of my views respecting the first principles of Algebra, in order to explain, and in some measure to justify, the course which I have followed in stating them:

I had imagined that it would be possible to exhibit them in a form, particularly when illustrated by a series of carefully selected examples, which would make them perfectly accessible to a Student who possessed sufficient firmness of purpose and steadiness of attention to labour cautiously through them: and I continued to indulge the same hope until a considerable part of the work was printed: I had not, however, when I formed this expectation, sufficiently considered the difficulty of uniting the characters of a controverter of old opinions and principles and of an expositor of new ones: nor had I made proper allowances for the irresistible tendency which an Author feels to adapt his arguments and reasonings, particularly upon topics of controversy, so as to satisfy the matured judgement of a reader who is familiar with the subject, rather than to guide the hesitating and uncertain steps of a novice in such studies, where every path appears dark and entangled, and where every form appears strange and unusual. But though I must abandon the hope of having reduced some of the Chapters of the following Work to a form which a Student may readily understand, yet there are many others in which he will experience no difficulty, whether he assumes and understands the first principles of Algebra, as stated in this Work, or contents himself with the more obvious and intelligible form in which they are given in common books on this subject.

The first Chapter contains the statement of the first principles of Algebra, not in their most abstract form, but modified by such references to their subsequent interpretations, as was considered requisite to make them more easily understood: the second Chapter contains the rules for performing the fundamental operations of Algebra, illustrated by a great number of examples of their application: the third Chapter contains a very lengthened exposition of the principles of Algebra in their most general form, of their connection with Arithmetic and arithmetical Algebra, of some of the most important general principles of mathematical reasoning to which they lead, and most particularly of the principles of interpretation of algebraical signs and operations: it is intended that this Chapter should be taken immediately in conjunction with the two first Chapters, in order to explain and justify the want of perfect logical sequence in some of the conclusions which would otherwise appear, in case the Articles were taken and *finally* considered in the precise order of succession in which they first present themselves.

The fourth and fifth Chapters consider the application of Algebra to the theory of numerical fractions, and to the reduction of algebraical fractions to their most simple forms, which form corresponding branches of arithmetical and symbolical Algebra: the processes in both these Chapters possess, in many cases, an analogy so very close and remarkable, as to have led to their being considered as resting upon a common foundation, and as justified by a common demonstration: of this kind are the rules for finding the greatest common measure of two numbers, and the highest common divisor of two algebraical expressions, the necessity for whose separate and independent consideration we have had particular occasion to point out.

The sixth Chapter is devoted to the complete developement of the general principle of indices, and to the reduc-

tion of expressions which involve them, whether in
diately or by interpretation, to their most simple for
the seventh Chapter is occupied with the theory of Dec
Fractions, with their generation from, and reconversion
both when finite or recurring, equivalent numerical f
tions: the eighth Chapter gives the rules for the extrac
of square and other roots in Algebra and in Arithme
together with some observations on the resolution of sym
trical and other expressions into their component fact
when such resolution is practicable without the introduct
of any other signs than + and — .

The ninth Chapter contains the exposition of
Theory of Permutations and Combinations, with
First Elements of the Theory of the Calculation
Chances: it includes also some of the most important p
positions in the Combinational Analysis, which has be
so extensively cultivated in Germany: I have not co
sidered it expedient however to enter upon the discussi
of the various *conventional* notations, the formation a
theory of which form so considerable, and, in fact,
essential a part of this species of Analysis: for though I a
not disposed to deny the comprehensive character of suc
notations, and the power which they afford of condensin
the most complicated formulæ into expressions which a
easily written, and when once understood, are easily inte
preted, yet inasmuch as they necessarily violate in
greater or less degree the general laws for the combinatio
and interpretation of algebraical quantities, they shoul
be sparingly and cautiously used, particularly in a Work
like the present, the object of which is to teach th
principles of Algebra and to follow out their consequences
according to their ordinary usage and acceptation.

The Binomial and Polynomial Theorems, with the properties of the coefficients of the resulting series, and the Theory of the Algebraical and Arithmetical Values of Series, form the subjects of the tenth Chapter. The existence and form of the binomial series, when the index is a general symbol, I have shewn, by means of the principle of the permanence of equivalent forms, to be dependent upon the general form of the series, when the index is a whole number: and I have pointed out on other occasions the necessity of this mode of proof in all cases where the peculiar connection between the primitive expression and its equivalent series was not capable of being previously defined: the theory of the arithmetical and algebraical values of series, which follows, forms one of the most embarrassing questions in Algebra; and though the distinction which I have pointed out between them is very important, yet I am by no means satisfied that I have altogether removed the difficulties which encumber this subject.

In the eleventh Chapter, I have endeavoured to point out the essential distinction which must exist between the definition of Proportion, as used in Arithmetic and in Geometry: in Arithmetic, there exists an absolute definition of ratio, which perfectly embodies the popular notions which are attached to the term, and the same definition may be assumed likewise in Algebra: it is sufficient therefore to say that a Proportion consists in the equality of ratios, in order completely to define it: in Geometry however there exists no absolute definition of a ratio, or rather there exists no geometrical mode of expressing its value; it follows therefore that in this science we must commence by an independent definition of proportion and subsequently determine our notions of ratios by asserting that are equal to each other, when they form the terms proportion.

The general views and opinions which are propose the very various subjects which are embraced in the tw Chapter are so different from those which have comm been admitted as well founded, that I feel it to be equ due to myself and to my reader to give some explana of the nature of those opinions, and of the reasons w have induced me to adopt them.

The practice of making the operations of Algebra the results obtained dependent upon a previous inter tation of their meaning, without any reference to essential generality of symbolical language, both in v and representation, leads constantly to results which ar variance with such interpretations: such results, if affe with a negative sign merely, were not considered impossi inasmuch as it was sometimes practicable to assign th a meaning, by a more or less forcible violation of primitive assumptions of the meanings of algebraical o rations; but a similar indulgence was never granted to results involving the square or any even roots of a negat quantity, inasmuch as their existence was apparently variance with the rule, whether assumed or proved, for incorporation of signs: such quantities therefore were c sidered as possessing a symbolical existence merely, in much as no really existing quantities could bear to a quantities affected with the signs + and −, the relatio which could satisfy the required conditions; and even t use of the term *impossible*, as characterizing such results h a powerful tendency to divert the attention both of autho and of readers, from any enquiries which might be directed to find a consistent interpretation of their meaning.

The first attempt which I can find of an interpretation of the meaning of such quantities was given by M. Buée in the *Philosophical Transactions* for 1806, in a Memoir which contains some original views on the use and signification of the signs of Algebra, though presented in a very vague and unscientific form: it was limited however to the interpretation of the sign $\sqrt{-1}$, as indicating *perpendicularity* in Geometry, and of its signification in one or two other cases, though accompanied by many other attempts which were either imperfect or altogether erroneous: and even in those cases in which his conclusions were correct, his reasonings were insufficient to establish them: I feel bound however to acknowledge my obligations to this paper, as having first directed my attention to this subject.

At a much later period, when the greatest part of the first three Chapters of this work were written and sent to the Press, though before I was in complete and satisfactory possession of my present views on this subject, the work of Mr. Warren* of Jesus College appeared, on the Geometrical Representation of the Square Roots of Negative Quantities; in this work which is distinguished for great originality and for extreme boldness in the use of definitions, Mr. Warren has completely succeeded in giving an interpretation of the roots of unity, when attached to symbols which denote lines in Geometry, or any quantities which such

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[*] "A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities," by the Rev. John Warren, M.A. Fellow and Tutor of Jesus College, Cambridge, 1828.

lines may represent: in so doing however he has adh strictly to the practice of all writers on Algebra, in n ing the interpretation govern the results and not the res the interpretation; thus he commences by defining *sum* of two lines making any angle with each other, tc the diagonal of the parallelogram which they include, their *difference* to be the *inverse* of the sum: and st therefore from the very proposition which is the conclus of my own researches on this subject, (Art. 513): he t proceeds to a definition of the proportion of lines to e other which are not in the same direction, which invol the determination of the meaning of the product or quoti of the signs of affection of two or more lines: he sub quently examines the consequences of these and other d nitions and shews their perfect accordance with the resu of the combinations of symbols according to the ordina definition of what is meant by *sum*, *difference*, *product* a *quotient:* his final conclusions agree generally with th which I have given, some of the more remarkable of whi I shall now proceed to notice

I commence by pointing out the distinction whi exists between arithmetical and symbolical values, ar therefore between arithmetical and symbolical roots, tl first being altogether independent of the signs of tl affection of the quantities whose roots are required: therefore the primitive and *recognized* signs of affectio be + and − or $+1$ and -1, considered as symbolica multipliers of the symbols or quantities they precede, assume $1^{\frac{1}{n}}$ or $(-1)^{\frac{1}{n}}$ or any other expressions which ar symbolically equivalent to them, to represent the appro priate signs of affection of the corresponding arithmetica roots: we shall thus no longer be confined to + and −

as the only signs of affection, inasmuch as we have created other signs, which are necessary to preserve the algebraical possibility of every operation whether of Addition, Subtraction and extraction of roots, or whether it be definable or not, to which we are conducted by the laws of symbolical combinations[*]: the theory of their origin and assumption being precisely similar to that of the independent existence of the signs + and −.

Again, in determining the explicit symbolical forms of such signs of affection, we consider the symbolical conditions which they must satisfy: thus, if $1^{\frac{1}{n}}$ be a sign of affection, it must be of such a kind that $\left(1^{\frac{1}{n}}\right)^n$ may be equal to $+1$: and whatever be the number of such forms, they are all of them equally the appropriate signs of affection, which the operation denoted by the index may signify: and inasmuch as it will be made to appear that if n be a whole number, there are n different symbolical values of $1^{\frac{1}{n}}$ and no more, there must therefore be n different symbolical values and no more of any such quantity as $a^{\frac{1}{n}}$; in a similar manner there are n different symbolical values of $(-1)^{\frac{1}{n}}$ and therefore n different symbolical values of $(-a)^{\frac{1}{n}}$.

Upon further examination it will likewise appear that any integral power of any one of these symbolical values of $1^{\frac{1}{n}}$ or $(-1)^{\frac{1}{n}}$, will reproduce a symbolical value of $1^{\frac{1}{n}}$ or $(-1)^{\frac{1}{n}}$ respectively, and therefore the symbolical values of $1^{\frac{m}{n}}$ and $(-1)^{\frac{m}{n}}$ where m is a whole number, are precisely the same in number and in form with those of $1^{\frac{1}{n}}$ and $(-1)^{\frac{1}{n}}$.

Having ascertained the symbolical properties of t signs of affection, we then proceed to their interpreta when attached to symbols with specific values, in any such interpretations can be discovered which are fectly conformable to such properties; we thus shew if a represent a line in a given position, that the diffe symbolical values of $(1)^{\frac{1}{n}}a$, when n is a whole number, represent a series of equal lines making angles equal to e other and to $\frac{1}{n}$th part of four right angles successively w each other: thus, if $n=4$, then $(1)^{\frac{1}{n}}=1$, or $\sqrt{-1}$, -1, or $-\sqrt{-1}$, and a, $\sqrt{-1}\,.\,a$, $-a$, $-\sqrt{-1}$ will denote four lines equal to a, of which a and $-a$ in opposite directions, and $\sqrt{-1}\,.\,a$ and $-\sqrt{-1}$ are in opposite directions likewise, but at right ang to the former: it is not however necessary to enter in any further statement in this place, of the interpretatio of such signs, when attached to symbols which den lines or other quantities, inasmuch as they are given great detail in the Chapter to which we are referring.

Having thus ascertained that such signs of affecti find their most appropriate interpretation in designating th positions of lines, our attention is naturally directed to th invention or determination of a sign which would designat position generally, or in other words, the position of a lin inclined at *any* angle to the direction of a primitive line for it is evident that the values of $(1)^{\frac{1}{n}}$ when n is a whol number, must correspond to determinate positions of line only, though if n be indefinitely great, there will alway exist values of $(1)^{\frac{1}{n}}$ which may correspond as nearly as w choose to any assigned position; such a sign is

$$\cos\theta + \sqrt{-1}\,\sin\theta,$$

where θ is the angle made by the line, whose position is to be designated, with a primitive line: but as the process by which I was led to the discovery and determination of this very comprehensive sign, by which Geometry is brought within almost entirely the dominion of Algebra, could not easily be made intelligible without a constant appeal to almost every step of the investigation, I feel it necessary to refer to it for that purpose.

The conclusions which follow from the introduction of this sign, are of very great importance, and are calculated to throw a new light upon the theory of the interpretation of algebraical results: and though they receive their most complete and satisfactory illustration in the position of lines and of planes, they are by no means confined to them, inasmuch as many other quantities are susceptible of affections which correspond to general or particular values of this sign: it enables us to consider, likewise, the signs + and − or + 1 and − 1, as particular symbolical values merely of a more general sign, and to shew that the operations which they denote, are merely the most simple cases of *affectations* which correspond to its different values.

The use of the sign $\cos\theta + \sqrt{-1}\,\sin\theta$, and the consideration of position as designated by it, have made it necessary for me to incorporate the science of Trigonometry with Algebra, assuming, in this instance, Geometry as the science of *suggestion*, instead of Arithmetic: we begin, therefore, with the geometrical definition of the sine and cosine of an angle, and determine their fundamental properties by means of it, and, subsequently, transfer them, by assumption, to the algebraical definition of quantities to which we give the same names: we from hence obtain the exponential expressions

$$\frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2} \text{ and } \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2\sqrt{-1}},$$

(where $\epsilon = e\sqrt{-1}$) for the cosine and sine of θ, and determine by means of them all the formulæ which may be said to constitute the science of Goniometry: we are thus enabled to confer upon such quantities a perfectly algebraical character, and also to interpret, when required, the results which are obtained, by an immediate reference to Geometry.

Inasmuch as the angle at the base of a right-angled triangle is determined by the ratio of the base or perpendicular to the hypothenuse, I have called the first the *cosine*, and the second the *sine* of the angle in question: these definitions form the proper foundation of the science of Goniometry when considered as an immediate branch of Geometry: and by considering sines and cosines as ratios and not lines, we are at once liberated from the embarrassment which is caused in formulæ which involve them by the radius of the circle to which they are commonly referred: it is by pursuing the same analogy, that we consider angles as measured by the ratio of the arc of *any* circle, which subtends them at its centre, to its radius, and, consequently, circles are merely used or introduced into this science as furnishing measures of angles: we are thus enabled to disencumber this science of the ordinary definitions of tangents and co-tangents, secants and co-secants, versed sines and suversed sines, as lines described in and about a circle in connection with the angle to which they correspond, and to consider them as merely abbreviated expressions for the ratios of the sine to the cosine, and of the cosine to the sine, for the reciprocal of the cosine and the reciprocal of the sine, for the algebraical difference or the algebraical sum of unity and the cosine: in whatever manner the formulæ of Goniometry are afterwards deduced, whether from the exponential expressions for the sine and cosine, as in Algebra, or from the fundamental formulæ for the sine and cosine of the sum and difference of two angles, as in Geometry, this modification of the definitions of the quantities which form the objects of the science, will greatly simplify the necessary investigations, and facilitate their adaptation to the purposes of calculation.

As one of the great objects of this Chapter has been to explain the nature of the connection between Geometry and Algebra, I have thought it necessary to enter into a very minute examination of the definitions and first principles of Geometry, with a view to determine the extent to which one science might be superseded by the other: some of the results of this examination are important, particularly with regard to the definitions of equality and of parallel lines: it is the definition of parallel lines which *must* be made use of in Algebra, which appears to me to furnish one of the most conclusive arguments in favour of the alteration which I have proposed.

I consider some apology due to the reader, for the introduction into a work on this subject, of an enquiry into the mathematical first principles of Statical Equilibrium: I was anxious, however, to give an example of *principles of suggestion*, as distinguished from assu and mathematical principles of reasoning, in one of branches of Natural Philosophy, and of the necessit separating them from each other: and it also furni me with an excellent illustration of my own systen Algebraic Geometry.

The thirteenth Chapter is occupied with the sub of Indeterminate Coefficients, and enters at some len likewise into the nature of the series which are or be equivalent to algebraical expressions when the nat of the operation which leads from one to the other is defined: it contains likewise, an investigation of Tayl Series, which is the foundation of the Differential Cal lus, upon principles which would appear to give a v close algebraical connection between the doctrine of lin and the method of Lagrange.

The fourteenth Chapter is devoted to the expla tion of the Theory of Logarithms, of the use and plications of Logarithmic Tables, of the reduction formulæ to Logarithmic Computation, of the series methods for calculating Logarithms, and lastly, of Theory of symbolical, as distinguished from arithmetic Logarithms; the enlarged view which I have taken the theory of signs of affection in the twelfth Chapt enables us to settle, very easily, the long disput question of the Logarithms of Negative Quantities, a to shew the particular cases in which they can poss an arithmetical logarithm.

The three last Chapters contain all that I consider myself allowed to say, in consequence of the great exte of the other parts of the Work, on the subject of Equations and their solution: I have entirely omitted the discussion of their general theory, confining myself to equations of the fourth degree, and such others as were reducible to them, in consequence of a particular relation of their coefficients: the first of these Chapters is confined to the Solution of Equations involving one unknown quantity only: the second considers the Properties of Simultaneous Equations, and gives the first elements of the theory of Elimination: the last considers the Solution of Problems which lead to equations, with a particular view to the interpretation of their roots. They may be considered as very imperfect sketches of very important subjects, and some apology is due to the Reader for the form in which they are presented to him: but I considered myself pledged to my Publishers for the early appearance of the Work, and I therefore felt myself embarrassed by the hasty manner in which I was compelled to prepare them, which did not enable me to confine all that I wished to say within the very narrow limits which were allowed to me. With this petition for indulgence with respect to a particular part of this Book, which I should most gladly extend to the whole of it, I beg leave to conclude this Preface, which, like the book itself, has already exceeded its proper limits.

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CHAP. I.

DEFINITIONS AND FIRST PRINCIPLES OF THE SCIENCE.

Definition of Algebra.

1. ALGEBRA may be defined to be, *the science of general reasoning by symbolical language.*

It is impossible however, by any simple definition, to express fully its objects and applications, which can only be clearly comprehended by a person acquainted with the science: it has been termed *Universal Arithmetic;* but this definition is defective, inasmuch as it assigns for the general object of the science, what can only be considered as one of its applications.

Of the Symbols of Algebra.

2. The symbols of Algebra may be made the representatives of every species of quantity, whether abstract or concrete: the operations to which they are subject are perfectly general, and are in no respect affected by the nature of the quantities which the symbols denote, being determined solely by the definitions and assumptions which constitute the first principles of the science.

Symbols most commonly used.

3. The symbols most commonly used are the letters of the alphabet great or small: as the choice of them is perfectly arbitrary, those are most commonly adopted, which are most easily written: in some cases, in order to exhibit to the eye the connection between the symbol and the thing signified, we make use of the initial letter of the term which designates the quantity represented: in other cases, when different quantities of the same kind are involved in an algebraical operation, we denote them, in order to indicate their connection with each other, by the same letter, with different accents, as a', a'', a''', a^{iv}, &c.; or by the same letter with numbers written underneath to the right hand, as a_1, a_2, a_3, a_4, &c.; or by the same letter in different alphabets, as a, A, α, a, &c.

Symbols of quantities considered as known or unknown.

4. In many of the operations of Algebra, it is ne sary to distinguish such quantities as are *known* and de minate, from such as are *unknown*, or whose values found by means of algebraical operations: it is usual make use of the first letters of the alphabet, such as a, b &c. to denote the former, and of the last letters of alphabet, such as v, u, x, y, &c. to denote the latter

Determinate or indeterminate.

It is also convenient sometimes to distinguish *indet minate* quantities, whose values are arbitrary and assignal at the pleasure of the operator, from such as are *det minate*, whether *known*, or *unknown:* they are common represented by the middle letters of the alphabet, such l, m, n, p, q, r, &c.

Variable and invariable.

Variable quantities, admitting of every value betwe given limits, when the variation is continuous, or of a ce tain number of such intermediate values, when the chan is discontinuous, are commonly distinguished from such are *invariable*, whether assigned or not, in the same mann as unknown quantities are distinguished from such as a known, by representing them respectively by the last an first letters of the alphabet.

Addition and Subtraction denoted by the signs + and −.

5. All quantities of the *same* kind admit of bein added to or subtracted from, each other: and the operation of addition and subtraction, which are of all others the mos used in considering the relations of quantity, are denote by the signs + and −: the first denoting Addition, an called *plus* or the *positive* sign: and the second denotin Subtraction, and called *minus* or the *negative* sign.

Further use and meaning of the signs + and −.

6. In symbols of *concrete* quantities of the *same* kind other relations besides those of greater or less, may be con sidered: thus, if the symbols represented lines, some ma represent lines drawn in one direction, and others line drawn in the direction opposite: if they represented por tions of time, some may represent time past, others time t come: if they denoted forces in the same direction, on symbol may designate a force which pushes, another a forc which pulls; and similarly, in other cases: in order tha

symbols may be capable of representing such relations, they are *in all cases* supposed to be *affected* by one of the signs $+$ or $-$, which are likewise used to denote the operations of Addition and Subtraction.

The sign $+$ is generally omitted before a symbol, which has no other symbol preceding it: but in such cases it is always understood, though not expressed, and is only suppressed in order to save trouble to the writer.

General rule of the signs $+$ and $-$.

7. Symbols may be *incorporated* into each other, so as to represent a new quantity of the same or a different kind, as in the operations of *Multiplication* and *Division:* in this case, the quantity which is the result of the operation, must have some determinate sign, dependent upon the signs of the symbols incorporated. In like manner, in *combining* symbols together by the operations of Addition and Subtraction, which are denoted by the same signs $+$ or $-$, by which the symbols themselves are affected, similar or dissimilar signs must come together, which it is expedient, in order to prevent confusion, to incorporate into one. In both cases, they are subject to the following general rule, which is *assumed* and not *proved*, and which may be considered as constituting one of the most important first principles of this science.

Whenever by the incorporation or combination of two symbols, two similar signs come together, whether $+$ *and* $+$ *or* $-$ *and* $-$, *they are replaced by the single sign* $+$: *but if the two signs are dissimilar, whether* $+$ *and* $-$ *or* $-$ *and* $+$, *they are replaced by the single sign* $-$.

Signs of the operations of *Multiplication* and *Division*.

8. The operations commonly called Multiplication and Division, are denoted by the signs $\times$ and $\div$ respectively: thus $a \times b$ means the *product* of a multiplied into b: and $a \div b$ means the *quotient* of a divided by b: it is more common to denote the first operation by interposing a dot between the symbols, as $a \, . \, b$; and still more common merely to write them consecutively as ab. The second operation is usually denoted by writing the *divisor* beneath the *dividend* with a line between them, as $\frac{a}{b}$.

It is indifferent in what order the symbols in an expression succeed each other.

9. In incorporating symbols such as a and b by the operation of multiplication, it is indifferent in what order the symbols succeed each other, whether ab or ba, though the alphabetical arrangement is most commonly adopted: the same remark applies to any number of symbols thus incorporated: and it may be assumed as a general principle, *that in all the four operations of addition, subtraction, multiplication and division, in which symbols are combined or incorporated with each other, it is indifferent in what order the symbols in the result are written, or when more operations than one are performed, in what order they succeed each other.*

Thus $a+b$ is identical with $b+a$ and $a-b$ with $-b+a$: $a+b-c$ is identical with $a-c+b$, or $b+a-c$, or $b-c+a$, or $-c+a+b$, or $-c+b+a$: abc is identical with bac, or acb, or cab, or cba or bca: a divided by b, multiplied by c, and then divided by d, or $a \div b \times c \div d$, or $\frac{a}{b} \times \frac{c}{d}$, is identical with $\frac{ac}{bd}$, or $\frac{ca}{bd}$, or $\frac{ac}{db}$, or $\frac{ca}{db}$: and similarly whatever be the number of symbols involved or of operations performed.

Inverse operations.

10. The operation of Division is the *inverse* of that of Multiplication; by which we mean, that if a symbol or quantity be first multiplied and then divided by the same symbol or quantity, or conversely, its value is not altered: thus a multiplied by b, and then divided by b, or $a \times b \div b$, or $\frac{ab}{b}$ is identical with a. In a similar sense, the operations of addition and subtraction are the inverse of each other: thus if to a we add b, and then subtract it, or conversely, its value is not altered: in other words $a+b-b$, or $a-b+b$ is identical with a.

Meaning of the terms square, cube, &c.

11. The quantity or algebraical expression $a \times a$, or aa is written a^2, and is called the *square* or the *second power* of a: the expression aaa, is written a^3, and is called the *cube* or *third power* of a: $aaaa$, where the symbol is

four times repeated, is written a^4, and is called the *fourth power* of a: and if the symbol a be repeated any number of times (n) in the expression $aaa\ldots$&c...., it is written a^n, and is called the n^{th} *power* of a, and the number n is called the *index* or *exponent* of the power.

General rule of indices.

12. When a^2 is multiplied into a^3, the result $a^2 \times a^3$ is identical with $aaaaa$ or a^5, since a is repeated 5 times in the product written at full length; the index 5 is the sum of the indices of the two factors a^2 and a^3 of the product: in the same manner, when a^4 is multiplied into a^7, the result $a^4 \times a^7$ is identical with a^{11}, since a is repeated 11 times in the product written at full length; the index 11 is the sum of the indices 4 and 7 of the two factors a^4 and a^7 of the product: more *generally*, if n and m represent any whole numbers (4), and if a^n be multiplied into a^m, the product $a^n \times a^m$, if written at full length, would exhibit the letter a repeated a number of times equal to $n + m$: it is therefore identical, upon the principle of the notation of indices stated in the last Article, with a^{n+m}: or, in other words, *the product of the* nth *power of a symbol* a, *into the* mth *power of the same symbol, is a power of* a *whose index is equal to the sum of the indices of the powers of* a, *which are incorporated together.*

Further generalization of the rule of indices.

As we have assumed an index to denote the continued product of the quantity above which it is placed, repeated as often as unity is contained in it, we may likewise assume other indices, both fractional and negative, to denote the result of other operations to which the same symbol or quantity is subjected: but in order that indices may never lead to results inconsistent with the rule already established with respect to indices which are *positive whole numbers*, we *assume* it to be true in its most general form, whatever those indices may be, whether whole numbers or fractions, positive or negative, or in short any quantities whatsoever: that is, $a^n \times a^m = a^{n+m}$, when n and m are general symbols; *or the product of any power whatever of a symbol into any other power of the same, is denoted by writing the same* *symbol with an index equal to the sum of the indices of quantities which are incorporated.*

Meaning of square, cube and other roots, and modes of denoting them.

13. The *square root* of any proposed quantity is t quantity whose *square* or *second power* gives the propo quantity as its result: it is denoted by prefixing the si $\surd$ to the quantity: thus $\surd a$ means the *square* root of $\surd a^3$ means the square root of a^3: and similarly in ot cases.

Another and more algebraical mode of denoting t square root by means of an index, is derived from t general principle of indices mentioned in the last Articl the symbol a may be considered as identical with a^1, who index is unity: if we assume therefore $a^{\frac{1}{2}}$ to represent $\surd$ the principle referred to would give us

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a:$$

the requisite condition is therefore satisfied, and $a^{\frac{1}{2}}$ co rectly represents the square root of a.

In the same manner $a^{\frac{3}{2}}$ represents $\surd a^3$, or the squa root of a^3: $a^{\frac{n}{2}}$ represents $\surd a^n$, or the square root of a^n $a^{\frac{1}{4}}$ represents $\surd a^{\frac{1}{2}}$, or $\surd \surd a$, or the square root of th square root of a: and similarly in other cases.

The *cube root* of any proposed quantity is that quantit whose cube or third power, gives the proposed quantity a its result: it is denoted by prefixing the sign $\sqrt[3]{\ }$ to th quantity: the general principle of indices however shew that $a^{\frac{1}{3}}$ is equivalent to $\sqrt[3]{a}$: for $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$ and $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$: and consequently sinc the cube of $\sqrt[3]{a}$, and of $a^{\frac{1}{3}}$, give the same quantity a fo the result, they may be considered as equally representin its cube root.

The n^{th} root of a, is that quantity which multiplied into itself n times, will produce a: it is denoted by $\sqrt[n]{a}$, or by $a^{\frac{1}{n}}$: for the result of the continued product of $a^{\frac{1}{n}}$ into itself, where $a^{\frac{1}{n}}$ is repeated n times, is represented by writing a with an index equal to n times $\frac{1}{n}$, which is equal to unity.

Meaning of the term coefficient.

14. The number which is prefixed to any symbol or expression, and by which it is multiplied according to the arithmetical meaning of the term, is called its *coefficient*: thus 6, 3 and 11 are the coefficients of $6a$, $3a^2$, and $11xy$ respectively: in the same manner, if a, b and c represent numbers, they may be considered as the coefficients of x, y^2, and xyz in the expressions ax, by^2, and $cxyz$ respectively: and one symbol is sometimes termed the coefficient of another, merely as a phrase of convenience, to distinguish one of them from the other, which may be the object of more particular consideration: thus a, b, c, d, &c. are termed the coefficients of x, x^2, x^3, x^4, &c. in the series

$$ax + bx^2 + cx^3 + dx^4 + \text{\&c.}$$

A mononomial.

15. A *mononomial* is any algebraical expression consisting of one term only; as

$$a, \ a^2, \ ab, \ -a^2b, \ -abc, \ \frac{a^3}{b^2}, \ \text{\&c.}$$

A binomial.

16. A *binomial* is any expression consisting of two terms connected by the signs + or − ; as $a+b$, $a-b$, a^2+b^2, $xy - z^2$, &c.

A trinomial, &c.

17. A trinomial consists of three terms, connected by the signs + and −, such as $a+b+c$, a^2-ab+b^2, $a^3-b^3-c^3$, $xy-xz-yz$, &c.: a *tetranomial* consists of four terms, and a *polynomial* consists of any number of terms connected together in a similar manner.

Use of vincula or brackets.

18. Compound algebraical expressions are further connected by means of *vincula* or *brackets*: thus $a+b$ is sometimes written $\overline{a+b}$ or $(a+b)$: such a connection indicates that the whole quantity beneath the *vinculum*, or between the brackets, is affected by the sign prefixed to it, whether +, −, ×, ÷, √, &c.: thus, in the expression $a-\overline{b-c}$, or $a-(b-c)$, the sign − affects both the quantities beneath the vinculum, or between the brackets, and therefore by following the general rule of the signs, it is equivalent to $a-b+c$: thus also $\overline{a+b}\times\overline{c+d}$, or

$(a+b)\,(c+d)$, means that the whole quantity $a+b$ is to be incorporated by multiplication into $c+d$.

The connection indicated by the vinculum, when not very distinctly written, is in some cases ambiguous, which is never the case with brackets, which are therefore generally adopted.

What quantities are said to be homogeneous.

19. Quantities are said to have the same dimensions when the sum of the indices of the symbols incorporated in each of their terms is the same, unity being supposed to be the index of each simple symbol such as a, b, c, &c.: such quantities are likewise said to be *homogeneous:* thus a is homogeneous with b, c, x, z: ab is homogeneous with a^2: abc is homogeneous with $a^2 b$ and a^3: $a^2 x - a x^2$ is homogeneous with $x^3 - 3xyz + z^3$, and similarly in other cases.

Quantities of one, two, or n dimensions.

20. We likewise speak of quantities as being of one, two, three or n dimensions, the dimension being measured by the sum of the indices of the symbols involved: thus a is a quantity of *one* dimension: xy and x^2 are quantities of *two* dimensions: xyz and x^3 are quantities of three dimensions, and x^n, $x^{n-1}y$, $x^{n-2}y^2$, &c. are quantities of n dimensions.

Not affected by numerical coefficients.

A numerical coefficient does not affect the dimensions of a quantity, since it alters its magnitude only, and not its nature: the same remark may be made with respect to any symbol, which is assumed to represent an abstract number.

Quantities like or unlike.

21. We speak of algebraical quantities as *like* or *unlike*, according as they involve the same or different symbols, without regarding their signs or numerical coefficients: thus $2a$ and $-3a$, $-x^2$ and $4x^2$, $5abc$ and $-7abc$, are pairs of *like* quantities: a, $-b$, $-c$, x^2, $4ab$, a^3, $7abc$, are *unlike* quantities.

Meaning of the sign $=$.

22. The sign $=$, placed between two quantities or expressions, indicates that they are equal or equivalent to each other: it may indicate the identity or absolute equality of the quantities between which it is placed: or it may shew that one quantity is equivalent to the other, that is,

if they are both of them employed in the same algebraic operation, they will produce the same result: or it may simply mean, as is not uncommonly the case, that one quantity is the result of an operation, which in the other is indicated and not performed.

Meaning of the signs > and <.

23. In order to indicate that one quantity is greater or less than another, we write them consecutively with the sign > between them in one case, and < in the other: thus $a > b$ means that a is greater than b: and $a < b$ means that a is less than b.

Other signs and terms.

24. There are other signs and terms, which we shall have occasion to use, the explanation of which may be most conveniently reserved for those parts of the subject where they first occur.

Amongst the various definitions, assumptions and propositions contained in the preceding Articles, there are many which are imperfectly stated, and which would require some acquaintance with the practice of Algebra, to make their complete developement and accurate limitation intelligible: it is on this account that the further discussion of them has been reserved for a more advanced part of the subject.

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CHAP. II.

ON THE METHODS OF COMBINING AND INCORPORATING ALGEBRAICAL QUANTITIES BY THE OPERATIONS OF ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

Addition.

25. THE operation of Addition is denoted by the sign $+$, which, when combined with the signs of each of the quantities to be added, leaves them the same as before (Art. 7.).

RULE. *Algebraical quantities, therefore, whether simple or compound, are added together by simply connecting them with their proper signs.*

Like algebraical quantities (Art. 21.), *must be collected into one term, whose coefficient will be the difference with its proper sign of the sums of the coefficients of the positive and negative terms respectively.*

In performing this operation, the quantities to be added are either written in one continued line, or placed underneath each other, as in the addition of numbers in arithmetic: the like quantities are then collected severally into one term, and the whole result written in one line.

What is meant by the *sum* of two quantities in Algebra.

26. By the *sum* of two quantities in Algebra, we mean the result of their addition to each other, according to the preceding rule: thus the sum of a and b is $a+b$: of a and $-b$, is $a+(-b)=a-b$: of $-a$ and $-b$, is $-a+(-b)=-a-b$: and similarly in other cases.

27. Examples.

$$(1)\quad \begin{array}{r} a \\ a \\ \hline 2a \\ \hline \end{array} \qquad\qquad (2)\quad \begin{array}{r} -\ a \\ -\ a \\ \hline -\ 2a \\ \hline \end{array}$$

The first of these examples might be written thus, $a + a = 2a$: or preserving all the signs, $+a + +a$: the first $+$ is omitted as being understood, without being written: the two signs $+ +$, which come together, are replaced by $+$, according to the rule (Art. 7.)

The second example $-a + -a$ is equivalent to $-a - a$, or $-2a$, the two signs $+ -$ being replaced by $-$ (Art. 7.).

$$(3)\quad \begin{array}{r} a \\ -\ a \\ \hline 0 \\ \hline \end{array} \qquad\qquad (4)\quad \begin{array}{r} -\ a \\ +\ a \\ \hline 0 \\ \hline \end{array}$$

These might be written $a - a$ and $-a + a$, respectively: and inasmuch as addition and subtraction are *inverse* operations (Art. 10.), the results are severally equal to zero.

$$(5)\quad \begin{array}{r} 3a \\ 5a \\ \hline 8a \\ \hline \end{array} \qquad (6)\quad \begin{array}{r} -3a \\ -5a \\ \hline -8a \\ \hline \end{array} \qquad (7)\quad \begin{array}{r} -3a \\ 5a \\ \hline 2a \\ \hline \end{array} \qquad (8)\quad \begin{array}{r} 3a \\ -5a \\ \hline -2a \\ \hline \end{array}$$

In the two first of these examples, the coefficients of the *like* quantities $3a$ and $5a$ have the same sign, and their arithmetical sum must, therefore, be taken with its proper sign: in the two last, the signs of $3a$ and $5a$ are different, and the arithmetical difference of their coefficients must be taken with the sign of the greater.

$$\begin{array}{lrlrlr} (9) & 3a & (10) & 3x^2 & (11) & -abc \\ & -5a & & -\ x^2 & & 12abc \\ & 7a & & -7x^2 & & 13abc \\ & -4a & & -4x^2 & & -20abc \\ \hline & a & & -9x^2 & & 4abc \end{array}$$

In these three examples, the quantities to be added are severally *like quantities* (Art. 21.): the first example is equivalent to $3a + 7a - 5a - 4a$, and, therefore, to $10a - 9a$ or a, the sum of the *positive* coefficients being 10, and of the negative coefficients being 9: in a similar manner, the second example is equivalent to $3x^2 - 12x^2$, or $-9x^2$: and the third example to $25abc - 21abc$, or $4abc$.

$$\begin{array}{lc} (12) & a + b \\ & a - b \\ \hline & 2a \end{array}$$

The symbol b disappears, being both added and subtracted (Ex. 3 and 4.); the result in this case, when expressed in words, gives the following general proposition: "If to the sum of any two quantities we add their difference, the result is equal to twice the greater."

$$\begin{array}{lclc} (13) & a+b-c & (14) & a^2+ab+b^2 \\ & a-b-c & & a^2-ab+b^2 \\ \hline & 2a-2c & & 2a^2+2b^2 \end{array}$$

The quantity b in the first of these examples, and ab in the second, disappear in the results.

(15) $$\begin{array}{r} a^3-3a^2b+3ab^2-b^3 \\ a^3+3a^2b+3ab^2+b^3 \\ \hline 2a^3+6ab^2 \\ \hline \end{array}$$

(16) $$\begin{array}{r} a-b \\ b-c \\ c-d \\ d-e \\ \hline a-e \\ \hline \end{array}$$

The quantities a^2b and b^3 in the first of these examples, and b, c, d in the second, disappear by *opposition* of signs.

(17) $$\begin{array}{r} a+b-c \\ a-b+c \\ a-b+c \\ \hline 3a-b+c \\ \hline \end{array}$$

(18) $$\begin{array}{r} 7a-5b+3c \\ 2a-3b-7c \\ a+2b+3c \\ \hline 10a-6b-c \\ \hline \end{array}$$

(19) $$\begin{array}{r} x^2-xy \\ -xy+y^2 \\ \hline x^2-2xy+y^2 \\ \hline \end{array}$$

(20) $$\begin{array}{r} x^2-7xy+5y^2 \\ xy+2x^2-3y^2 \\ \hline 3x^2-6xy+2y^2 \\ \hline \end{array}$$

The *like* (Art. 21.) quantities must in all cases be combined together, whether they appear underneath each other or not: and it is generally expedient to arrange the quantities in the result as much as possible in alphabetical order, though they may not appear so, in the expressions to be added.

(21) $$\begin{array}{l} -3a-4b+5c \\ -\ \ a+2b-3d \\ \quad\ \ 3b-4c+6e \\ \quad\ \ 7c-8d-9e \\ \hline -4a+b+8c-11d-3e \\ \hline \end{array}$$

(22)
$$\begin{array}{l}
3ab^2 - 4a^2b + a^3 \\
-\ 4ac^2 + 5ab^2 - c^3 \\
-\ 7b^3 \quad + 2a^2b - 6ac^2 \\
5a^3 \quad - 11ab^2 - 12ac^2 \\
\hline
6a^3 - 2a^2b - 3ab^2 - 7b^3 - 22ac^2 - c^3 \\
\hline
\end{array}$$

(23)
$$\begin{array}{l}
a^2 - \frac{ax}{2} + \frac{x^2}{3} \\
\frac{-a^2}{3} + \frac{ax}{4} - \frac{x^2}{5} \\
\hline
\frac{2a^2}{3} - \frac{ax}{4} + \frac{2x^2}{15} \\
\hline
\end{array}$$

In this case, we must subtract the fraction $\frac{1}{3}$ from $\frac{1}{4}$ from $\frac{1}{2}$ and $\frac{1}{5}$ from $\frac{1}{3}$, in order to get the respect coefficients of a^2, ax and x^2 in the result.

(24)
$$\begin{array}{l}
a - \frac{b}{2} + \frac{c}{3} - \frac{d}{4} \\
-\frac{a}{4} + \frac{b}{3} - \frac{c}{2} + d \\
\frac{a}{5} - \frac{b}{4} + \frac{c}{3} - \frac{d}{2} \\
\hline
\frac{19a}{20} - \frac{5b}{12} + \frac{c}{6} + \frac{d}{4} \\
\hline
\end{array}$$

In this example, we must subtract $\frac{1}{4}$ from the sum o and $\frac{1}{5}$, $\frac{1}{3}$ from the sum of $\frac{1}{2}$ and $\frac{1}{4}$, $\frac{1}{2}$ from $\frac{2}{3}$, and the s of $\frac{1}{2}$ and $\frac{1}{4}$ from 1, in order to get the respective coefficie of a, b, c and d.

28. The operation of subtraction is denoted by the sign $-$, which, when combined with the signs of each of the quantities to be subtracted, will change them all (Art. 7.). Subtraction.

RULE. *In order, therefore, to subtract one or more sets of algebraical quantities from each other, we must change the signs of all the quantities to be subtracted, and then proceed as in addition.*

In performing the operation of subtraction, we usually place the quantity to be subtracted beneath the other, as in the subtraction of numbers in arithmetic: and then *tacitly* supposing the signs of all the terms in the subtrahend to be changed, we proceed as in addition.

29. When we speak of the *difference* of two quantities in algebra, we mean the result of the subtraction of the second from the first, according to the algebraical meaning of the term: thus, the difference of a and b, is $a-b$: of a and $-b$, is $a - -b$, or $a+b$: of $-a$ and b is $-a-b$: of $-a$ and $-b$, is $-a - -b$, or $-a+b$: and similarly in other cases. What is meant by the *difference* of two quantities in Algebra.

30. When the sign $-$ is placed before a set of quantities, included between brackets, and when those brackets are removed, it must be combined with the sign of every term so included, and must consequently change them all: thus $a+b-(a-b)$ is equivalent to $a+b-a+b=2b$; $a-(-3a+7b-4c)=a+3a-7b+4c=4a-7b+4c$: $a-(x+y)=a-x-y$, and similarly in other cases. Effect of removing the *vinculum* or brackets, when preceded by a *minus* sign.

The *vinculum* is sometimes used instead of brackets: thus $a-(x+y)=a-\overline{x+y}$: if the *vinculum* in this case be produced too far towards the a, as when written thus, $a\overline{-x+y}$, the expression may be confounded with $\overline{a-x+y}$, or $(a-x+y)$; an ambiguity which may very easily arise from indistinct writing: it is on this account, that the *vinculum* is rarely used by modern algebraical writers.

Double *brackets.*

31. We sometimes use *brackets* within *brackets*, as the expression $a - \{b+(c-d)\}$: when we remove the e ternal brackets, this expression becomes $a-b-(c-$ which is equivalent to $a-b-c+d$: it may be readi seen from this example, in what manner the brackets m be removed in more complicated cases.

Examples.

32.

$$\begin{array}{llllllll} (1) & a & (2) & -a & (3) & a & (4) & -a \\ & \underline{a} & & \underline{-a} & & \underline{-a} & & \underline{a} \\ & \underline{0} & & \underline{0} & & \underline{2a} & & \underline{-2a} \end{array}$$

These examples are equivalent to $a-a$, $-a+a$, $a+$ and $-a-a$, respectively: and correspond to the followin examples in addition: (27).

$$\begin{array}{llllllll} (\alpha) & a & (\beta) & -a & (\gamma) & a & (\delta) & - \\ & \underline{-a} & & \underline{+a} & & \underline{a} & & \underline{-} \end{array}$$

$$\begin{array}{llllllll} (5) & 7a & (6) & -7a & (7) & 7a & (8) & -7 \\ & \underline{3a} & & \underline{-3a} & & \underline{-3a} & & \underline{3} \\ & \underline{4a} & & \underline{-4a} & & \underline{10a} & & \underline{-10} \end{array}$$

These examples are equivalent to $7a-3a$, $-7a+3a$ $7a+3a$ and $-7a-3a$ respectively, changing the sig of the quantity to be subtracted.

$$\begin{array}{llllllll} (9) & a+b & (10) & a-b & (11) & a+b & (12) & -a-b \\ & \underline{a-b} & & \underline{a+b} & & \underline{-a+b} & & \underline{a-b} \\ & \underline{2b} & & \underline{-2b} & & \underline{2a} & & \underline{-2a} \end{array}$$

These examples are equivalent to $a+b-(a-b)$, $a-b-(a+b)$, $a+b-(-a+b)$, and $-a-b-(a-b)$, or to $a-b-a+b$, $a-b-a-b$, $a+b+a-b$, and

$-a-b-a+b$, respectively: the quantity to be subtracted being included between brackets, with the negative sign preceding in one case, and the brackets being removed and the signs of the terms included between them changed, in the other.

$$\begin{array}{rl} (13) \quad & a-b+c-d \\ & a+b-c+d \\ \hline & -2b+2c-2d \quad \text{or } -2(b-c+d). \end{array}$$

In this case we have put down $-2(b-c+d)$ as equivalent to $-2b+2c-2d$, which will readily appear, if we remove the brackets, change the signs of the terms which they include, and multiply them by 2: such a change of form is frequently convenient, and even important, when several terms have a common coefficient, such as 2 in this case.

$$\begin{array}{rl} (14) \quad & a+b-c-d \\ & a-b-c+d \\ \hline & 2b-2d \quad \text{or } 2(b-d). \end{array}$$

In this case, $2(b-d)$ is equivalent to $2b-2d$.

$$\begin{array}{rc} (15) \quad & a^2+ax+x^2 \\ & x^2+a^2-ax \\ \hline & 2ax \end{array} \qquad \begin{array}{rc} (16) \quad & a^3+3a^2x+3ax^2+x^3 \\ & a^3-3a^2x+3ax^2-x^3 \\ \hline & 6a^2x+2x^3 \end{array}$$

Compare with these examples, Ex. 14 and 15 (Art. 27).

$$\begin{array}{rl} (17) \quad & 2(a+b)-3(c-d) \\ & a+b\ -4(c-d) \\ \hline & a+b\ +c-d \end{array}$$

In this example, we consider $a+b$, and $c-d$ as simple terms: if the brackets were entirely removed, it would stand as follows:

$$\begin{array}{r} 2a+2b-3c+3d \\ a+b-4c+4d \\ \hline \end{array}$$

$$(18) \qquad \begin{array}{r} a-\frac{3x}{2} \\ b-\frac{x}{2} \\ \hline a-b-x \\ \hline \end{array}$$

In this case, we have to subtract the coefficient $\frac{1}{2}$ from the coefficient $\frac{3}{2}$: the remainder is 1, and

$$-\frac{3x}{2}+\frac{x}{2}=-x.$$

$$(19) \qquad \begin{array}{r} a-\frac{b}{2}+\frac{c}{3} \\ 2a-b-c \\ \hline -a+\frac{b}{2}+\frac{4c}{3} \\ \hline \end{array}$$

$$(20) \qquad \begin{array}{r} \frac{a}{2}-\frac{5x}{2}-\frac{3a}{4}+\frac{x}{3} \\ 3b+\frac{11x}{4}-\frac{2a}{3} \\ \hline \frac{5a}{12}-3b-\frac{59}{12}x \\ \hline \end{array}$$

In the last example, from the sum of $\frac{1}{2}$ and $\frac{2}{3}$ we must subtract $\frac{3}{4}$ in order to get the coefficient of a: and from the sum of $\frac{5}{2}$ and $\frac{11}{4}$ we must subtract $\frac{1}{3}$ in order to get the coefficient of x: the quantities in the result are then arranged in alphabetical order.

(21) $3x-7y-(x+2y)-(4y-7x)$: or removing the brackets

$$3x-7y-x-2y-4y+7x=9x-13y.$$

Or thus: write $x+2y$ and $4y-7x$, successively beneath $3x-7y$, as in addition, their signs being previously changed.

$$\begin{array}{r} 3x-7y \\ -x-2y \\ -4y+7x \\ \hline 9x-13y \\ \hline \end{array}$$

This would be the answer to the following question: "It is required to subtract $x+2y$ and $4y-7x$ from $3x-7y$."

(22) $a+b-(2a-3b)-(5a+7b)-(-13a+2b)$
$=a+b-2a+3b-5a+7b+13a-2b$
$=7a-5b.$

Or thus:

$$\begin{array}{r} a+b \\ -2a+3b \\ -5a-7b \\ 13a-2b \\ \hline 7a-5b \\ \hline \end{array}$$

(23) $x^2+2xy+y^2-\{x^2+xy-y^2-(2xy-x^2-y$

$=x^2+2xy+y^2-x^2-xy+y^2+(2xy-x^2-y$

$=xy+2y^2+(2xy-x^2-y^2)$

$=xy+2y^2+2xy-x^2-y^2$

$=-x^2+3xy+y^2.$

In this case, we first remove the exterior brackets, a reduce the quantities which are external to those whi remain: we then remove the remaining brackets and r duce the quantities by collecting like quantities togethe and arranging them in the last result in alphabetical ord

(24) $a-\left(a+b-\{a+b+c-(a+b+c+d)\}\right)$

$=a-a-b+\{a+b+c-(a+b+c+d)\}$

$=-b+\{a+b+c-(a+b+c+d)\}$

$=-b+a+b+c-(a+b+c+d)$

$=a+c-(a+b+c+d)$

$=a+c-a-b-c-d$

$=-b-d,\quad \text{or} -(b+d).$

In this case, we have a triple set of brackets, which ar successively removed, the quantities external to them being previously reduced to their most simple form.

Multiplication.

33. The method of denoting the operation of multiplication has been explained in Art. 8.

It will be convenient to consider three cases of this operation, according as the quantities to be incorporated by means of it, consist of one or more terms.

Case 1.

34. When the quantities to be incorporated are monomials, (15).

In determining the product of two or more algebraical quantities, each consisting of one term, we must first consider its *sign*; secondly, its *numerical coefficient*; and thirdly, the *symbols* involved in it.

Rule. *The sign of the product is determined by the general principle stated in Art.* 7.

The numerical coefficient is found by multiplying the numerical coefficients of all the quantities together, as in common arithmetic.

To this coefficient with its proper sign we must subjoin the symbols in alphabetical order, incorporating the same letter when it occurs more than once or powers of the same letter, by the general principle of indices, which is given in Art. 12.

35. (1) $a \times -b = ab$. (2) $-a \times -b = ab$. Examples.

(3) $a \times -b = -ab$. (4) $-a \times b = -ab$.

These four results express the *rule of signs* as applied to the *incorporation* of algebraical quantities by the operation of multiplication.

(5) $3a \times 5b = 15ab$. (6) $-7a \times -9b = 63ab$.

(7) $2x \times -11y = -22xy$. (8) $-13x \times 15y = -195xy$.

In these examples, we first determine the sign, secondly, the numerical coefficient, and, lastly, the product of the symbols.

$$(9) \qquad \frac{1}{3}x \times \frac{4}{5}y = \frac{4}{15}xy.$$

$$(10) \qquad \frac{7}{8}z \times -\frac{3}{11}u = -\frac{21}{88}uz.$$

The fractional coefficients $\frac{1}{3}$ and $\frac{4}{5}$ in the first example, and $\frac{7}{8}$ and $\frac{3}{11}$ in the second, are multiplied together as in the multiplication of fractions in common arithmetic.

(11) $$-\frac{4}{5}xy \times -\frac{3}{16}xyz = \frac{3}{20}x^2y^2z.$$

The product of the fractions $\frac{4}{5}$ and $\frac{3}{16}$ is reduced its lowest terms: the symbols x and y severally oc twice in the product, and their products are theref denoted by x^2 and y^2, (12).

(12) $$\frac{11}{12}ab^2c^3 \times -\frac{13}{44}a^2b^3c^4 = -\frac{13}{48}a^3b^5c^7.$$

The product of $\frac{11}{12}$ and $\frac{13}{44}$, or $\frac{143}{528}$, reduced to lowest terms is $\frac{13}{48}$: the product of a and a^2 is equivalc to a^3: of b^2 and b^3 is equivalent to b^5: and of c^3 and is equivalent to c^7 (12).

(13) $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$: the meaning of this notation a the principle upon which this result is founded, a explained in Art. 11.

(14) $$\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb} = \frac{a^2}{b^2}.\quad \text{(Art. 12.)}$$

(15) $$\frac{a}{b} \times \frac{a^2}{b^2} = \frac{a \times a^2}{b \times b^2} = \frac{a^3}{b^3}.\quad \text{(Art. 12.)}$$

(16) $\frac{a}{b} \times b = \frac{ab}{b} = a$: in this case a is both multiplied and divided by the same symbol, and therefore it value is not altered. (Art. 11.)

(17) $\frac{a}{b} \times \frac{b}{c} = \frac{ab}{bc} = \frac{a}{c}$: for $\frac{ab}{b} = a$: and therefor $\frac{ab}{bc} = \frac{a}{c}$.

It follows from hence, that a symbol which is common to the numerator and denominator of an expression under a fractional form, may be struck out without altering its value or signification.

(18) $\frac{ax}{b} \times \frac{1}{x} = \frac{ax}{bx} = \frac{a}{b}.$

(19) $x^2 \times \frac{1}{x} = \frac{x^2}{x} = \frac{xx}{x} = x.$

(20) $\frac{a}{b} \times \frac{b^2}{c^2} = \frac{ab^2}{bc^2} = \frac{abb}{bc^2} = \frac{ab}{c^2}.$

(21) $\frac{ax}{y^2} \times \frac{by}{cx^3} = \frac{abxy}{cx^2y^2} = \frac{abxy}{cxyxy} = \frac{ab}{cxy}.$

(22) $\frac{a^2x^3}{b^3y^2} \times \frac{cy^3}{ax^4} = \frac{a^2cx^3y^3}{ab^3x^4y^2} = \frac{acx^3y^2y}{b^3x^3y^2x} = \frac{acy}{b^3x}.$

The results in these five last examples are explained upon the principle mentioned in Example 17.

(23) $3a \times -5b \times -7c = 105abc.$

We first determine the sign, which is +, since the sign of the product of the two first factors is −; secondly, the coefficient by multiplying 3, 5 and 7 together; and lastly, the literal product, which is abc.

(24) $3a \times -4ab \times -5abc \times -6abcd$
$= -360a^4b^3c^2d.$

(25) $\frac{1}{a} \times \frac{3ac}{x} \times \frac{c}{x} = \frac{3ac^2}{ax^2} = \frac{3c^2}{x^2}$; since a is common to the numerator and denominator of the expression $\frac{3ac^2}{ax^2}$.

(26) $\frac{a^2}{2x^2} \times -\frac{ax}{by} \times -\frac{b^2xy}{c^2z} = \frac{a^3b^2x^2y}{2bc^2x^2yz} = \frac{a^3b}{2c^2z}$: since bx^2y is common to the numerator and denominator of $\frac{a^3b^2x^2y}{2bc^2x^2yz}$.

(27) $x^{\frac{1}{2}} \times x^{\frac{1}{3}} = x^{\frac{1}{2}+\frac{1}{3}} = x^{\frac{5}{6}}$: for when any powers whatever of the same symbol are multiplied together, the result is found by writing the symbol with an index equal to the sum of the indices of the factors (Art. 12, 13).

(28) $x^{\frac{1}{2}} \times x^{\frac{1}{3}} \times x^{\frac{1}{4}} = x^{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = x^{\frac{13}{12}}$: upon the principle stated in the last Example.

We shall devote a separate chapter to the consideration of indices and the incorporation of quantities in which they are involved: the examples just given and an attention to the general principle stated in Art. (12, 13.) will be quite sufficient to enable us to treat them in ordinary cases.

We have taken it for granted that the operation denoted by ×, is identical with arithmetical multiplication, when the quantities incorporated are numerical: we shall consider the relations of the operations of multiplication in Algebra and Arithmetic more at length in the next chapter.

Case 2. 36. When one of the quantities to be incorporated consists of more than one term.

Rule. *Multiply successively the mononomial or mononomials into every term of the polynomial, and connect the results together with their proper signs.*

Examples. 37. (1) $a \times (a + b) = a(a + b) = a^2 + ab$.

(2) $a(b + c) = ab + ac$.

(3) $xy(x - y) = x^2y - xy^2$.

(4) $\frac{a}{b}\left(\frac{a}{b} - \frac{b}{a}\right) = \frac{a^2}{b^2} - \frac{a\,b}{a\,b} = \frac{a^2}{b^2} - 1.$

(5) $a\,b\left(\frac{a}{b} + \frac{b}{a}\right) = \frac{a^2 b}{b} + \frac{a\,b^2}{a} = a^2 + b^2.$

(6) $a^2 x^2 \times -(a^2 - a\,x + x^2) = -a^4 x^2 + a^3 x^3 - a^2 x^4.$

(7) $a^{\frac{1}{2}} x^{\frac{1}{2}} \{a - a^{\frac{1}{2}} x^{\frac{1}{2}} + x\} = a^{\frac{3}{2}} x^{\frac{1}{2}} - a\,x + a^{\frac{1}{2}} x^{\frac{3}{2}}.$

For the product of $a^{\frac{1}{2}}$ and a is $a^{1+\frac{1}{2}}$ or $a^{\frac{3}{2}}$, and of $x^{\frac{1}{2}}$ and x is $x^{1+\frac{1}{2}}$ or $x^{\frac{3}{2}}$. (12, 13.)

(8) $\sqrt{a}\,(\sqrt{a} - \sqrt{x}) = a^{\frac{1}{2}}(a^{\frac{1}{2}} - x^{\frac{1}{2}}) = a - a^{\frac{1}{2}} x^{\frac{1}{2}}.$

In this case, we replace $\sqrt{a}$ and $\sqrt{x}$ by the equivalent quantities $a^{\frac{1}{2}}$ and $x^{\frac{1}{2}}$ (13).

(9) $a \times -a\,b \times -(a\,b^2 - a^2 b)$

$= a^2 b\,(a\,b^2 - a^2 b) = a^3 b^3 - a^4 b^2.$

(10) $-a^2 x \times -a\,x^2 \times -(a^2 + a\,x + x^2)$

$= -a^3 x^3 (a^2 + a\,x + x^2)$

$= a^5 x^3 - a^4 x^4 - a^3 x^5.$

(11) $\frac{a^2}{b^2} \times -a\,b \times -\frac{b^3}{a^3} \times \left(\frac{a}{b} - 2 + \frac{b}{a}\right)$

$= b^2 \times \left(\frac{a}{b} - 2 + \frac{b}{a}\right) = a\,b - 2\,b^2 + \frac{b^3}{a}.$

(12) $a^{\frac{3}{2}} x^{\frac{3}{2}} \times -a\,x \times -a^{\frac{1}{2}} x^{\frac{1}{2}} (a^{\frac{1}{2}} x^{\frac{1}{2}} - a - x)$

$= a^3 x^3 (a^{\frac{1}{2}} x^{\frac{1}{2}} - a - x) = a^{\frac{7}{2}} x^{\frac{7}{2}} - a^4 x^3 - a^3 x^4.$

In this case $a^3 \times a^{\frac{1}{2}} = a^{3+\frac{1}{2}} = a^{\frac{7}{2}}$ and $x^3 \times x^{\frac{1}{2}} = x^{3+\frac{1}{2}} = x^{\frac{7}{2}}$. (12.)

Case 3. 38. When two or more of the quantities to l incorporated have more than one term.

Rule. *Multiply successively each term of on polynomial into every term of the other, and add tl resulting products together.*

If there are three factors to be incorporated, w must multiply the third into the product of the othe two: and so on, whatever be the number of factor which are to be multiplied together.

In performing this operation, we generally plac the factors underneath each other, as in the multiplication of numbers in arithmetic, and commence wit the terms on the right hand: the PARTIAL *products ar then arranged underneath each other,* LIKE *terms bein placed as much as possible under* LIKE *terms, in orde to make their subsequent addition more easy and expeditious.*

This is the general rule for the multiplication o compound algebraical quantities: in many cases howeve there are methods by which the process may be greatly shortened, some of which will be noticed amongst the examples which follow:

Examples. 39. (1) Multiply $a + b$ into $a + b$.

$$\begin{array}{lll} & a + b & \\ & a + b & \\ \hline \text{Multiply } a+b \text{ by } a & a^2 + a\,b & \\ \text{Multiply } a+b \text{ by } b & \quad + a\,b + b^2 & \\ \hline & a^2 + 2\,a\,b + b^2 & \\ \hline \end{array}$$

Placing the like terms $(a b)$ underneath each other.

This is the square of $a+b$ (Art. 11), and the result may be expressed in words, as follows: "The square of the sum of two quantities is equal to the sum of the squares of the two quantities, together with twice their product."*

(2) To find the square of $a+b+c$.

$$\begin{array}{lrrrrrrr}
 & a & +b & +c & & & & \\
 & a & +b & +c & & & & \\
\hline
\text{Multiply } a+b+c \text{ by } a & a^2 & +ab & +ac & & & & \\
\text{.........} \ a+b+c \text{ by } b & & +ab & & +b^2 & +bc & & \\
\text{.........} \ a+b+c \text{ by } c & & & +ac & & +bc & +c^2 & \\
\hline
 & a^2 & +2ab & +2ac & +b^2 & +2bc & +c^2 &
\end{array}$$

$$a^2+2ab+b^2+2ac+2bc+c^2$$

Or $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$.

Mode of forming the square of a trinomial from that of a binomial.

The following process of reasoning would shew in what manner this result may be deduced from that obtained in the last Example.

In the first place $(a+b+c)^2 = \{(a+b)+c\}^2$: the result required involves therefore $(a+b)^2$, or the square of the first term $(a+b)$, and therefore it involves $a^2+2ab+b^2$.

In the second place, the result required must consist entirely of quantities, such as a^2 and ab, or the squares and products of a, b and c: for the process of multiplication in this case can produce quantities of two dimensions only.

In the third place, $(a+b+c)^2 = (a+c+b)^2$ $=(b+a+c)^2=(b+c+a)^2=(c+a+b)^2=(c+b+a)^2$ (Art. 9): in other words the result is the same, in whatever order a, b, c succeed each other; it must therefore be

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* Vide Euclid, Book II. Prop. 4.

symmetrical with respect to each of those quantities, they must be equally and similarly involved, so that change may take place in the result by their interchan with each other.

One term of the result is a^2: b^2 and c^2 must be equall so, otherwise it would not be *symmetrical* with respe to a, b, c.

One term of the result is $2ab$: therefore $2ac$ an $2bc$ must be equally so, in order that it may b *symmetrical* with respect to a, b and c: suppress one o them or alter their coefficients and it is no longer sc The result can involve no negative term; we may fron hence conclude that

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

Method of forming the square of any number of terms.

The same reasoning leads necessarily to the following rule for the formation of the square of any number of terms connected by the sign +.

The square consists of the sum of the squares of all the terms, together with the sum of twice the several products which are formed by multiplying those terms two and two with each other.

Thus, $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$.

Also, $(a+b+c+d+e)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + 2ab + 2ac + 2ad + 2ae + 2bc + 2bd + 2be + 2cd + 2ce + 2de$.

In forming the products, it is important, in order that none may be omitted, to adhere strictly to the alphabetical order: multiplying $2a$ into b, c, d, e, which follow: secondly, $2b$ into c, d, e: thirdly, $2c$ into d, e: and, lastly, $2d$ into e.

(3) $(a + ax + ax^2)^2 = a^2 + a^2x^2 + a^2x^4 + 2a^2x + 2a^2x^2 + 2a^2x^3 = a^2 + 2a^2x + 3a^2x^2 + 2a^2x^3 + a^2x^4$

$$= a^2(1 + 2x + 3x^2 + 2x^3 + x^4).$$

In this case, we form the square of the trinomial $a + ax + ax^2$, by the general rule investigated in the last Example; first writing down the squares of the several terms, and then twice their products two and two successively: the next step is to arrange the terms according to the powers of x, collecting also like terms into one: the last form (whose identity with the one before it may be ascertained by multiplying a^2 into the expression between the brackets), is shorter, and therefore generally more convenient than the one before it, where the same quantity a^2 is repeated five times.

(4) $(1 + 2x + 3x^2 + 4x^3)^2 = 1 + 4x^2 + 9x^4 + 16x^6 + 4x + 6x^2 + 8x^3 + 12x^3 + 16x^4 + 24x^5 = 1 + 4x + 10x^2 + 20x^3 + 25x^4 + 24x^5 + 16x^6$.

(5) To find the square of $a - b$.

$$\begin{array}{lr}
 & a - b \\
 & a - b \\
\hline
\text{Multiply } a \text{ into } (a - b) & a^2 - ab \\
\text{..........} -b \text{ into } (a - b) & - ab + b^2 \\
\hline
 & a^2 - 2ab + b^2
\end{array}$$

This result may be deduced from the law of formation of the square of $a + b$ which is given in Ex. 1: it is merely necessary to put $a - b$ under the equivalent form $a + - b$, or for greater distinctness, under the form $a + (- b)$: we thus get $(a - b)^2 = \{a + (- b)\}^2 = a^2 + 2a(-b) + (-b)^2 = a^2 - 2ab + b^2$; since $2a(-b) = -2ab$, and $(-b)^2 = -b \times -b = b^2$.

Mode of forming the square of $a-b$ from that of $a+b$.

Upon a similar principle, we may find the square $(a-b-c)$ from that of $a+b+c$: for $(a-b-$
$= \{a+(-b)+(-c)\}^2 = a^2+(-b)^2+(-c)^2+2$
$(-b)+2a(-c)+2(-b)(-c) = a^2+b^2+c^2-2$
$-2ac+2bc$.

(6) $(a-bx+cx^2-dx^3)^2 = a^2+b^2x^2+c^2x^4+d^2$
$-2abx+2acx^2-2adx^3-2bcx^3+2bdx^4-2cd$
$= a^2-2abx+(2ac+b^2)x^2-2(ad+bc)x^3+(2bd+c^2)$
$-2cdx^5+d^2x^6$.

The square required is formed from the law of formation of $(a+bx+cx^2+dx^3)^2$, those products bein negative which involve one *only* of the two negativ terms bx and dx^3: the last result is arranged accordin to the powers of x, considering the other symbols as coefficients (Art. 14), and therefore treating those term which involve the same power of x as *like terms*, an consequently collecting them into one.

(7) Multiply $a+b$ into $a-b$.

$$\begin{array}{lll} a+b & & \\ a-b & & \\ \hline a^2+ab & & \\ \quad\;\; -ab & -b^2 & \\ \hline a^2-b^2 & & \end{array}$$

Rule for forming the product of the sum and difference of two quantities.

This result expressed in words gives the following proposition: "The product of the sum and difference of two quantities, is equal to the difference of their squares."

This result affords an easy and rapid method of forming the product of two expressions, which only differ from each other in the signs of one or more of their terms: the rule is as follows: "From the square of the sum of those terms which have the same signs in both factors, subtract the square of the sum of those terms whose signs are different: the result is the product required."

The following are examples:

(α) To find the product of $a+b+c$ and $a+b-c$.

The factors are the sum and difference of $a+b$ and c.

The product $=(a+b)^2-c^2=a^2+2ab+b^2-c^2$.

(β) To find the product of $a-b+c$ and $a+b-c$.

The factors are the sum and difference of a and $b-c$: for if we add them, we get $a+b-c$, and if we subtract them, we get $a-(b-c)$ or $a-b+c$.

Their product is $=a^2-(b-c)^2=a^2-(b^2-2bc+c^2)$ $=a^2-b^2+2bc-c^2$.

(γ) To find the product of $a-b+c-d$, and $a+b-c-d$.

The factors are the sum and difference of $a-d$ and $b-c$;

$$\begin{aligned}\therefore\ (a-b+c-d)\ (a+b-c-d) &= (a-d)^2-(b-c)^2\\ &=a^2-2ad+d^2-b^2+2bc-c^2\\ &=a^2-b^2-c^2+d^2-2ad+2bc.\end{aligned}$$

(δ) To find the product of a^2+ax+x^2 and a^2-ax+x^2.

The factors are the sum and difference of a^2+x^2 and ax,

$$\begin{aligned}\therefore\ (a^2+ax+x^2)\ (a^2-ax+x^2) &= (a^2+x^2)^2-(ax)^2\\ &=a^4+2a^2x^2+x^4-a^2x^2=a^4+a^2x^2+x^4.\end{aligned}$$

(ε) To find the product of $a^3+2a^2x+2ax^2+x^3$ and $a^3-2a^2x+2ax^2-x^3$.

The factors are the sum and difference of $a^3 + 2a$ and $2a^2x + x^3$;

$$\begin{aligned}
\therefore\ (a^3 + 2a^2x + 2ax^2 + x^3)\ (a^3 - 2a^2x + 2ax^2 - x^3 \\
&= (a^3 + 2ax^2)^2 - (2a^2x + x^3)^2 \\
&= a^6 + 4a^4x^2 + 4a^2x^4 - (4a^4x^2 + 4a^2x^4 + x^6) \\
&= a^6 + 4a^4x^2 + 4a^2x^4 - 4a^4x^2 - 4a^2x^4 - x^6 \\
&= a^6 - x^6.
\end{aligned}$$

The same product would arise from multiplying $+ na^2x + nax^2 + x^3$ into $a^3 - na^2x + nax^2 - x^3$, or $a^3 + a$ into $a^3 - x^3$, or $a^2 - x^2$ into $a^4 + a^2x^2 + x^4$, or $a - x$ int $a^5 + a^4x + a^3x^2 + a^2x^3 + ax^4 + x^5$, or $a + x$ into $a^5 - a^4$ $+ a^3x^2 - a^2x^3 + ax^4 - x^5$.

(8) Multiply $x + a$ into $x + b$.

$$\begin{array}{l}
x + a \\
x + b \\
\hline
x^2 + ax \\
\qquad + bx + ab \\
\hline
x^2 + ax + bx + ab, \quad \text{or } x^2 + (a + b)\,x + ab.
\end{array}$$

The result is arranged according to powers of x, and consequently ax and bx being considered as *like* quantities are collected into one term, which is $(a + b)\,x$.

The following three results are connected with the above and deducible from it, by a change of the signs of a and b, in the factors and their corresponding products.

(a) $(x - a)\ (x - b) = x^2 - ax - bx + ab$

$$\begin{aligned}
&= x^2 - (a + b)\ x + ab: \text{ since } -ax - bx \\
&= -(a + b)\,x.
\end{aligned}$$

(β) $(x+a)\ (x-b) = x^2 + ax - bx - ab$

$= x^2 + (a-b)\,x - ab$: since $ax - bx = (a-b)\,x$.

(γ) $(x-a)\ (x+b) = x^2 - ax + bx - ab$

$= x^2 - (a-b)\,x - ab$: since $-ax + bx = -(a-b)\,x$.

These four results merit particular attention, from their importance in the theory of quadratic equations. They furnish also the law of formation of all products like the following.

(δ) $(x+3)\ (x+5) = x^2 + 8x + 15$.

(ε) $(x-4)\ (x-11) = x^2 - 15x + 44$.

(ζ) $(x+7)\ (x-1) = x^2 + 6x - 7$.

(η) $(x-10)\ (x+9) = x^2 - x - 90$.

9. To find the cube of $a+b$,

$$\begin{array}{l}
a+b \\
a+b \\
\hline
a^2+2ab+b^2 \quad = (a+b)^2.\ \text{Ex. 1.} \\
a+b \\
\hline
a^3+2a^2b+ab^2 \\
\quad\ +a^2b+2ab^2+b^3 \\
\hline
a^3+3a^2b+3ab^2+b^3 \quad = (a+b)^3. \\
\hline
\end{array}$$

In this case $(a+b)^3 = (b+a)^3$, and the result is *symmetrical* with respect to a and b.

Formation of $(a+b+c)^3$.

In a similar manner $(a+b+c)^3$ must be *symmetrical* with respect to a, b, c: and since $(a+b+c)^3 = \{(a+b)+c\}^3$, the result involves $(a+b)^3$ and therefore

a^3 and $3a^2b$: it consequently involves b^3 and c^3, and ever different form of $3a^2b$ which can arise from putting a i the place of b or c, b in the place of a and c, and c i the place of a and b: it also involves $3(a+b)^2c$, an therefore $6abc$: consequently

$$(a+b+c)^3 = a^3+b^3+c^3+3a^2b+3a^2c+3ab^2$$
$$+3ac^2+3b^2c+3bc^2+6abc.$$

$$\begin{aligned}(a)\quad (a-b)^3 &= \{a+(-b)\}^3\\ &= a^3+3a^2(-b)+3a(-b)^2+(-b)^3\\ &= a^3-3a^2b+3ab^2-b^3;\ \text{since } (-b)^2=b^2 \text{ and}\\ &\quad (-b)^3=-b^3.\end{aligned}$$

The cube of $(a-b)$ differs from the cube of $a+b$, merely in having the signs of those terms negative, which involve b and its odd powers.

(10) To find the fourth power of $(a+b)$.

$$\begin{array}{ll} (a+b)^3 = & a^3+3a^2b+3ab^2+b^3 \\ & a+b \\ \hline & a^4+3a^3b+3a^2b^2+ab^3 \\ & \quad +a^3b+3a^2b^2+3ab^3+b^4 \\ \hline (a+b)^4 = & a^4+4a^3b+6a^2b^2+4ab^3+b^4. \\ \hline \end{array}$$

By the same reasoning as in the last example we shall find

$$(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4.$$

Formation of $(a+b+c)^4$

By a similar process of reasoning to that which was employed in deducing the cube of $a+b+c$ from that of $a+b$, we should find the fourth power of $a+b+c$ from that of $a+b$: thus $(a+b+c)^4 = a^4+b^4+c^4+4a^3b+4a^3c+4ab^3+4ac^3+4b^3c+4bc^3+6a^2b^2+6a^2c^2+6b^2c^2+12a^2bc+12ab^2c+12abc^2$.

(11) $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

Laws observable in the formation of powers of $a+b$.

It is evident from the mode of forming the powers of $a+b$, that the number of terms in the result will exceed by unity the index of the binomial: for there are two terms in $a+b$, three in $(a+b)^2$, four in $(a+b)^3$, and so on, one term being added by every successive multiplication by $a+b$.

It is likewise evident that the numerical coefficients of the terms taken from the beginning and the end, are identical with each other: for otherwise, if we put a in the place of b, and b in the place of a, the result would no longer be the same as before, which it ought to be, since a and b are equally involved.

The odd powers of b are involved in the second, fourth, and other terms, whose places are designated by even numbers: the signs, therefore, of the *even* terms only will be changed, when we put $-b$ in the place of b.

The law of formation of any power of $a+b$, such as $(a+b)^m$, constitutes the celebrated *binomial theorem*, and will form the subject of a lengthened discussion in a subsequent chapter.

(12) To find the product of $x+a$, $x+b$, and $x+c$.

$(x+a)(x+b)=x^2+(a+b)x+ab$. Ex. 6.

$$\begin{array}{l}
x+c \\
\hline
x^3+(a+b)x^2+abx \\
\qquad\qquad +cx^2+(a+b)cx+abc \\
\hline
\end{array}$$

$$+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.$$

In deducing this last result, we must observe that $(a+b)$

$x^2 + cx^2 = \{(a+b)+c\}x^2 = (a+b+c)x^2$: and also th
$abx + (a+b)cx = \{ab + (a+b)c\}x = (ab+ac+bc)$
the last form being adopted as being more symmetric
than the one before it.

By attending to the form of this product, we a
enabled without much difficulty to form the product
$x+a$, $x+b$, $x+c$, and $x+d$; for it is evident tha
the result must be symmetrical with respect to a, b, c,
The first term of the product is clearly x^4. It also involve
ax^3, and therefore bx^3, cx^3, and dx^3; and consequentl
the second term is $(a+b+c+d)x^3$. It involves abx
and therefore every other combination of a, b, c, d, take
two and two: the third term is therefore $(ab+ac+a$
$+bc+bd+cd)x^2$. It involves $abcx$, and therefor
every other similar combination of a, b, c, d: the fourt
term is therefore $(abc+abd+acd+bcd)x$: and th
last term is $abcd$; we thus get the complete product
or $(x+a)(x+b)(x+c)(x+d)$

$$=x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x$$
$$+(abc+abd+acd+bcd)x+abcd.$$

By a similar process of reasoning, we should find

$$(x+a)(x+b)(x+c)(x+d)(x+e)$$
$$=x^5+(a+b+c+d+e)x^4$$
$$+(ab+ac+ad+ae+bc+bd+be+cd+ce+de)x^3$$
$$+(abc+abd+abe+acd+ace+ade+bcd+bce+bde$$
$$+cde)x^2+(abcd+abce+abde+acde+bcde)x$$
$$+abcde.$$

Law of formation of product of $x+a$, $x+b$, $x+c$, &c.

If we venture to generalize the law of formation of these products of binomial factors with the same first term, such as $x+a$, $x+b$, &c. we should find the first term would be a power of x, whose index is the number of factors, and that the successive inferior powers of x

would appear in the following terms: the coefficient of the second term would be the sum of the second terms of the binomial factors; the coefficient of the third term would be the sum of their products two and two; the coefficient of the fourth term the sum of their products three and three; and so on to the last term, which is their continued product. The formal demonstration of this proposition, which is of great importance in the theory of equations, will be given in a subsequent chapter.

(α) $$(x-a)\,(x+b)\,(x+c) \\ = x^3-(a-b-c)x^2-(ab+ac-bc)\,x-abc.$$

This result is deducible from Ex. 12, by making all those terms in that result negative in which a appears: we then change

$$(-a+b+c) \text{ into } -(a-b-c) \text{ and } (-ab-ac+bc) \text{ into } -(ab+ac-bc).$$

(β) $$(x+3)\,(x+5)\,(x+7)=x^3+15x^2+71x+105.$$

This is deduced from the general case: the coefficient of the second term being $3+5+7=15$: of the third term, being $3\times5+3\times7+5\times7=71$: and the last term being $3\times5\times7=105$.

(γ) $$(x-10)\,(x+1)\,(x+4)=x^3-5x^2-46x-40.$$

In this case, we have

$$\begin{aligned} &-10+1+4 &&=-5 \\ &-10\times1-10\times4+1\times4 &&=-46 \\ &-10\times1\times4 &&=-40. \end{aligned}$$

(δ) $$(x-4)\,(x-6)\,(x+10)=x^3-76x+240.$$

In this case, we have

$$\begin{aligned} &-4-6+10=0 \\ &-4\times-6-4\times10-6\times10=-76 \\ &-4\times-6\times10=240. \end{aligned}$$

(e) $(x+2)(x+6)(x+10)(x+14)$
$= x^4 + 32x^3 + 344x^2 + 1408x + 1680.$

In this case, we have

$$2 + 6 + 10 + 14 = 32,$$

$$2 \times 6 + 2 \times 10 + 2 \times 14 + 6 \times 10 + 6 \times 14 + 10 \times 14 = 344$$

$$2 \times 6 \times 10 + 2 \times 6 \times 14 + 2 \times 10 \times 14 + 6 \times 10 \times 14 = 1408$$

$$2 \times 6 \times 10 \times 14 = 1680.$$

(13) $(x-a)(x^2 + ax + a^2) = x^3 - a^3.$

(14) $(x-a)(x^3 + ax^2 - a^2x + a^3) = x^4 - a^4.$

(15) $(x+a)(x^4 - ax^3 + a^2x^2 - a^3x + a^4) = x^5 + a^5.$

(16) $(x-a)(x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5)$
$= x^6 - a^6.$

(17) $(x-1)\{x^4 - (p-1)x^3 + (q-p+1)x^2 - (p-1)$
$x + 1\}$
$= x^5 - px^4 + qx^3 - qx^2 + px - 1.$

(18) $(a^2 + b^2 + c^2 - ab - ac - bc)(a+b+c)$
$= a^3 + b^3 + c^3 - 3abc.$

(19) $(a+b+c)(b+c-a)(a+c-b)(a+b-c)$
$= \{(b+c)^2 - a^2\}\{a^2 - (b-c)^2\}$ Ex. 5.
$= (b^2 + 2bc + c^2 - a^2)(a^2 - b^2 + 2bc - c^2)$
$= (2bc)^2 - (b^2 + c^2 - a^2)^2$ Ex. 5.
$= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4.$

In this example, as well as in the one before it, a, b, c are symmetrically involved.

(20) $(x^2 + ax + b)(x^2 + ax + c)$
$= x^4 + (b + c - a^2)x^2 - a(b-c)x + bc.$

$$(21) \qquad (x^2 + a x + b)(x^2 + a' x + b') = x^4 + (a + a') x^3 + (a a' + b + b') x^2 + (a b' + a' b) x + b b'.$$

In this example, the connection existing between a and a', as coefficients of x in both the factors, is indicated by using the same letter a, merely distinguishing it from the first by an accent; and the same remark applies to b and b': by means of this notation, the relation existing between the product and its component factors is more clearly exhibited to the eye, than if distinct and independent symbols had been used for the same purpose.

$$(22) \qquad (x^3 + a x^2 + b x + c)(x^3 + a' x^2 + b' x + c') = x^6 + (a + a') x^5 + (a a' + b + b') x^4 + (a b' + a' b + c + c') x^3 + (a c' + a' c + b b') x^2 + (b c' + b' c) x + c c'.$$

Products of this kind are most conveniently formed in the following manner: The first term is x^6: the term involving x^5, is formed by multiplying x^3 into $a' x^2$, and $a x^2$ to x^3, no other combinations producing the same power of x: the second term is therefore $(a + a') x^5$: the term involving x^4 is formed by multiplying x^3 into $b' x$, $a x^2$ into $a' x^2$, and $b^2 x$ into x^3, and is therefore $(a a' + b + b') x^4$: the term involving x^3 is formed by multiplying x^3 into c', $a x^2$ into $b' x$, $b x$ into $a' x^2$, and c into x^3, which is therefore $(a b' + a' b + c + c') x^3$: the term involving x^2 is formed by multiplying $a x^2$ into c', $b x$ into $b' x$, c into $a' x^2$, and it is therefore $(b b' + a c' + a' c) x^2$: the term involving x is formed by multiplying $b x$ into c', and c into $b' x$; it is therefore $(b c' + b' c) x$: the last term is the product of c and c'.

Formation of products where the terms in the factors and product are ranged according to the powers of one letter.

In some cases, a term of the product of two quantities involving a given power of some letter, according to which they are arranged is, all that is required: under such circumstances, it would be superfluous to go through the whole operation of multiplying the quantities together,

Formation of any assigned term of such products.

inasmuch as the single term required may be determined by the process just described.

Thus, suppose it was required to find the term involving x^4 in the product of

$$x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4$$
$$x^4 + A_1 x^3 + A_2 x^2 + A_3 x + A_4.$$

By collecting together those combinations of terms in one factor with terms in the other, which each involve x^4 when multiplied together, we find them to be $x^4 \times a_4$, $A_1 x^3 \times a^3 x$, $A_2 x^2 \times a_2 x^2$, $A_3 x \times a_1 x^3$, and $A_4 \times x^4$, and therefore the term required is

$$(a_4 + a_3 A_1 + a_2 A_2 + a_1 A_3 + A_4)\, x^4.$$

The notation which we have adopted in this instance is frequently extremely convenient, since the numbers attached to each coefficient serve the double purpose of distinguishing the symbols from each other, and of pointing out their position in each expression: it is only necessary, however, to resort to refinements of notation of this kind when several different quantities are involved, and when it would be otherwise difficult to remember the connection existing amongst a great number of independent symbols.

(23) $$\left(\frac{5}{2} x^2 + 3 a x - \frac{7 a^2}{3}\right) \left(2 x^2 - a x - \frac{a^2}{2}\right)$$
$$= 5 x^4 + \frac{7}{2} a x^3 - \frac{107}{12} a^2 x^2 + \frac{5}{6} a^3 x + \frac{7}{6} a^4.$$

(24) $$\left(\frac{a^2}{b^2} + 2 + \frac{b^2}{a^2}\right) \left(\frac{a}{b} + \frac{b}{a}\right) = \frac{a^3}{b^3} + \frac{3 a}{b} + \frac{3 b}{a} + \frac{b^3}{a^3}$$
$$= \left(\frac{a}{b} + \frac{b}{a}\right)^3.$$

(25) $$\left(\frac{a^2}{b^3} + \frac{2c^3d^4}{b^5} - \frac{7c^2}{2a^4b^2}\right)\left(\frac{a^2}{b^3} - \frac{2c^3d^4}{b^5} + \frac{7c^2}{2a^4b^3}\right)$$

$$= \frac{a^4}{b^6} - \left(\frac{2c^3d^4}{b^5} - \frac{7c^2}{2a^4b^3}\right)^2 \quad \text{Ex. 5.}$$

$$= \frac{a^4}{b^6} - \frac{4c^6d^8}{b^{10}} + \frac{14c^5d^4}{a^4b^8} - \frac{49c^4}{4a^8b^6}.$$

(26) $(a^m + b^p - 2c^n)\ (2a^m - 3b) = 2a^{2m} + 2a^m b^p$
$- 4a^m c^n - 3a^m b - 3b^{p+1} + 6bc^n.$

40. The mode of denoting the operation of division has been explained in Art. 7. Division.

In performing this operation, we must keep in mind, that it is the inverse of multiplication (Art. 10): upon this principle the rules for Division in all cases are founded.

The *quotient* multiplied into the *divisor* must produce the *dividend*: this is only another mode of expressing the principle just referred to: it supplies the test by which the truth of the operation in any case may be ascertained.

There are three cases to be considered in Division.

41. When both the divisor and the dividend are monomials. Case 1.

Rule. *Omit all the quantities which are common to the divisor and dividend: underneath what remains of the dividend, write what remains of the divisor, and the resulting quantity is the quotient.*

If no part of the divisor remains except unity, the remaining part of the dividend is the quotient.

We have already had examples of this case of Division, in the reduction of some expressions under a fractional form to more simple forms: (Art. 35. Ex. 16, 17, 18, 19, 20, 21, 22, 25, 26).

Examples.

42. (1) $a\div b=\frac{a}{b}$. (Art. 7.)

(2) $a\div -b=\frac{a}{-b}=\frac{-a}{b}$: this is evident, since $\frac{a}{-b}\times -b=a$ (Art. 10.); and $\frac{-a}{b}\times -b=\frac{-a\times -b}{b}=\frac{ab}{b}=a$ (Art. 10.); that is, $\frac{a}{-b}$ and $\frac{-a}{b}$, being multiplied into the same quantity, produce the same result: they are therefore equivalent quantities (Art. 22).

The signs of the dividend and divisor may be changed without affecting the quotient.

(3) $-a\div b=\frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b}$: for the reason given in Ex. 2.

(4) $-a\div -b=\frac{-a}{-b}=\frac{a}{b}$: for $\frac{-a}{-b}\times -b=-a$, (Art. 10): and $\frac{a}{b}\times -b=\frac{-ab}{b}=-a$ (Art. 10): consequently $\frac{-a}{-b}$ and $\frac{a}{b}$ are equivalent expressions.

From these examples, we may conclude generally, "That we may change the signs of *both* the divisor and dividend, without altering the value or signification of the quotient."

$$(5)\quad a\div\frac{1}{b}=\frac{a}{\frac{1}{b}}=\frac{ab}{1}=ab.$$

It may be necessary to explain this notation, and the result which is just given.

When a is divided by b, the result is written $\frac{a}{b}$, which is identical with $a \times \frac{1}{b}$: or in other words, a divided by b, is equivalent to a multiplied by $\frac{1}{b}$: and since these operations are the inverse of each other, and therefore reciprocal, it follows that to divide by $\frac{1}{b}$ is the same thing as to multiply by b: or since $a \div \frac{1}{b} \times \frac{1}{b} = a$ (Art. 10), and $ab \times \frac{1}{c} = \frac{ab}{b} = a$: it follows that $a \div \frac{1}{b} = ab = a \times b$.

(6) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{bc} \div \frac{1}{d} = \frac{a}{bc} \times d = \frac{ad}{bc}$: this conclusion follows immediately from the notation explained and the result given in the last example.

Rule i divisio algebr fractio

Since $\frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c} = \frac{a}{b} \div \frac{c}{d}$, it follows that division by $\frac{c}{d}$ is equivalent to multiplication by its reciprocal $\frac{d}{c}$: we from hence derive a general rule for the division of algebraical quantities, when the divisor is under a fractional form: which is, "To invert the divisor, and then proceed as in multiplication."

(7) $a^3 \div a = \frac{a^3}{a} = a^2$: for $\frac{a^3}{a} = \frac{a^2 \times a}{a} = a^2$.

(8) $a^7 \div a^3 = \frac{a^7}{a^3} = a^4$: for $\frac{a^7}{a^3} = \frac{a^4 \times a^3}{a^3} = a^4$.

(9) $-12abcde \div -8aod = \frac{12bc}{8} = \frac{3bc}{2}$, dividing 12 and 8 severally by their common measure 4.

(10) $$\frac{3ac}{4} \div \frac{5abd}{6} = \frac{3ac}{4} \times \frac{6}{5abd} = \frac{18ac}{2abd}$$

$$= \frac{9c}{10bd}.$$

(11) $$\frac{2ay}{5bx^2} \div 3ac = \frac{2y}{15bcx^2}.$$

(12) $$27a^3b^2cfg \div -18abcghk = \frac{-27a^2bf}{18hk}$$

$$= -\frac{3a^2bf}{2hk},$$ dividing 27 and 18 by 9.

(13) $$\frac{3afx}{bc} \div \frac{2fx^2}{5cde} = \frac{3afx}{bc} \times \frac{5cde}{2fx^2}$$

$$= \frac{15ade}{2bx}.$$

(14) $\frac{a^4}{b^4} \div \frac{a^5}{b^5} = \frac{a^4}{b^4} \times \frac{b^5}{a^5} = \frac{b}{a}$: since a^4b^4 is common to numerator and denominator.

(15) $\frac{a}{b} \div \sqrt{\frac{a}{b}} = \frac{a}{b} \div \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ (Art. 12.) $= \frac{a}{b} \times \frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$:

since $\frac{ab^{\frac{1}{2}}}{ba^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times b^{\frac{1}{2}}}{b^{\frac{1}{2}} \times b^{\frac{1}{2}} \times a^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$.

Case 2. 43. When the divisor only is a mononomial.

Rule. *Divide each term of the polynomial separately, as in Case* 1, *and combine the results with their proper signs.*

Examples. 44. (1) $(ax + bx) \div x = \frac{ax + bx}{x} = a + b.$

(2) $$(a x^3 + a^2 x^2 + a^3 x) \div a x = \frac{a x^3 + a^2 x^2 + a^3 x}{a x}$$
$$= x^2 + a x + a^2.$$

(3) $$\left(3ac - 2ade + \frac{c}{d}\right) \div 2a = \frac{3c}{2} - de + \frac{c}{2ad}.$$

(4) $$(8a^2 - 6ab + 4c + 1) \div 4a^2 = 2 - \frac{3b}{a} + \frac{c}{a^2} + \frac{1}{4a^2}.$$

(5) $$(12acfg - 4af^2g + 3fg^2h) \div 4a^2b^2fg$$
$$= \frac{3c}{ab^2} - \frac{f}{ab^2} + \frac{3gh}{4a^2b^2}.$$

(6) $$\left(\frac{a}{b} + \frac{b}{a}\right) \div \frac{a}{b} = \left(\frac{a}{b} + \frac{b}{a}\right) \times \frac{b}{a} = 1 + \frac{b^2}{a^2}.$$

(7) $$\left(\frac{a}{b} - \frac{df}{2c} - 3ac + 7\right) \div \frac{3c}{d}$$
$$= \frac{ad}{3bc} + \frac{d^2f}{6c^2} - ad + \frac{7d}{3c}.$$

(8) $$\left(\frac{a^3}{b^3} + \frac{2a^2c}{b^2d} + \frac{ac^2}{bd^2}\right) \div \frac{ac}{bd} = \frac{a^2d}{b^2c} + \frac{2a}{b} + \frac{c}{d}.$$

(9) $$(a^{\frac{5}{2}} x^{\frac{1}{2}} + a^{\frac{3}{2}} x^{\frac{3}{2}} + a^{\frac{1}{2}} x^{\frac{5}{2}}) \div a^{\frac{1}{2}} x^{\frac{1}{2}} = a^2 + a x + x^2.$$

(10) $$(a\sqrt{b} + b\sqrt{a}) \div \frac{\sqrt{b}}{\sqrt{a}} = a\sqrt{a} + b\sqrt{b};$$

if we use indices,

$$(a b^{\frac{1}{2}} + a^{\frac{1}{2}} b) \div \frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}} = a^{\frac{3}{2}} + b^{\frac{3}{2}}.$$

Case 3.

45. When the divisor contains more than one term.

Rule. *Arrange the divisor and dividend, if possibl according to the powers of some one letter, and plac them in one line in the same manner as in the divisio of numbers in arithmetic: find the quantity whic multiplied into the first term of the divisor, will produc the first term of the dividend; this is the first term o the quotient: multiply this term into the divisor, an subtract the result from the dividend; consider the re-mainder, if any, as a new dividend, and proceed as before.*

If there be any quantity which is common to every term of the dividend and divisor, it is generally convenient to divide them both by it in the first instance, and afterwards to proceed with the resulting quantities according to the preceding rule.

Quotient complete and incomplete.

By this process, we subtract from the dividend, the product of the divisor and all the terms of the quotient, which is clearly *complete* when there is no remainder after the last operation: as long as a remainder exists, the quotient is *incomplete;* and in those cases in which the remainder can never disappear, however often the operation is repeated, the quotient is necessarily *incomplete* and *interminable*, as there is no algebraical expression consisting of a finite number of terms, which multiplied into the divisor, can produce the dividend.

Examples.

46. (1) Divide $6a^2-9ab$ by $2a-3b$.

$$\begin{array}{l} 2a-3b)\ 6a^2-9ab\ (3a \quad \text{the quotient.} \\ \underline{6a^2-9ab} \end{array}$$

$$\text{or } \frac{6a^2-9ab}{2a-3b}=3a.$$

(2) $$\frac{70a^2bd-150ab^2c-160a^2bcd}{7ad-15bc-16acd}=10ab.$$

In both these examples, there is no remainder after the first operation, and the quotient is complete in one term.

(3) Divide $a^2+2ab+b^2$ by $a+b$.

$$\begin{array}{lr} & a+b)\ a^2+2ab+b^2\ (a+b \\ \text{Multiply } a+b \text{ into } a, & \underline{a^2+\ ab\qquad\quad} \\ & ab+b^2 \\ \text{Multiply } a+b \text{ into } b, & \underline{ab+b^2} \\ & \cdot\quad\ \cdot \end{array}$$

By the first operation, we take away the product of a and $a+b$ from the dividend: by the second we take away from it, the additional product of b and $a+b$: we have thus taken away altogether from the dividend, the entire product of $a+b$ and $a+b$, and there is no remainder: consequently $a+b$ is the *complete* quotient of $a^2+2ab+b^2$ divided by $a+b$.

If we reverse the order of the letters in the divisor and dividend, as follows,

$$b+a)\ b^2+2ab+a^2,$$

we should find $b+a$ for the quotient, by a process similar to the preceding: the arrangement in this case, as in the former, being alphabetical, according to the powers of some one letter: but if we pay no attention to the alphabetical arrangement of the terms of the dividend and divisor, though we may sometimes find the *complete* quotient after a greater number of operations than are otherwise necessary, yet in most cases the process will never terminate, and consequently the complete quotient will never be determined.

For suppose the divisor and dividend arranged follows:

$$\begin{array}{l}
a+b)\ 2ab+a^2+b^2\ (2b+a-b=a+b \\
\qquad\ \ 2ab+2b^2 \\
\hline
\qquad\qquad\quad a^2-b^2 \\
\qquad\qquad\quad a^2+ab \\
\hline
\qquad\qquad\quad -ab-b^2 \\
\qquad\qquad\quad -ab-b^2 \\
\hline
\qquad\qquad\qquad \cdot\quad \cdot
\end{array}$$

But if we place them in the following order, and adher to it throughout, the process would never terminate:

$$\begin{array}{l}
a+b)\ 2ab+b^2+a^2\ \left(2b-\frac{b^2}{a}+\frac{b^3}{a^2}-,\ \&c.\right. \\
\qquad\ \ 2ab+2b^2 \\
\hline
\qquad\qquad -b^2+a^2 \\
\qquad\qquad -b^2-\frac{b^3}{a} \\
\hline
\qquad\qquad\qquad \frac{b^3}{a}+a^2 \\
\qquad\qquad\qquad \frac{b^3}{a}+\frac{b^4}{a^2} \\
\hline
\qquad\qquad\qquad -\frac{b^4}{a^2}+a^2.
\end{array}$$

The second term $\frac{-b^2}{a}$ in the quotient, is the quantity which, when multiplied into a, produces $-b^2$: the third term $\frac{b^3}{a^2}$, is the quantity which, when multiplied into a, produces $\frac{b^3}{a^2}$: and so on, the terms of the quotient, according to the rule, being such quantities as, when multiplied into the first term of the divisor, necessarily produce the first terms of the remainders: it is quite clear that by this process the remainder can never disappear, and consequently the quotient can never become complete.

(4) **Divide** $x^2+(a+b)x+ab$ by $x+a$. (Art. 39. Ex. 8.)

$$\begin{array}{l}
x+a)\ x^2+(a+b)\,x+ab\ (x+b \\
\qquad\quad \underline{x^2+ax\qquad\qquad\quad} \\
\qquad\qquad\qquad bx+ab\ ;\ \text{for}\ (a+b)\,x-ax=bx \\
\qquad\qquad\qquad \underline{bx+ab} \\
\qquad\qquad\qquad\ \cdot\quad\cdot
\end{array}$$

(5) **Divide** $x^2-(a-b)x-ab$ by $x+b$. (Art. 39. Ex. 8. γ.)

$$\begin{array}{l}
x+b)\ x^2-(a-b)\,x-ab\ (x-a \\
\qquad\quad \underline{x^2+bx\qquad\qquad} \\
\qquad\qquad -ax-ab;\ \text{for}\ -(a-b)x-bx=-ax \\
\qquad\qquad \underline{-ax-ab} \\
\qquad\qquad\quad \cdot\quad\cdot
\end{array}$$

(6) **Divide** $x^2+3x-28$ by $x-4$.

$$\begin{array}{l}
x-4)\ x^2+3x-28\ (x+7 \\
\qquad\quad \underline{x^2-4x\qquad\quad} \\
\qquad\qquad\quad 7x-28 \\
\qquad\qquad\quad \underline{7x-28} \\
\qquad\qquad\qquad \cdot\quad\cdot
\end{array}$$

(7) Divide $x^3-86x-140$ by $x-10$.

$$\begin{array}{l}
x-10)\ x^3-86x-140\ (x^2+10x+14\\
\qquad\quad x^3-10x^2\\
\hline
\qquad\qquad 10x^2-86x\\
\qquad\qquad 10x^2-100x\\
\hline
\qquad\qquad\qquad\quad 14x-140\\
\qquad\qquad\qquad\quad 14x-140\\
\hline
\qquad\qquad\qquad\quad\ \cdot\quad\ \cdot
\end{array}$$

In this case, it is not necessary to bring down -140 to form a part of the first remainder, as it would not be affected by the second operation: it is omitted therefore to save superfluous writing.

(8) Divide $x^4-4x^3-34x^2+76x+105$ by $x-7$.

$$\begin{array}{l}
x-7)\ x^4-4x^3-34x^2+76x+105\ (x^3+3x^2-13x-15\\
\qquad\ x^4-7x^3\\
\hline
\qquad\quad 3x^3-34x^2\\
\qquad\quad 3x^3-21x^2\\
\hline
\qquad\qquad -13x^2+76x\\
\qquad\qquad -13x^2+91x\\
\hline
\qquad\qquad\qquad -15x+105\\
\qquad\qquad\qquad -15x+105\\
\hline
\qquad\qquad\qquad\quad \cdot\quad\ \cdot
\end{array}$$

If the complete remainders were brought down after each subtraction, the process would stand as follows:

$$\begin{array}{l}
x-7)\;x^4-4x^3-34x^2+76x+105\;(x^3+3x^2-13x-15\\
\qquad x^4-7x^3\\
\hline
\qquad\qquad 3x^3-34x^2+76x+105\\
\qquad\qquad 3x^3-21x^2\\
\hline
\qquad\qquad\qquad -13x^2+76x+105\\
\qquad\qquad\qquad -13x^2+91x\\
\hline
\qquad\qquad\qquad\qquad\quad -15x+105\\
\qquad\qquad\qquad\qquad\quad -15x+105\\
\hline
\qquad\qquad\qquad\qquad\qquad \cdot\quad\cdot
\end{array}$$

(9) **Divide** $3a^5+16a^4b-33a^3b^2+14a^2b^3$ by a^2+7ab.

In the first place, it is evident from inspection that a is common to every term of the divisor and dividend: we begin therefore by dividing them both by a, and then proceed with the results as follow:

$$\begin{array}{l}
a+7b)\;3a^4+16a^3b-33a^2b^2+14ab^3\;(3a^3-5a^2b+2ab^2\\
\qquad\quad 3a^4+21a^3b\\
\hline
\qquad\qquad -\;5a^3b-33a^2b^2\\
\qquad\qquad -\;5a^3b-35a^2b^2\\
\hline
\qquad\qquad\qquad\qquad 2a^2b^2+14ab^3\\
\qquad\qquad\qquad\qquad 2a^2b^2+14ab^3\\
\hline
\qquad\qquad\qquad\qquad\quad \cdot\quad\cdot
\end{array}$$

(10) $$\frac{a^2+ab+2ac-2b^2+7bc-3c^2}{a+2b-c}=a-b+3c.$$

(11) $$\frac{x^6-140x^4+1050x^3-3101x^2+3990x-1800}{x^3+12x^2-43x+30}$$

$$=x^3-12x^2+47x-60.$$

(12) Divide $x^4 - \frac{19a^2x^2}{6} + \frac{a^3x}{3} + \frac{a^4}{6}$ by $x^2 - 2ax + \frac{a^2}{2}$.

$$x^2 - 2ax + \frac{a^2}{2}\Big)\ x^4 - \frac{19a^2x^2}{6} + \frac{a^3x}{3} + \frac{a^4}{6}\ \Big(x^2 + 2ax + \frac{a^2}{3}$$

$$x^4 - 2ax^3 + \frac{a^2x^2}{2}$$

$$2ax^3 - \frac{11a^2x^2}{3} + \frac{a^3x}{3}$$

$$2ax^3 - 4a^2x^2 + a^3x$$

$$\frac{a^2x^2}{3} - \frac{2a^3x}{3} + \frac{a^4}{6}$$

$$\frac{a^2x^2}{3} - \frac{2a^3x}{3} + \frac{a^4}{6}$$

$$\cdot \quad \cdot \quad \cdot$$

(13)

$$\frac{-\frac{5x^2}{9} + \frac{11xy}{3} - \frac{10xz}{3} + \frac{15y^2}{4} + 25yz}{-\frac{2x}{3} + 5y} = \frac{5x}{6} + \frac{3y}{4} + 5z.$$

(14) Divide $x^3 - 2ax^2 + (a^2 - ab - b^2)\,x + a^2b + ab^2$ by $x - a - b$.

$$x - a - b)\ x^3 - 2ax^2 + (a^2 - ab - b^2)\,x + a^2b + ab^2\ \big(x^2 - (a-b)\,x -$$

$$x^3 - (a+b)\,x^2$$

$$-(a-b)\,x^2 + (a^2 - ab - b^2)\,x$$

$$-(a-b)\,x^2 + (a^2 - b^2)\,x; \text{ for } (a+b)(a-b) = a^2 - b^2$$

$$-abx + a^2b + ab^2$$

$$-abx + a^2b + ab^2$$

$$\cdot \quad \cdot \quad \cdot$$

The divisor and dividend being arranged according o powers of x, the divisor must be considered as a binomial, under the form $x-(a+b)$.

(15) $$\frac{y^6+(a^2-2b^2)\,y^4-(a^4-b^4)\,y^2-a^6-2a^4b^2-a^2b^4}{y^2-a^2-b^2}$$

$$=y^4+(2a^2-b^2)\,y^2+a^4+a^2b^2.$$

(16) $$\frac{x^3-a^3}{x-a}=x^2+ax+a^2.$$ (Art. 39. Ex. 13.)

(17) $$\frac{x^3+a^3}{x+a}=x^2-ax+a^2.$$

This result is deducible from Ex. 16, by putting $-a$ in the place of a.

(18) $$\frac{x^4-a^4}{x-a}=x^3+ax^2+a^2x+a^3.$$ (Art. 39. Ex. 14.)

If we put $-a$ in the place of a, we get

$$\frac{x^4-a^4}{x+a}=x^3-ax^2+a^2x-a^3.$$

(19) $$\frac{x^5-a^5}{x-a}=x^4+ax^3+a^2x^2+a^3x+a^4.$$

If we put $-a$ in the place of a, we get

$$\frac{x^5+a^5}{x+a}=x^4-ax^3+a^2x^2-a^3x+a^4.$$ (Art. 39. Ex. 15.)

(20) To divide $x^3 - a^3$ by $x + a$.

$$
\begin{array}{l}
x + a)\ x^3 - a^3\ \left(x^2 - a x + a^2 - \dfrac{2a^3}{x} + \dfrac{2a^4}{x^2} - \text{\&c.}\right. \\
\qquad\quad \underline{x^3 + a x^2} \\
\qquad\qquad - a x^2 - a^3 \\
\qquad\qquad \underline{- a x^2 - a^2 x} \\
\qquad\qquad\qquad\quad a^2 x - a^3 \\
\qquad\qquad\qquad\quad \underline{a^2 x + a^3} \\
\qquad\qquad\qquad\qquad - 2a^3 \\
\qquad\qquad\qquad\qquad \underline{- 2a^3 - \dfrac{2a^4}{x}} \\
\qquad\qquad\qquad\qquad\qquad\quad \dfrac{2a^4}{x} \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{\dfrac{2a^4}{x} + \dfrac{2a^5}{x^2}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad - \dfrac{2a^5}{x^2}
\end{array}
$$

The remainder after the third operation is $-2a^3$, and the next term in the quotient is therefore $-\frac{2a^3}{x}$, since $-\frac{2a^3}{x} \times x = -2a^3$; or the product of the new term in the quotient and of the first term of the divisor is the first, and, in this case, the only term, in the remainder: the next remainder is $\frac{2a^4}{x}$, and the corresponding term in the quotient is therefore $\frac{2a^4}{x^2}$, since $\frac{2a^4}{x^2} \times x = \frac{2a^4}{x}$: as the nature of the process in this case leaves necessarily new **remainder** after every operation, the quotient ay evidently be continued without limit, and will consist f a series of terms alternately positive and negative, e index of a in the numerator and of x in the deno- inator, increasing by unity in every successive term.

We have here an example of an indefinite quotient, nd the theory of its formation may be easily under- tood, by reference to the rule for division. As the erms in the quotient are successively determined by nding a quantity, which multiplied into the first term f the divisor produces the first term of the remainder, t is evident that the process may be continued as long s the remainder exists, and consequently may be con- nued *indefinitely*, when the remainder never disappears, hich must always be the case when the divisor is not *factor* of the dividend: such quotients, therefore, may e considered as originating in the rule for division, hich, like all other operations in Algebra, is perfectly neral, and the application of which is not limited by e practicability of determining the quotient in any nite number of terms.

(21) To divide 1 by $1+x$.

$$\begin{array}{rl}
1+x)\;1 & (1-x+x^2-, \text{\&c....} \\
\underline{1+x} & \\
-x & \\
\underline{-x-x^2} & \\
x^2 & \\
\underline{x^2+x^3} & \\
-x^3 &
\end{array}$$

Three terms are sufficient to ascertain the law of the ies of terms in the quotient, which are alternately negative and positive, the index of x increasing by in each successive term.

If we reverse the terms of the divisor, a very dif series would appear in the quotient: let it be req to divide 1 by $x+1$.

$$
\begin{array}{r}
x+1\,\big)\;1\quad\Big(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-,\ \&c. \\
1+\frac{1}{x}\qquad\qquad\qquad\quad \\
\hline
-\frac{1}{x}\qquad\qquad\qquad\quad \\
+\frac{1}{x}-\frac{1}{x^2}\qquad\qquad\quad \\
\hline
\frac{1}{x^2}\qquad\qquad\quad \\
\frac{1}{x^2}+\frac{1}{x^3}\qquad\quad \\
\hline
-\frac{1}{x^3}\qquad\quad
\end{array}
$$

(22) $$\frac{1}{1-x}=1+x+x^2+,\ \&c.$$

(23) $$\frac{a}{1-x}=a+ax+ax^2+,\ \&c.$$

(24) $$\frac{a}{a-x}=1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+,\ \&c.$$

(25) $$\frac{x}{a-x}=\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+,\ \&c.$$

(26) Divide $x+a$ by $x+b$.

$$x+b\,\Big)\;x+a\;\Big(1+\frac{(a-b)}{x}-\frac{(a-b)b}{x^2}+\frac{(a-b)b^2}{x^3}-\text{\&c.}$$

$$\begin{array}{l}
x+b \\ \hline
(a-b) \\
(a-b)+\dfrac{(a-b)b}{x} \\ \hline
-\dfrac{(a-b)b}{x} \\
-\dfrac{(a-b)b}{x}-\dfrac{(a-b)b^2}{x^2} \\ \hline
\dfrac{(a-b)b^2}{x^2}
\end{array}$$

We include $a-b$ between brackets, considering it as a single term, since the result is to be arranged according o powers of x.

If we reverse the order of the terms in the divisor nd dividend, putting x in the second place, the process ould stand as follows:

$$b+x\,\Big)\;a+x\;\Big(\frac{a}{b}-\frac{(a-b)}{b^2}x+\frac{(a-b)x^2}{b^3}-\text{\&c.}$$

$$\begin{array}{r}
a+\dfrac{ax}{b} \\ \hline
\left(1-\dfrac{a}{b}\right)x=\dfrac{(b-a)x}{b}=-\dfrac{(a-b)x}{b} \\
-\dfrac{(a-b)x}{b}-\dfrac{(a-b)x^2}{b^2} \\ \hline
\dfrac{(a-b)x^2}{b^2} \\
\dfrac{(a-b)x^2}{b^2}+\dfrac{(a-b)x^3}{b^3} \\ \hline
-\dfrac{(a-b)x^3}{b^3}
\end{array}$$

(27) $$\frac{x-a}{x-b}=1-\frac{(a-b)}{x}-\frac{(a-b)b}{x^2}-\frac{(a-b)b^2}{x^3}-\text{\&c.}$$

(28) $$\frac{a-x}{b-x}=\frac{a}{b}+\frac{(a-b)x}{b^2}+\frac{(a-b)x^2}{b^3}+\frac{(a-b)x^3}{b^4}+\text{\&c.}$$

(29) $$\frac{a+x}{b-x}=\frac{a}{b}+\frac{(a+b)x}{b^2}+\frac{(a+b)x^2}{b^3}+\frac{(a+b)x^3}{b^4}+\text{\&c.}$$

(30) $$\frac{x-a}{x+b}=1-\frac{(a+b)}{x}+\frac{(a+b)b}{x^2}-\frac{(a+b)b^2}{x^3}+\text{\&c.}$$

In these four examples, as well as in Ex. 22, 23, 24 25, we have used the sign $=$, to signify that the series on one side of it, is merely the result of the operation which, on the other side, is indicated and not performed we shall examine in the next chapter, under what circumstances an *arithmetical* equality can be considered as existing between them.

(31) Divide $a'+b'x$ by $a+bx$.

$$a+bx)\ a'+b'x\left(\frac{a'}{a}+\frac{(ab'-a'b)x}{a^2}-\frac{(ab'-a'b)bx^2}{a^3}+\text{\&c.}\right.$$

$$a'+\frac{a'b}{a}x$$

$$b'x-\frac{a'b}{a}x=\left(b'-\frac{a'b}{a}\right)x=\frac{(ab'-a'b)x}{a}$$

$$\frac{(ab'-a'b)x}{a}+\frac{(ab'-a'b)bx^2}{a^2}$$

$$-\frac{(ab'-a'b)bx^2}{a^2}$$

$$-\frac{(ab'-a'b)bx^2}{a^2}-\frac{(ab'-a'b)b^2x^3}{a^3}$$

$$\frac{(ab'-a'b)b^2x^3}{a^3}$$

If we should make $k = (ab' - a'b)$, the process of vision, as well as the form of the resulting series, ould be considerably simplified.

(32) $$\frac{b'x + a'}{bx + a} = \frac{b'}{b} - \frac{k}{b^2x} + \frac{ak}{b^3x^2} - \frac{a^2k}{b^4x^3} + \&c.$$
where $k = ab' - a'b$.

(33) $$\frac{a' - b'x}{a - bx} = \frac{a'}{a} - \frac{kx}{a^2} - \frac{bkx^2}{a^3} - \frac{b^2kx^3}{a^4} - \&c.$$
where $k = ab' - a'b$.

(34) $$\frac{b'x - a'}{bx - a} = \frac{b'}{b} + \frac{k}{b^2x} + \frac{ak}{b^3x^2} + \frac{a^2k}{b^4x^3} + \&c.$$
where $k = ab' - a'b$.

(35) Divide $a + \beta x + \gamma x^2$ by $a + bx$.

$$a + bx\Big)\, a + \beta x + \gamma x^2 \Big(\frac{a}{a} + \frac{kx}{a^2} + \frac{k'x^2}{a^3} - \frac{bk'x^3}{a^4} + \&c.$$

$$a + \frac{abx}{a}$$

$$\frac{kx}{a} + \gamma x^2, \text{ where } k = a\beta - ab$$

$$\frac{kx}{a} + \frac{bkx^2}{a^2}$$

$$\frac{k'x^2}{a^2}, \text{ where } k' = a^2\gamma - bk$$
$$= a^2\gamma - ab\beta + ab^2$$

$$\frac{k'x^2}{a^2} + \frac{bk'x^3}{a^3}$$

$$-\frac{bk'x^3}{a^3}$$

$$-\frac{bk'x^3}{a^3} - \frac{b^2k'x^4}{a^4}$$

$$\frac{b^2k'x^4}{a^4}$$

(36) $$\frac{1-3x-2x^2}{1-4x}=1+x+2x^2+2.4.x^3+2.4^2.x^4+\&c.$$

(37) $$\frac{x^2-px+q}{x-a}=x-(p-a)+\frac{k}{x}+\frac{ak}{x^2}+\frac{a^2k}{x^3}+\&c.$$

where $k=a^2-pa+q$.

(38) Divide x^3-px^2+qx-r by $x-a$.

$$\begin{array}{l}
x-a)\;x^3-px^2+qx-r\;(x^2+(a-p)x+a^2-pa+q+\frac{k}{x}+\&c.\\
\qquad\underline{x^3-ax^2\qquad}\\
\qquad\qquad(a-p)x^2+qx\\
\qquad\qquad\underline{(a-p)x^2-a(a-p)x}\\
\qquad\qquad\qquad(a^2-pa+q)x-r\\
\qquad\qquad\qquad\underline{(a^2-pa+q)x-a(a^2-pa+q)}\\
\qquad\qquad\qquad\qquad\qquad a^3-pa^2+qa-r\text{ or }k\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\underline{k-\frac{ak}{x}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\frac{ak}{x}
\end{array}$$

The third remainder a^3-pa^2+qa-r or k, is a quantity identical with the dividend, putting a in the place of x; and it is obvious that the quotient will not be finite, unless $k=0$.

(39) $$\frac{x^4-px^3+qx^2-rx+s}{x-a}=x^3+(a-p)x^2+(a^2-pa+q)x$$

$$+a^3-pa^2+qa-r+\frac{(a^4-pa^3+qa^2-ra+s)}{x}+\&c.$$

In this case, the division does not terminate unless the fourth remainder, or $a^4-pa^3+qa^2-ra+s$, be equal to zero.

CHAP. III.

OBSERVATIONS UPON THE FIRST PRINCIPLES AND FUNDAMENTAL OPERATIONS OF ALGEBRA.

47. THE operations of Algebra have been named from those operations in Arithmetic, with which they are analogous, or partly identical, and most of the terms made use of in this science have had a similar origin: the consequence has been, that the peculiar and in some respects limited meaning which those operations and terms possessed in Arithmetic, has been attached to them when used in a much more general signification: and hence also has arisen the custom of considering Algebra as merely such a generalization of the processes of Arithmetic as was derived from the use of symbolical language.

The operations and terms used in Algebra derived from Arithmetic.

48. In order to ascertain to what extent, or in what sense, Arithmetic may be considered as the basis of the science of Algebra, it will be expedient to examine the nature of the symbols employed in Arithmetic, the extent of their representation and the meaning and limits of the operations to which they are subject.

The extent and nature of the symbols and operations of Arithmetic.

49. The symbols of Arithmetic are the nine digits and zero, and no others.

Symbols.

The quantities which they represent are numbers, whether abstract or concrete. As far, however, as the operations of Arithmetic are concerned, we may consider them as abstract only, inasmuch as the relative magnitude of quantities denoted by numbers is alone considered, without any reference to the affections or specific properties of the quantities themselves.

Their representation.

The position of the symbols determines their numerical value: there is nothing arbitrary, either in the symbols themselves, or in the notation by means of them.

Fundamental operations.

50. There are four *fundamental* operations in Arithmetic; Addition, Subtraction, Multiplication, and Division.

Addition.

51. Addition is the union of two or more numbers into one sum.

Subtraction.

Subtraction is equivalent to the determination of a number (the remainder) which added to another (the subtrahend) will produce a given number for their sum: the operation is of course limited to those values of the subtrahend, which are less than the number from which they are to be subtracted.

Inverse operations.

Addition and Subtraction are inverse operations in the following sense: if to one number another be first added and then subtracted, or conversely, its value will remain unaltered.

Multiplication.

52. Multiplication is equivalent to the perpetual addition of one number, which is called the *multiplicand*, to itself, as often as unity is contained in another number, which is called the *multiplier*. The multiplicand and the multiplier are likewise convertible quantities; that is, it is indifferent whether we make the multiplier the multiplicand, and the multiplicand the multiplier, or conversely, the product in both cases being the same.

Division.

Division may be considered as equivalent to finding a number (the quotient) which multiplied into another (the divisor) will produce a given number (the dividend).

Inverse operations.

Division is the inverse of multiplication in the following sense: if a number be first multiplied and then divided by the same number, or conversely, its value is not altered.

Fractions.

53. Another class of quantities considered in Arithmetic, are *fractions* or broken numbers: different explanations have been given of their origin and meaning: the following, however, is in all cases consistent with their actual representation.

Their meaning.

A fraction consists of a numerator and denominator: if we suppose unity (a *concrete unit*, if such a supposition be necessary for logical accuracy) divided into as many parts as there are units in the denominator, the fraction denotes a quantity which comprehends as many of those parts as there are units in the numerator.

Reduction of different fractions to the same denominator.

54. Upon this principle, we multiply or divide the numerator and denominator of a fraction by the *same* number, without altering its value: for if we double the number of units in the numerator, we *halve* their value, and conversely; and similarly when we multiply or divide by any other numbers: we are thus enabled to reduce different fractions to others equivalent to them, which may have the same denominator, and consequently to make the numerators, numbers of the *same kind*, whose units represent the same magnitudes.

Addition and subtraction of fractions.

55. We add and subtract fractions, when reduced to the same denominator, by adding or subtracting their numerators: it is of course necessary to subscribe the common denominator to the sum or difference, inasmuch as it determines what portion of the *primary* unit, the *secondary* units in the numerator are.

Multiplication of fractions.

56. When we multiply one fraction by another, our meaning is, that the first fraction must be multiplied by the numerator of the second and divided by its denominator: thus to multiply $\frac{3}{4}$ by $\frac{5}{7}$, we first multiply $\frac{3}{4}$ by 5, which gives $\frac{15}{4}$, the operation being equivalent to adding $\frac{3}{4}$ five times to itself: we then divide the result $\frac{15}{4}$ by 7, which gives $\frac{15}{28}$, an operation which is equivalent

to making each unit in the numerator of $\frac{15}{4}$, one-seventh of its former value.*

Division of fractions.

In dividing $\frac{3}{4}$ by $\frac{5}{7}$, we multiply 3 into 7 for the numerator, and 4 into 5 for the denominator, of the quotient: the correctness of this process is proved, by considering division as the inverse of multiplication, and that consequently, if we multiply by $\frac{5}{7}$ the quotient of $\frac{3}{4}$ divided by $\frac{5}{7}$, the result must be the same quantity $\frac{3}{4}$, or equivalent to it: which will be found to be the case, since $\frac{3}{4} \times \frac{7}{5} \times \frac{5}{7} = \frac{3}{4}$.

It is indifferent in what order operations succeed each other.

57. It is not very difficult to shew from the preceding principles and statements, that it is indifferent in what order operations in Arithmetic succeed each other: thus, when several numbers are to be added together, the result is the same, in whatever order they are taken: when one number is to be added to, and another to be subtracted from, another, it is indifferent with which operation we begin: when two or more numbers are to be multiplied together, they may be taken in any order: when a number is to be divided by one number, and multiplied by another, we may either divide first and multiply afterwards, or conversely: and similarly in all other cases.

The particular numbers employed in arithmetical operations are not traceable in the results.

58. It is a necessary consequence of arithmetical notation, that the results of operations leave no traces of the original numbers, combined or incorporated: thus the sum of 28 and 7 is 35: their difference 21; their

---

* The determination of the value of a fraction of a fraction, may be deduced as a necessary consequence of the explanation we have given of the meaning of a fraction: thus $\frac{3}{4}$ of $\frac{5}{7}$ would be shewn to be equivalent to $\frac{15}{28}$, or to the product of $\frac{3}{4}$ and $\frac{5}{7}$, in the following manner: the *primary* unit is divided into 7 equal parts, of which 5 are taken: each of these 5 *secondary* units is divided in 4 equal parts, of which 3 are severally taken: the whole number of the *ternary* units is 3 × 5 or 15, and each of them is the 4 × 7th or 28th part of the primary unit: it is only however by *defining*, or in other words *assuming* the meaning of the *product* of $\frac{3}{4}$ and $\frac{5}{7}$, that we shew it to be equivalent to $\frac{3}{4}$ of $\frac{5}{7}$.

product 196; their quotient 4: and in none of the quantities 35, 21, 196 and 4, are there any traces left of the numbers 28 and 7.

Arithmetical Algebra.

59. Such are the principles and fundamental operations of Arithmetic, which will be found sufficient for the explanation of the most complicated processes which are required in the different applications of the science: we shall now proceed to consider the nature and extent of a system of Algebra, which should be strictly confined to those principles and operations as its foundation.

Symbols.

60. The symbols of Algebra, entirely founded upon such a basis, would denote numbers, whether whole or fractional, or such magnitudes only, as admitted of numerical representation, without embracing their affections or properties.

The signs + and −.

61. The signs + and − would denote addition and subtraction only: and there would be no meaning in symbols with either of these signs attached to them, such as $+a$ and $-b$, considered independently of their connection with other symbols.

Sum and difference.

62. The *sum* of a and b would be denoted by $a+b$; their *difference* by $a-b$: and there would be no other more general sense in which those terms would be used.

Rule for subtraction.

63. The difference of a and $a-b$ would be equal to b: for it is obvious that a quantity less than a by b, being subtracted from a, will leave b for a remainder: in other words, $a-(a-b)=a-a+b=b$: it is upon this principle that we should arrive at the algebraical rule for subtraction, and for the *combination* of the signs + and −, when like or unlike signs come together in this operation.

Limits of the operation.

If we proposed to subtract $a+b$ from a, which would be denoted by $a-(a+b)$, the proper answer would be, that the question proposed is *impossible*, there being no *prototype* for such an operation in Arithmetic: in other words, there is no such quantity as $-b$.

When we make use therefore of expressions such as $a-b$, we must consider them as having no meaning, or rather we must term them impossible or imaginary, unless a is greater than b: under such circumstances, the symbols have limits to their relative values, and are no longer therefore arbitrary even in magnitude.

Multiplication and Division. 64. The operations denoted by × and ÷, or by equivalent modes of representation, must strictly coincide with the arithmetical operations of Multiplication and Division.

Rule of signs. 65. The rule of the signs would be proved, and not assumed, to be true, by means of processes like the following:

(1) $a \times (c-d) = ac + ad$, being the product of a into c, added to the product of a into d.

(2) $a \times (c-d) = ac - ad$, being the difference of the products of a into c and of a into d.

(3) $$(a+b)\,(c+d) = (a+b)c + (a+b)d$$
$$= ac + bc + ad + bd.$$

In this case, we find b preceded by the sign +, multiplied into d likewise preceded by the sign +, and the result is bd preceded by the same sign +.

(4) $$(a-b)\,(c-d) = a \times (c-d) - b \times (c-d)$$
$$= ac - ad - (bc - bd) = ac - ad - bc + bd.$$

In this case, we find b preceded by the sign −, multiplied into d likewise preceded by the sign −, and the result is bd preceded by the sign +.

(5) $(a+b)(c-d)=(a+b)\times c-(a+b)d$
$=ac+bc-ad-bd.$

In this case, we find b preceded by the sign $+$, multiplied into d preceded by the sign $-$, and the result is bd preceded by the sign $-$.

(6) $(a-b)(c+d)=(a-b)\times c+(a-b)\times d$
$=ac-bc+ad-bd.$

In this case, we find b preceded by the sign $-$, multiplied into d preceded by the sign $+$, and the result is bd preceded by the sign $-$.

It thus appears, that two quantities preceded by *like* signs, produce, when incorporated by multiplication, a result which is preceded by the sign $+$, and that when they are preceded by *unlike* signs the result is preceded by the sign $-$.

66. Fractions such as $\frac{a}{b}$ admit of the same explanation as in Arithmetic: and the rules, with their demonstrations, for their addition, subtraction, multiplication and division would be precisely the same in both sciences.

Fractions.

67. Upon the same principles likewise we should infer, that it is indifferent in what order different operations succeed each other.

Order of operations.

68. The doctrine of *indices*, which is applicable to arithmetical quantities expressed by means of arithmetical symbols, would be equally applicable therefore to this system of Arithmetical Algebra: fractional indices would admit of the same interpretation as in Symbolical Algebra, but negative indices must be rejected as impossible or imaginary.

Indices.

69. Inasmuch as quantities such as $-a$ have no existence in this system, and expressions such as $a-b$ are considered as impossible, unless a be greater than b,

No quantities such as $\sqrt{-a}$.

we must of course reject altogether such quantities as $\sqrt{-a}$, which indeed could never occur, when the extent of algebraic operations is limited in the manner which this system renders necessary.

Zero the absolute *minimum.*

70. In Arithmetical Algebra, as well as in Arithmetic, zero is the *absolute minimium:* thus the least value of $a-b$, or the inferior limit of its variation, would be when $a=b$, or when $a-b=0$.*

Principal differences between arithmetical and symbolical Algebra.

71. In order to determine to what extent the principles and operations of Algebra, as stated in our first chapter, and such as they really exist in practice, can be deduced from the principles and operations of Arithmetical Algebra, as stated in the preceding articles, we will point out briefly the chief differences between them, and afterwards proceed to their separate examination.

(α) In one system, the symbols represent numerical quantities only: in the other, they are perfectly general in their representation.

(β) In one system, the signs + and − denote addition and subtraction only: in the other, they not only denote operations which are the inverse of each other, but are likewise used independently, one or other of them being prefixed to all symbols.

---

* Such would be the legitimate principles of Algebra founded entirely upon Arithmetic, where the terms and operations of one science are transferred to the other, without any change in their extent or meaning: such also are the principles which some modern algebraists have adopted in their strictest sense, rejecting altogether the independent use of the signs and every other consequence which has not its prototype in Arithmetic, and which is not a fair and legitimate result of the difference of notation: however much of its beauty, symmetry and extent the science must lose by being viewed in this partial light, at all events its principles and rules would admit of strict demonstration, without imposing the necessity of deducing conclusions more general than the premises upon which they are founded.

(γ) In one system, the rule of signs is *proved:* in the other it is *assumed.*

(δ) In one system, it is required to be proved that it is indifferent in what order operations succeed each other: in the other it is assumed to be so.

(ϵ) In one system, all operations are limited by the possibility of interpreting the results, consistently with arithmetical *prototypes:* in the other, the operations are perfectly unlimited, there being a symbolical result in all cases.

(η) In one system, *zero* is the absolute *minimum:* in the other, the *maximum* and the *minimum* are equally unlimited.

(ζ) In one system, the general rule of indices is *proved* to be a consequence of the first assumption of them, and limited to arithmetical values: in the other, it is *assumed* in its most general form.

(ι) In one system, the sign = means arithmetical equality or identity: in the other, it means symbolical identity or symbolical equivalence.

72. If Symbolical Algebra is to be considered as a science admitting of complete demonstration, it must be founded, like all other sciences possessing demonstrative evidence, upon its own definitions and assumptions: the first question, therefore, which we have to consider, is, whether those definitions and assumptions are deducible from assumptions as well as propositions demonstrated in Arithmetical Algebra, by any legitimate process of generalization.

Are the definitions and assumptions of Symbolical Algebra deducible from those of Arithmetical Algebra?

73. In the first place, we may correctly generalize the representation of the symbols from numerical quantities which they must denote in one system, to quantities which are unlimited, both in their nature and their magnitude,

Generalization of symbols.

which they may denote in the other: for we may consider all generalizations in language and notation as admissible, which are antecedent to any consequences to be deduced from them.

Manner in which symbols are limited in Arithmetical Algebra.

74. In Arithmetical Algebra, we have considered $a-b$ as an impossible quantity, when a was less than b, inasmuch as the operation denoted by $-$ would, under such circumstances, be destitute of meaning (Art. 63): we were therefore compelled to consider a and b, though general symbols, as *limited* in relative magnitude, when placed in such a position with respect to each other.

The generalization, however, which we have assumed with respect to these symbols, would preclude all limitation of value, which was not conveyed by the symbols themselves: and as the symbols a and b convey no limitation of *greater* or *less*, we must therefore consider $a-b$ as equally possible, in Symbolical Algebra, whether a be greater than b or b greater than a.

Generalization of the operation denoted by $-$.

75. If, however, we *generalize* the operation denoted by $-$, so that it may admit of application in all cases, we shall then find the independent existence of this sign will follow as a necessary consequence, and we shall thus introduce a class of quantities, whose existence was never contemplated in Arithmetic or Arithmetical Algebra, and to which, in those sciences, no proper interpretation could be given.

Its consequences.

76. In order to shew that such consequences would follow, let us resume the consideration of $a-b$, both when a is greater and less than b: in the first case, let a exceed b by a quantity c, or in other words, let $a=b+c$, or $c+b$: then $a-b$ is equivalent to $b+c-b$, or $c+b-b$: if we now *obliterate* b, (since the operations denoted by $+$ and $-$ are inverse operations) the two results are $+c$ and c, which are therefore equivalent to each other. In the second case, let b exceed a by a quantity c, or let $b=a+c$:

then $a-b$ is equivalent to $a-(a+c)$, and by the generalization of the operation denoted by $-$, it becomes equivalent to $a-a-c$, and, therefore, by obliterating a, it is equivalent to $-c$.

Equivalent to the assumption of the rule for the concurrence of signs.

77. This *generalization* of the operation denoted by $-$, is in reality an *assumption*, inasmuch as it is not a consequence deducible from the operation of subtraction as defined and used in Arithmetic and Arithmetical Algebra: for as the operations denoted by $+$ and $-$, which are defined to be the inverse of each other, (Art. 10.) to which the names of Addition and Subtraction are given, (Art. 5.) are further defined by rules, (Art. 25 and 28.) which are the mere expression in words of the *rule of the concurrence of like and unlike signs*, we must consider the generalization or assumption in question, as merely another form of the assumption of the *rule of signs:* the existence of independent signs, as we have just seen, must be *admitted* as a joint consequence of the universal application of the rule, and of the unlimited representation of the symbols.

Independent signs.

Algebra in its most general sense.

78. Algebra may be considered, in its most general form, as *the science which treats of the combinations of arbitrary signs and symbols by means of defined though arbitrary laws:* for we may *assume* any laws for the combination and incorporation of such symbols, so long as our assumptions are independent, and therefore not inconsistent with each other: in order, however, that such a science may not be one of useless and barren speculations, we choose some subordinate science as the guide merely, and not as the foundation of our assumptions, and frame them in such a manner that Algebra may become the most general form of that science, when the symbols denote the same quantities which are the objects of its operations: and as Arithmetic is the science of calculation, to the dominion of which all other sciences, in their application at least, are in a greater or less degree subject,

How and why limited by making Arithmetic a subordinate science to it.

it is the one which is usually, because most usefully, selected for this purpose.

Under what circumstances the operations denoted by + and − are identical with those of Addition and Subtraction in Arithmetic.

79. In Arithmetic and Arithmetical Algebra, which is its most general form, we find the two operations of Addition and Subtraction, which are the *inverse* of each other (Art. 51 and 61): in Symbolical Algebra, we assume the existence of two operations, denoted by + and −, which are likewise defined to be the inverse of each other: if we now assume the same signs + and −, to denote Addition and Subtraction in Arithmetical Algebra (Art. 62), we shall deduce, as a necessary consequence, the rule for the concurrence of like and unlike signs (Art. 63), from the previously understood and defined nature of these operations: in order to keep up the parallel between the two sciences, we *assume* the same law or rule of signs in Symbolical Algebra, which likewise further defines their meaning: so long, therefore, as the symbols denote arithmetical quantities, we are allowed to *interpret* the meaning of the operations denoted by + and −, as in every respect identical with Arithmetical Addition and Subtraction: when the quantities denoted by the symbols are no longer arithmetical, whether they be numbers with negative signs attached to them, or lines or areas on any other quantities, their interpretation, when they admit of any, must be made in perfect accordance with the assumptions previously made respecting them.

Under what circumstances the operations denoted by × and ÷ are identical with Multiplication and Division in Arithmetic.

80. Again, in Arithmetic and Arithmetical Algebra, we have likewise the operations of Multiplication and Division, which are the inverse of each other (Art. 52 and 64): in Symbolical Algebra, we *assume* the existence of two operations, denoted by × or ÷, or by the position of the symbols with respect to each other, which are likewise the inverse of each other (Art. 8 and 10): in Arithmetical Algebra, when the quantities incorporated are compound, we deduce a rule for the incorporation of the signs + and −, as a necessary consequence of their previously ascertained meaning (Art. 65): in Symbolical Algebra, such signs come together, in performing these

operations, whether the expressions are compound or simple: we *assume*, therefore, as we are at liberty to do, the same law of the incorporation of signs (Art. 7.):* we are thus enabled to assume the operations denoted by × and ÷, or equivalent modes of notation, which we call Multiplication and Division, to be identical with Arithmetical Multiplication and Division, when the quantities incorporated by means of them are arithmetical: in all other cases the interpretation of their meaning, when such is possible, must depend partly upon the assumptions made respecting them, and partly upon the specific nature of the quantities incorporated.

Order of succession of operations indifferent.

81. In Arithmetic and Arithmetical Algebra, it is *proved* that it is indifferent in what order the operations are performed: in order that the operations may be the same in Symbolical Algebra, when the quantities employed are arithmetical, the same proposition is *assumed* to be true in all cases (Art. 9): this additional assumption serves still further to define and limit the interpretation of their results, when the symbols are any quantities whatsoever.

Rule of indices.

82. The assumption of indices (Art. 11) in the first instance, may be considered as common to Arithmetic and to Arithmetical and Symbolical Algebra, as a convenient abbreviation of notation: and the law of indices, when they are whole numbers, is deducible as a necessary consequence from it: this law, however, is itself an assumption, when the indices are fractional numbers, or any other quantities whatever, and is made in accordance with the former law deduced, in order that they may coincide with each other, when those indices become whole

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* The reasoning in Art. 65, from which we deduced the rule for the incorporation of signs in Arithmetical Algebra, not only assumes the nature of the operations denoted by +, − and ×, to be identical with Arithmetical Addition, Subtraction and Multiplication, but is evidently not applicable to such quantities as $+a \times +b$, $-a \times -b$, $+a \times -b$, and $-a \times +b$; or $a \times b$, $-a \times -b$, $a \times -b$, and $-a \times b$, where the signs are used independently.

numbers: the ***interpretation*** of all other indices, of which we have already given examples, must be made in entire conformity with the general law which is assumed.

Further considerations on the parallelism between Arithmetic and Algebra.

83. The parallel which we have drawn in the preceding articles, between Arithmetical and Symbolical Algebra, will shew in what sense one may be considered as the foundation of the other: considering Arithmetic, which is the science of calculation, as the most important of all the subordinate sciences which Algebra may be supposed to comprehend; and Arithmetical Algebra as the mode of exhibiting its laws in their most general form, at the same time that their mutual dependence and connection is most strictly preserved; we assume its laws and operations, as the guide for those ***assumptions*** in Symbolical Algebra, which constitute its real foundation, and which alone can give it the dignity and the character of a demonstrative science: such ***assumptions***, though suggested by Arithmetical Algebra, and bearing the closest analogy to its principles and laws, are not on that account the less arbitrary: they are chosen and not deduced, in order that the two sciences may have common laws and common operations, as long as the quantities which are subject to them are common; and that we may thus be enabled to transfer the conclusions deduced by the more general science to the one which is less so.

Under what circumstances the conclusions of one science may be transferred to the other.

84. Inasmuch as the symbolical laws and processes of Arithmetical and Symbolical Algebra, without reference to their origin and establishment, are the same, with one important exception, which is the independent use of the signs, the conclusions must be the same in both sciences, with that exception only: so far, therefore, as they agree, we may correctly transfer the conclusions of one science to the other, without any limitation or modification, inasmuch as it would be perfectly indifferent which of the two sciences we made use of in the investigation of them: but whenever an example of the excepted case presents

itself in the result, or whenever the use of it is essential to the operation, we must then apply other principles to the interpretation of the conclusion, or to explain the influence and effect of its use in the process, towards bringing about the conclusion itself.

Necessity of principles of interpretation.

85. The conclusions deduced by Algebraical operations being in the first instance purely symbolical, and founded upon general laws for the combinations of symbols, which have no reference to their specific nature, it becomes an enquiry of the greatest importance to determine or interpret their meaning, when such specific values are assigned to the symbols employed: we shall thus find that a different interpretation of the same result will be necessary for every variation of the nature of the symbols, whether they denote numbers, or lines, or areas, or forces, or any other quantities whatsoever: as such interpretations must be made in accordance with the assumed laws of the combinations of the symbols, and also with their arithmetical character, when the quantities which they denote are represented by numbers, there are some general laws by which they are always regulated, which we shall now proceed to determine, and to shew their application to such cases as are of most common occurrence: we shall commence with the signs + and −, when used independently.

Principles of the interpretation of the signs + and − when used independently.

86. The general principles of the interpretation of the meaning of quantities affected by the signs + and −, whose specific nature is assigned, must be founded upon the essential connection between the independent existence of those signs and the laws by which the operations denoted by them are defined.

The first is, that inasmuch as the operations denoted by + and − are the inverse of each other, the properties which they designate, when used independently, must bear a similar relation.

The second is, that whilst the operation of Addition preserves, the operation of Subtraction changes the properties or affections of quantities to which it is applied.

The first of these principles will be sufficient to ascertain the relation, as well as the nature of the affections which *may* be designated by the signs + and −: whilst the second will enable us to determine the specific meaning of the operation of Subtraction in the particular case which may be under consideration.

With abstract numbers.

87. The use of abstract numbers excludes the consideration of specific properties, and so far as affections are concerned, numbers such as 3 and -5 admit of no interpretation, as distinguished from each other.

In what sense the terms *greater* and *less* are symbolized by + and −.

88. The terms *greater* and *less*, or rather the phrases *greater by* and *less by*, are symbolized by + and −, such expressions indicating that the numbers before which those signs are respectively placed, are to be added to or subtracted from some *other number* in the ordinary arithmetical sense of Addition or Subtraction. Thus, 7 *greater by* 3 is denoted by $7+3$, and 7 *less by* 3 is denoted by $7-3$: or in other words, $7+3$ is *greater than* 7 by 3, and $7-3$ is *less than* 7 by 3.

Numbers less than zero.

89. It would bear a close analogy to the phraseology used in this example, if we should speak of *zero greater by* 3, and *zero less by* 3, and denote them by $0+3$ or 3, and $0-3$ or -3.

In what sense.

By such a notation, however, we merely express the relation of Addition and Subtraction, as inverse operations: and when we speak of a number less than *zero*, which we denote by prefixing a negative sign to it, we mean that the same number being added to it, the final result is equal to *zero*.

It is in this sense, and in this sense only, that we speak of numbers as greater or less than *zero:* thus the series

$$5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5,$$

would represent the *symbolic* results of the successive subtraction of 1 from 6, and its successive remainders, which are the terms of the series: or the terms -1, -2, -3, -4, &c. would denote, that if 1, 2, 3, 4, &c. were added to them respectively, the several results would be equal to *zero.*

In what sense *zero* is not the absolute minimum.

90. In Arithmetic and Arithmetical Algebra, zero is the absolute *minimum:* in a more extended sense, however, zero may be considered as the *common limit* of *negative* and *positive* quantities, and the *maximum* and *minimum* are therefore equally unlimited.

When the symbols are not abstract numbers.

91. When abstract numbers or quantities are concerned, the only interpretation which the signs admit of when used independently, has reference to the operations denoted by them: when concrete numbers or quantities, however, are affected by those signs, they will in many cases admit of a specific interpretation dependent upon their specific properties.

Examples where the signs denote affections of property.

Thus, if property possessed or due be denoted by a number or symbol with a *positive* sign, a *debt* would be indicated by a number or symbol with a negative sign, or *conversely:* such affections of property are correctly *symbolized* by the signs + and −, since they possess the *inverse* relation to each other, which those signs require: for if to a person *A* there be given a certain property or sum of money, combined with or *added* to a debt of equal amount, his wealth or property remains the same as before.*

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* We shall afterwards find the sign $\sqrt{-1}$ made use of to denote a modified affection of quantity, equally related to those denoted by + and −: thus, if a denoted property possessed, and $-a$ a debt, $\sqrt{-1}.a$ might denote property neither possessed nor owed, such as a mere deposit would be.

Since the operation of *Subtraction*, denoted by the sign −, changes the sign of the quantity *subtracted*, it therefore changes its affection, and consequently the *subtraction* of a debt, in algebraical language, does not mean its *obliteration* or *removal*, but its conversion into property.

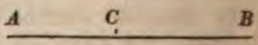
We should also be enabled to *symbolize* the affections of property considered as *lost* or *gained*, *bought* or *sold*, by the signs + and −, and conversely, since such quantities would possess the requisite inverse relations to each other: *subtraction* would in these cases mean the conversion of loss into gain, of property bought into property sold, and conversely.

It thus appears that the signs + and − may denote different modifications of the same concrete quantity: thus property may be considered as possessed, lost or sold: and a symbol with the sign + attached to it, or understood may be assumed to represent any one of the three: it is merely necessary that the modification of property designated by the same symbol, with the sign − attached to it, may possess such a relation to the former as may satisfy the general laws to which those signs are subjected.

Meaning of the signs + and − in Geometry.

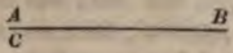
92. We shall now proceed to the interpretation of the signs + and −, when the symbols to which they are attached represent lines in Geometry.

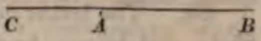
If a body moves from A to B, and back again from B to C, its final distance from A is equal to AC, the geometrical *difference* of AB and BC.



If we denote AB by a, and BC by b, without reference to their signs, then this final distance is denoted by $a-b$, which is the *algebraical* difference or sum of the symbols which denote AB and BC, according as their signs are the same or different.

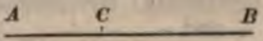
If $a=b$, or $AB=BC$, then the body returns to A: if, however, we suppose a less than b, or AB less than BC, the body on its return passes through A, and is found at C: it follows, therefore, that the distance AC, still represented by $a-b$, is equal to $a-(a+c)$ or $-c$, if $b=a+c$.



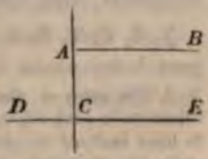


It follows, therefore, that if a line AB be denoted by a symbol with a positive sign, a line AC drawn in the opposite direction is denoted by a symbol with a negative sign.

It appears, therefore, that the signs + and −, when applied to symbols representing lines, indicate *direction:* for if a line in one *direction* be considered as positive, a line in the *direction* opposite would be considered as negative, and conversely: and this would be the case, without reference to the point from which they were reckoned, provided they were upon the same line: for if AB be denoted by a, BC reckoned from B to C, is *algebraically* represented by $-b$, and their *algebraical sum* or AC by $a-b$.



Again, if lines on one side of a line be considered as positive, lines parallel to them on the opposite side must be considered as negative: for as far as *direction* and *magnitude* are concerned, the equal and parallel lines AB and CE, are in every respect identical, and therefore may be represented by the same symbol, with the same sign: under such circumstances, therefore, the line CD must be denoted by a symbol with a different sign, for it is drawn in a direction opposite to CE.



General conclusion.

We are thus enabled to conclude generally, that if *parallel lines in one direction be considered as positive, all lines parallel to the former, but drawn in a direction opposite to them, must be considered as negative.*[*]

Other notions symbolized by the signs + and −.

93. The terms *heighth* and *depth*, *ascent* and *descent*, *above* and *below*, *backwards* and *forwards*, *motion with* and *motion against*, as in the case of a boat in a stream, and many others of a similar kind, involve in common the notion of *opposite directions* in the *same* or *parallel* lines, and are consequently all equally symbolized by the signs + and −, applied to the symbols by which they are denoted: the same remark applies to *forces* considered with reference to the direction of their action.

Time past and future expressed by different signs.

94. Time also, which may be compared to motion in one continuous stream in the same line, is susceptible of similar relations expressible by the signs + and −: for if we reckon it either from the present moment, or from any given epoch, considered in either case, as a point in the line by which its course is denoted, the parts of the line on one side of this point may represent time to come, and those on the other side time past, or conversely: and these are correctly symbolized by the signs + and −.

Again, if time be expressed in numerical units, whether seconds, minutes, hours, days, weeks, years or centuries, and if zero correspond to the moment from which it is reckoned, the series

$$7, 6, 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6, -7,$$

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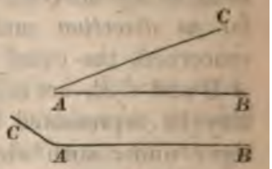
[*] The signs + and −, considered with reference to each other, could not be applied to lines *inclined* to each other like *AB* and *AC*: the directions in this case being different but not opposite, and therefore not satisfying the inverse relation which their theory requires: the relation of their position with respect to each other, would require a much more complicated sign ($\cos\theta + \sqrt{-1}\sin\theta$, where $\theta = \angle BAC$), the consideration of which hardly comes within the compass of this work.



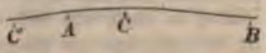
continued indefinitely both ways, would represent the successive instants, separated by the interval which unity denotes, in time past and time to come: thus, if twelve o'clock of the day was the zero point, and time was expressed in hours, 3 would express three hours before noon, and -3 three hours after, or conversely: if the zero point was the Birth of Christ, and time was expressed in years, -753 might express the epoch of the foundation of Rome, and 1688 that of our Revolution, it being of no consequence which of the two signs we adopt to represent time past or future, so long as different signs are employed for that purpose.

Heat and cold, as measured on thermometric scales, expressed by + and −.

95. A case, strictly analogous to the preceding, presents itself in the graduation of thermometric scales: if the *zero* point be the *heat* or *cold* at which water freezes, as in some, or at a certain elevation above it, as in others; and if the numbers expressing states of temperature *above* the zero point be considered as positive, those which express states of temperature below the zero point must be considered as negative, or conversely: if the term *hot* was invariably applied to the former, and *cold* to the latter, the ideas of *heat* and *cold* would be correctly symbolized in such a case by the signs + and −.

Direction expressed by signs not necessarily in straight lines.

96. In the preceding examples, the use of the signs is connected with the notion of direction in a straight line, whether regarding position or motion, which is made the foundation of their interpretation: but their application, even in Geometry, is not confined to straight lines, as may very easily be shewn by the following example and others analogous to it.

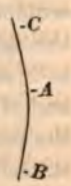


A traveller moves along the crooked road $C'AB$, from A to B, and then returns from B to C: if the distance travelled over from A to B be denoted by a, and the distance travelled over in return from B to C, be denoted by b, without reference to their signs, the distance *upon the road* from the point of starting would be denoted by their *geometrical* difference $a-b$, which would have been their algebraical sum, if the distance BC in return, had been denoted by $-b$.

If the traveller returns through A to C', and if $BAC' = a + c$, then the original expression for the distance from A or $a-b$, becomes $a-(a+c)$ or $-c$: in other words, if distances in one direction be denoted by symbols with positive signs, those in the direction opposite, whether upon straight lines or curved, will be denoted by symbols with the negative sign: the reasoning by which this conclusion is deduced in the two cases is precisely the same.

Positive and negative arcs.

97. If A be a point in a circular arc, it would follow exactly as in the case just considered, that if the arc AB on one side of A be considered as positive, the arc AC on the other would be negative, and conversely: or more generally, if any arc of a circle be considered as positive, any other arc *measured* in the opposite direction may be considered as negative.



The same observation would apply to the arcs of any other curve, whether *symmetrical* with respect to the zero point or not, it being merely necessary that the distances should be measured upon the curve.*

It would be very easy to produce many other examples of the meaning and application of the signs + and −, if we felt ourselves at liberty to select them from the various branches of natural philosophy: it will be found,

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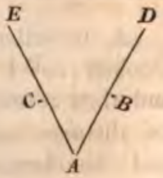
* Under this view of the meaning of opposite directions, distances upon two lines AD and AE, making an angle with each other, might be denoted by different signs: in this case, A is the zero point, and the motion is from B through A to C, or conversely.



however, that most of the cases which occur, bear a very close analogy to those which we have already considered: and at all events the general principles by which our interpretations have been regulated, will be quite sufficient for the explanation of any case which can present itself, when the nature of the quantities or the conditions of the questions render such an explanation possible.

Interpretation of Algebraic products without reference to their signs.

98. When symbols are incorporated by the operation of Multiplication, the sign of the product is determined by the rule of signs; or in other words, the signs of the factors determine the sign of the product: when, therefore, the factors of any product are assigned, the sign of the product is no longer arbitrary, and its interpretation must, therefore, be regulated by that of the signs of the factors themselves: before entering, however, upon this enquiry, it is necessary to interpret the meaning of the product itself, independently of its sign: in other words, to determine what specific quantity this product may and does represent, without the knowledge of which no meaning can be given to the signs + and −.

When one of the two symbols denotes an abstract number.

99. When one of the quantities involved in $\pm ab$, is an abstract number, the product simply means, that the concrete quantity which the other symbol denotes must be multiplied by that abstract number in its ordinary arithmetical sense: and as no alteration of its sign or representation can in this case take place in consequence of that operation, the interpretation of the meaning of its signs with reference to each other, must take place according to the principles already investigated and exemplified: the same remark applies to the quotient resulting from the division of a concrete quantity by an abstract number.

When both the symbols are concrete.

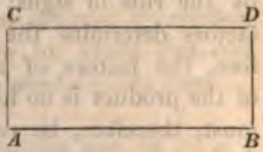
100. When both the quantities incorporated are concrete, whether of the same nature or not, the interpretation of the meaning of the result must be made in accordance with the two following principles.

1st. The product is always the same in meaning and quantity, when the factors incorporated are so, without reference to their order:

2d. The operation must coincide with Arithmetical Multiplication, when the quantities incorporated are represented by numbers.

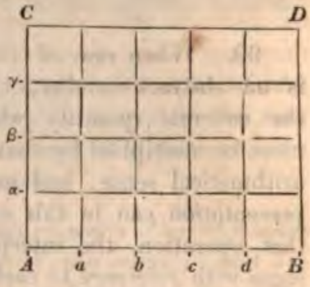
When the two symbols represent lines.

101. The most important case which we have to consider, is that in which the symbols incorporated by this operation are geometrical lines.



If the side AB of the rectangle $ABDC$, be represented by a, and the side AC by b, the product ab will correctly represent the area of the rectangle.

For in the first place, this area is always the same for the same values of a and b: and in the second place, this area will represent the arithmetical product of a and b, in the only sense in which it can do so, when a and b become numbers.



For let us suppose the lines a and b in the proportion of the numbers 5 and 4, and also represented by them: if we divide AB into 5 equal parts in the points a, b, c, d, and AC in 4 equal parts in the points α, β, γ, each of those parts will be a line which is represented by *unity* or 1: through the several points of division, let lines be drawn parallel to AC and AB respectively, and the whole rectangular area will thus be divided into 5 × 4 or 20 equal squares, constructed upon one of the linear units: in this case, therefore, $ab = 5 \times 4 = 20$, as it ought to be, but the units in the factors are equal lines, and those in the product are squares constructed on them: the analogy, however, between ab and the rectangular area is complete, as far as it is required to be so, and one *may*, therefore, be correctly taken to represent the other.

It does not follow, however, from the preceding investigation, that a rectangular area is the only area which may be correctly represented by ab: for any parallelogram whose adjacent sides are a and b, might upon similar principles be denotod by ab: for if the sides of such a parallelogram be denoted by numbers, its area might be divided into a number of equal rhombs equiangular with it, which would be equal to the product of the linear units in each of the sides: we should thus find, that the analogy upon which our interpretation of the meaning of such products is founded, would be equally complete in both cases.

We have supposed, however, that when a and b were given in magnitude, the value of their product, however interpreted, would always be the same: if, therefore, ab be supposed to represent a parallelogram, whether rectangular or not, whatever angle between the sides a and b we select in one case, we must adhere to in all others: there are many reasons for choosing a right angle in preference to any other.

For in the first place, it is the mean between all acute and obtuse angles, being equally related to both, and on that account most naturally selected. In the second place, in the science of calculation, the square is taken as the measure of area: thus different areas are compared by first forming squares equal to them, and then comparing the number of squares of the same linear units which they severally contain*: in order, therefore, to keep up the accordance between Arithmetic and Algebra, in this as well as in other and more essential particulars, we consider ab to denote a rectangular area only, and no other, when a and b are lines.

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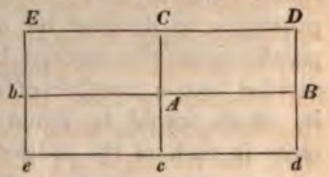
* It is for this reason that we speak of the *quadratures* of areas as equivalent to the determination of their values.

Meaning of the sign of ab, when a and b are lines.

102. Having thus determined the meaning of ab, without reference to its signs, or the signs of its factors, it remains to determine the additional interpretation to be given to it, when those signs are taken into consideration: for this purpose, if we assume, in the accompanying figure, $AB = a$, and $AC = b$, and produce the lines BA and CA in a *contrary direction* to that of AB and AC, to b and c respectively, making $Ab = AB$ and $Ac = AC$; then it follows from what we have already determined (Art. 92), that Ab is denoted by $-a$ and Ac by $-b$: complete the rectangles $ABDC$, $ABdc$, $ACEb$ and $Aceb$: the first corresponds to the product of a and b, or of $+a$ and $+b$, and therefore to $+ab$ or ab: the second to the product of a and $-b$, and therefore to $-ab$: the third to the product of $-a$ and b, and therefore to $-ab$: the fourth to the product of $-a$ and $-b$, and therefore $+ab$ or ab: the signs of these products, and the interpretation of their signs, is therefore completely determined by the signs of the factors themselves.



Ambiguity in passing from the product to the factors.

103. There are two rectangles which correspond to the same product ab, and two which equally correspond to the same product $-ab$: the two first correspond to the product of $+a$ into $+b$, and of $-a$ into $-b$: the two last correspond to the product of $+a$ into $-b$, and of $-a$ into $+b$: in passing, therefore, from the simple factors, the product and its representation are in every respect determined: but in passing from the product to the component factors, there are two different pairs of factors which equally correspond to it, and its determination is therefore ambiguous, even supposing the absolute magnitude of the factors to be given.

Signs of ab considered as a simple symbol.

104. If the product ab be considered as a simple symbol, without any reference to its factors, representing any rectangle such as $ABCD$; and if we suppose two equal and similar rectangles $ACEb$ and $ABdc$, constructed upon its sides AC and AB: then if the first of these rectangles be considered as positive, each of the others must be considered as negative; for rectangles of the same altitude bear the same relation to each other as the lines upon which they are constructed, and therefore may be analogically represented both in relative magnitude and affection by those lines: the rectangle $ABDC$ being therefore positive, being constructed upon the *positive* line AB, the rectangle $ACEb$ must be negative, being similarly constructed upon the *negative* line Ab: in the same manner, the rectangles $ABDC$ and $ABdc$, being considered with reference to each other, if the first be positive, being constructed upon the line AC, the second will be negative, being similarly constructed upon the negative line Ac: the fourth rectangle $Abec$, being equal to $ABDC$, and similarly related to $ACEb$ and $ABdc$ respectively, is therefore equally represented by the same symbol with the same sign.

Product of three symbols representing lines.

105. If we extend our enquiry to the determination of the meaning of the product abc of three symbols a, b, c, which represent lines, we shall find that it will correctly represent the volume or solid content of a rectangular parallelopipedon, of which a, b and c are three adjacent edges: for if we supposed that a, b and c were denoted by numbers, and that the edges which they severally represented were divided into linear units and planes parallel to three adjacent planes of the parallelopipedon were passed through the points of division, the solid would be divided into a number of equal cubes, constructed upon a linear unit, which would be equal to the continued product of the numbers which denoted the number of linear units in each edge respectively: the requisite arithmetical condition is therefore satisfied, it being merely

necessary to keep in mind that the units in the product are equal cubes, whilst those in the factors are equal lines.

The solid is the same, whatever arrangement of the symbols or edges corresponding is made, or in whatever order they are taken: and the product abc denotes a rectangular and not an oblique parallelopipedon, since all solids are compared together by making cubes equal to them in value; and when we come to arithmetical calculation, by comparing the cubes of the linear units which are contained in their sides.

Signs of the product of a, b and c.

106. There are only two signs of the product of three factors, though they may arise from *eight* different modifications of those factors, which are as follows.

(1) $+a \times +b \times +c = abc.$

(2) $+a \times -b \times -c = abc.$

(3) $-a \times -b \times +c = abc.$

(4) $-a \times +b \times -c = abc.$

(5) $+a \times +b \times -c = -abc.$

(6) $+a \times -b \times +c = -abc.$

(7) $-a \times +b \times +c = -abc.$

(8) $-a \times -b \times -c = -abc.$

Their interpretation.

These eight products would correspond to eight different, though equal and similar, rectangular parallelopipedons, having a common angle, and constructed upon edges which form severally one of each of the pairs of lines corresponding to $+a$ and $-a$, $+b$ and $-b$, $+c$ and $-c$, the lines a, b, c, and therefore $-a$, $-b$, $-c$, being at right angles to each other: those solids which touch by their planes, and have, therefore, two edges in common, such as (1) and (5), (1) and (6), (1) and (7): or (2) and (8), (3) and (8), (4) and (8), will partake (upon the principle mentioned in Art. 104) of the signs of the third edges of each, which are not in contact, and which are in the same straight line but in opposite directions: those which have one edge only in common, and two other edges of each in opposite directions with different signs, such as (1) and (2), (1) and (3), (1) and (4), or (5) and (8), (6) and (8), (7) and (8), will, for the same reason, have the same sign: whilst those pairs of solids, which have one point only in common and all the three edges with different signs, such as (1) and (8), (2) and (7), (3) and (5), (4) and (6), will have severally different signs.

Products of four or more symbols.

107. If we should attempt the explanation of the product of four or more symbols representing lines, or of two or more symbols representing areas, or of any other combination of symbols representing lines, areas, or solids, which should exceed three dimensions, there is no prototype in Geometry with which such products can be compared, and consequently no interpretation can be given to them: in other words, the existence of such products is possible in symbols only.

In interpreting products or quantities resulting from Algebraical operations, the results are alone to be attended to.

108. The process of the solution of a question, where the symbols are lines or areas, will frequently in the course of the operation, and before its completion, lead to powers or combinations which exceed three dimensions: this, however, is of no consequence, so long as the results present themselves under a form admitting of interpretation: for the operations of Algebra take no account of the specific magnitude or nature of the quantities involved, the symbolical results being the same in all cases, and those results, with the originally assumed values and representations of the symbols, are alone concerned when we come to the business of interpretation.

Geometry may be considered as a symbolical science.

109. We have traced the analogy which exists between the products ab and abc, when the symbols involved are numbers and geometrical lines: and the great principle

which we have assumed as our guide for the interpretation of those products when the symbols were any quantities whatsoever, was that of their degenerating into arithmetical products, when the symbols were represented by numbers: if we may suppose, therefore, that lines may represent numbers (and there is no relation of magnitude which they may not denote), they may likewise represent any magnitudes which numbers denote, and therefore any magnitudes which symbols denote, of which those numbers are the representatives: under these circumstances, if a and b are represented by lines, the rectangle contained by them will represent any quantity which ab can denote; and if a third symbol or line is to be introduced, the rectangular parallelopipedon abc will be equally general in its representation, whether a, b, c are lines or symbols: we thus give to Geometry the character of a symbolical science.

Space represented by a rectangular area.

110. Thus, if v represented the uniform velocity of a body's motion, and t the time during which it was continued, the product vt would represent the space over which the body had moved in that time: if we assume one line to denote v, and another to denote t, the rectangle formed by them would represent the same space equally with the product vt.

Linear units perfectly arbitrary.

When, however, lines represent quantities like v and t, which are different in their nature, and therefore admit of no comparison in respect of magnitude, the first assumption of them must be perfectly arbitrary: thus, if v denoting a certain velocity, be represented by an assumed line, v' denoting any other velocity, would be represented by another line bearing the same proportion to the former that v' bears to v: in a similar manner, one line may represent a time t, and another line any other time t', if they bear to each other the relation of t and t': but the quantities v and t admit of no comparison with each other, and therefore the line which represents an assigned magnitude of one quantity, can bear no determinate relation to the line which represents an assigned magnitude of the other: in other words, the lines which represent them may be assumed at pleasure.

The same remark applies, and for the same reasons, to the representation of essentially different quantities by means of numbers, the values of the primary units being perfectly arbitrary: thus the unit of time may be a second, a minute, an hour, &c. whilst the unit of space or velocity (for one is the measure of the other) may be a foot, a yard, a furlong, &c.: thus, if the units be assumed to denote a second of time and a foot in space: we may speak of a velocity denoted by 1, 2, 3, 10 or 20, being such as would cause a body to move uniformly over 1, 2, 3, 10 or 20 feet of space in one second, twice those spaces in two seconds, three times those spaces in three seconds, and therefore through a space which would be denoted by vt, if the body moved with a velocity equal to v, during any number of seconds equal to t: if we now pass from Arithmetic to Geometry, we may assume a line to represent a second of time, whilst an equal or any other line represents a foot in space, or a velocity of one foot: such primary units being thus assumed, all other values of those quantities will be represented by lines bearing the proper relation to them.

111. The following example, where three dimensions are involved, will illustrate this subject more fully.

Interest of money represented by a solid.

If p represent the principal or sum of money lent or forborne, r the rate of interest (of £1. for one year), and t the number of years, then the interest accumulated or due will be represented by prt: for if r be the interest of £1. for one year, pr will be the interest of a sum of money denoted by p for one year, and therefore prt will be the amount of this interest in t years, no interest being reckoned upon interest due: such would be the result according to the principles of Arithmetical Algebra.

If we now suppose p, r, t represented respectively by lines, which form the adjacent edges of a parallelopipedon, the solid thus formed will represent the interest accumulated or due: in other words, it will represent whatever is represented by the general formula prt, when specific values and significations are given to its symbols: for in whatever manner we may suppose any one of the symbols of prt to vary, the solid will vary in the same proportion.

The lines which we assume to represent units of p, r and t, are perfectly arbitrary, whether they are made equal to each other or not: this is clearly the case with p and t, which are quantities of a different nature: and the third quantity is likewise different from the other two, being an abstract numerical quantity: for it expresses the relation between the interest of £1. and £1., or between the interest of £100. and £100., which is the quotient of the division of one quantity by another of the same nature: thus, if the interest be 5 per cent., then $r = \frac{5}{100}$ or $\frac{1}{20}$: if 4 per cent., then $r = \frac{4}{100}$ or $\frac{1}{25}$: and similarly in other cases: the line, therefore, which is assumed to represent the abstract unit to which r is referred, is independent of the lines which represent units of p and of t, and may therefore be assumed at pleasure, equally with those lines.

The lines which represent p and t form a rectangular area, which is the geometrical representation of their product: the third quantity r, being merely numerical, may either be represented by a line as in the case just considered, when a solid parallelopipedon is made the representative of prt: or we may consider the area pt as representing the product prt when $r = 1$, and that this product in any other case is represented by a rectangle which bears to the rectangle pt the ratio of r to 1: this may be effected by increasing or diminishing one of the sides of the rectangle in the required ratio: the product prt may therefore be correctly represented either by a solid or an area, when one of the factors is an abstract number.

The examples which we have given, and the observations connected with them, would shew that such geometrical representations are coextensive with algebraical products as far as three dimensions: when more factors than three are introduced, the analogy of such products with Geometrical areas or solids ceases to exist, and it is only by supposing one or more of them to be abstract numbers, that they can be brought within the limits of such representations.*

It is not always necessary to interpret the products of heterogeneous factors.

112. In the solution of questions in the Rule of Three, in the many different forms under which that rule presents itself, we have frequently products of heterogeneous factors, to which it is not necessary to give any interpretation: for in such cases those products present themselves in pairs, each containing the same number of factors, which are severally homogeneous with each other, and whose relative magnitude is alone considered, without any reference to their specific nature: the following examples will more fully explain our meaning.

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* Before the introduction and general use of general symbols in mathematical reasonings, such representations were necessarily adopted as the most general form at that time known of symbolical language: and long after the use of Algebraical symbols was fully understood, the influence of ancient habits and associations induced mathematicians to reduce the results obtained by means of them to Geometrical forms: this practice continued to a very late period in this country, in consequence of its having been sanctioned by the authority of Newton and the most illustrious of his contemporaries. It was contended that such representations conveyed a sensible image of magnitudes to the eye as well as to the mind, and that we were thus enabled more readily and more clearly to seize the relations existing among them: but it is obvious that such representations give no greater aid to the interpretation of their meaning than when expressed in general symbols: and that as far as the business of calculation is concerned, we are farther removed from the result in one case than in the other.

"If a number of men (m) dig a ditch, whose length is a, breadth b, and depth c, in a given time, what number of men (m') will dig a ditch whose depth is a', breadth b', and depth c', in the same time?"

The ordinary rule or process for the solution of such questions, leads to the proportion

$$m : m' :: abc : a'b'c',$$

or to the equivalent fractions,

$$\frac{m}{m'} = \frac{abc}{a'b'c'} = \frac{a}{a'} \times \frac{b}{b'} \times \frac{c}{c'}.$$

The ratios of a to a', of b to b', of c and c', which are severally numerical, and totally independent of the specific nature of the quantities a', b', c', are alone required: we from hence get the ratio of m to m', and are thus enabled to determine m', which is the quantity required: for if $\frac{a}{a'} \times \frac{b}{b'} \times \frac{c}{c'} = e$, then $m' = \frac{m}{e}$.

If we should form two solid rectangular parallelopipedons, of which a, b, c and a', b', c' were respectively the adjacent edges, the ratio of their magnitudes would be that of m to m': they would represent respectively the ditches to be excavated, and their magnitudes would therefore be the measure of the work to be done in each case: the very principle of this representation would shew that abstract numbers might be substituted for a and a', b and b', and for c and c'.

We will now state the same question with some additional conditions.

"If a number of men (m) dig a ditch whose length is a, breadth b, and depth c, in t days, working for h hours each day, how many men (m') will dig a ditch whose length is a', breadth b', and depth c', in t' days, working for h' hours each day?

In this case, we arrive at the proportion

$$m : m' :: \frac{abc}{th} : \frac{a'b'c'}{t'h'}$$

or the equivalent fractions,

$$\frac{m}{m'} = \frac{abct'h'}{a'b'c'th} = \frac{a}{a'} \times \frac{b}{b'} \times \frac{c}{c'} \times \frac{t'}{t} \times \frac{h'}{h},$$

and it is clearly a matter of indifference, when the proportion is once established, what are the quantities denoted by a and a', b and b', c and c', t and t', h and h' respectively.

Algebraical quotients.

113. The meaning of Algebraical products, when the factors are any assigned quantities, being once determined, we experience no difficulty in interpreting the meaning of Algebraical quotients, when the dividend and divisor are assigned both in signification and value: the general principle of such interpretations being, that "*the operation of Division is in all cases the inverse of Multiplication:* in other words, the quotient or result of the division must be such a quantity, that when multiplied into the divisor, it will produce the dividend: we will enumerate a few cases.

Their meaning in different cases.

114. If the dividend and divisor be both abstract numbers, the quotient is an ordinary number or numerical fraction.

If the dividend be concrete, and the divisor numerical and abstract, the quotient is a concrete quantity of the same nature with the dividend.

If the dividend and divisor be both concrete and of the same nature, the quotient is an abstract number or a numerical fraction: for no change is produced in the nature of the divisor by being multiplied into the quotient.

If the dividend be an area, rectangular or not, and the divisor a line, the quotient is a line which contains with the divisor, a rectangular area equal to the dividend: if the dividend be a rectangle, and one of its sides the divisor, the other side is the quotient.

If the dividend be a solid and the divisor a line, the quotient is an area whether rectangular or not, forming with the divisor a solid equal to the dividend.

If the dividend be a solid and the divisor an area, the quotient is a line, which contains with the divisor a solid equal to the dividend.

If the dividend be a space passed over, and the divisor the uniform velocity with which it is described, the quotient is the time of describing it.

It is not necessary, however, to multiply examples of such interpretations, when the principle which connects them with the corresponding products admits of such easy and immediate application.

Indefinite quotients: their theory.

115. The rule for the division of compound algebraical expressions, leads necessarily, as we have already seen (Art. 46. Ex. 19), to the formation of incomplete and indefinite quotients: it is an enquiry of considerable importance to ascertain the meaning of the sign =, when placed between such a series, and the Algebraical fraction from which it arises: in order to make our observations upon this subject more easily understood, we will connect them with a particular example.

Let us consider the quotient of the division of a by $1-x$, which gives

$$\frac{a}{1-x} = a + ax + ax^2 + \ldots \frac{R}{1-x}.$$

If we retain the remainder R, the two parts on each side of the sign = are reducible to identical expressions, and they are therefore equal and equivalent to each other in the ordinary sense of those terms.

Thus supposing the division continued to n terms, when we find

$$\frac{a}{1-x} = a + ax + ax^2 + \dots ax^{n-1} + \frac{R}{1-x}:$$

if we multiply both sides by $1-x$, we get

$$a = a - ax^n + R = a,$$

since $R = ax^n$.

116. There are two questions which may be proposed with respect to such divisions, which are the reciprocal or converse of each other. Questions to be considered.

The first is, "Given the dividend and divisor, to find the indefinite quotient."

The second or converse question is, "Given the divisor and the indefinite quotient, to find the dividend."

117. The series or indefinite quotient equally results from the operation of division, whether we retain the remainder or not: when, therefore, we say that First meaning of the sign =.

$$\frac{a}{1-x} = a + ax + ax^2 + ax^3 + \&c.$$

the sign = merely indicates, that the indefinite series on one side of it is the result of the operation which on the other side of it is indicated and not performed.

118. There are two modes in which the second or converse question may be answered. Given the divisor and the indefinite quotient with the remainder represented and not expressed, to find the dividend.

First, retaining the remainder and multiplying $1-x$ into $a + ax + ax^2 + ax^3 + \&c. + \frac{R}{1-x}$, we necessarily get the dividend at whatever term of the series the original division was made to terminate: if the division was supposed, however, to be indefinite, and consequently the actual remainder not assigned nor assignable, but merely represented by a general symbol such as *R*, the result of this multiplication would still produce the original dividend.

Given the divisor and indefinite quotient without the remainder either represented or expressed, to find the dividend.

119. In the second case, we take the indefinite series or quotient without noticing the remainder: if we multiply this series into the divisor, we shall get the original dividend, with an indefinite term or terms which are neither exhibited nor capable of being so: so far, however, as the *discovery* of the dividend is concerned, we detect it equally by multiplying the divisor into the indefinite quotient, *with or without a remainder*, omitting in the second case all notice of such parts of the product as are not capable of actual Algebraical expression.

Second meaning of the sign $=$.

120. When viewed in this light, the sign $=$, placed between $\frac{a}{1-x}$ and the series $a+ax+ax^2+ax^3+$ &c. means not only that the series results from the fraction by the ordinary operation of division, but likewise that it is equivalent to it, when employed as a factor in multiplication: for those quantities are considered as equivalent in Algebra, which produce the same result, when employed in the same operation or operations.

Third meaning of the sign $=$.

121. So long as the remainder was retained, the parts on each side of the sign $=$ were reducible to indentical expressions, and were consequently *equal* as well as *equivalent:* and such an equality cannot of necessity exist between Algebraical expressions under any other circumstances.

Arithmetical equality when indicated by the sign $=$ placed before such series.

122. But if we pass from the use of symbols in their general form, to assigned or relative values of them, the fraction and the resulting series without its remainder, may become Arithmetically equal to each other, at least in a certain qualified sense.

Thus, if x be a proper fraction or less than unity, the terms a, ax, ax^2, ax^3, &c. and therefore the successive remainders, become less and less after every operation: by proceeding far enough, therefore, we may arrive at a remainder less than any fraction which may be assigned: in other words, the series may be made to differ from the true quotient by a fraction so small as to be incapable of Arithmetical expression: under such circumstances, as far as the purposes of calculation are concerned, one may be assumed for the other.

And when not.

If, however, the series diverges, and its terms therefore increase continually, the remainder is greater than all its terms, and the fraction and the resulting series are neither Arithmetically equal nor equivalent to each other.

Two series resulting from the same fraction.

123. The fractions $\frac{a}{1+x}$ and $\frac{a}{x+1}$ are equivalent to each other, but the series which results from one is, $a - ax + ax^2 - ax^3 +$ &c.; whilst that which, in conformity with the rule for division, corresponds to the other, is $\frac{a}{x} - \frac{a}{x^2} + \frac{a}{x^3} - \frac{a}{x^4} +$ &c.: we thus find two series essentially differing from each other in all their terms, which result from equivalent Algebraical fractions: and several examples of a similar nature may be seen in Art. 46.

In what sense are they Algebraically equivalent to each other.

124. The dividend is *discovered* by multiplying the first series by $1+x$, and the second by $x+1$, taking no notice of such quantities in each as are incapable of being actually expressed or exhibited.

Not equivalent in an Arithmetical sense.

125. If we pass from general symbols to Arithmetical or defined relative values of them, if one series converges, the other diverges, and one only of the two series, though Algebraically equivalent, in the sense attached to the term, to each other, is Arithmetically equal to $\frac{a}{1+x}$ or $\frac{a}{x+1}$.

General conclusion from the preceding theory.

126. The following may be mentioned as the most important conclusions which have been deduced from the preceding examination of the theory of indefinite quotients.

(α) Indefinite quotients are considered as Algebraically equivalent to the Algebraical fractions from which they are derived, whether with or without the remainder.

(β) The dividend may always be determined from the actual product of the divisor and the indefinite quotient, including the remainder; or even without the remainder, if no attention be paid to a term or terms which are incapable of actual Algebraical expression.

(γ) The sign $=$, when it signifies Algebraical equivalence, has no reference to the actual magnitude of the quantities between which it is placed.

(δ) The same sign $=$ implies Arithmetical or ordinary equality only, when Arithmetical or relative values are given to the symbols, and when the terms of the indefinite series are convergent.

Renewed consideration of the different meanings of the sign $=$.

127. The sign $=$, as we have seen above, will have different significations, like other signs in Algebra, according to the peculiar circumstances of its position and use: and as the interpretation which must be given to it, in any given case, is connected with some of the most important theories in Algebra, it may be proper in this place to give it a more particular examination.

Its most common signification, which may be rendered by the phrase *gives as its result*, is when it is placed between two expressions, one of which is the result of an operation which in the other is indicated and not performed.

Thus, the expression $\frac{a^2 - x^2}{a - x} = a + x$, may be otherwise expressed as follows: the division of $a^2 - x^2$ by $a - x$, *gives as its result* $a+x$. Again, it is the same thing whether we say $a^m \times a^n = a^{m+n}$, or the product of a^m into a^n *gives as its result* a^{m+n}: and the same interpretation may be given to it, whenever it is used in the examples given in the last Chapter.

In all such cases, it may be expressed likewise by the term *equivalent*, inasmuch as both members of such equations may be used *indifferently* in Algebraical operations: and again, the term *equivalent* may be replaced by *equal* in its ordinary sense, except in certain cases of indefinite quotients, where the remainders are not noticed.

Identical equations.

128. In all such *identical* equations, where the members on each side of the sign = are either identical or reducible to identity, by means of Algebraical operations, the symbols preserve their general and arbitrary values, no limitation being imposed upon them by the conditions of the equation: in other words, they are subject to the *law of continuity*[*], and admit of every possible value from infinite and negative to infinite and positive.

Equations not identical.

129. But if one member of the equation is not reducible by any Algebraical operations to identity with the other, the sign = can no longer be interpreted in its first or second sense, since one member cannot be replaced by the other in Algebraical operations: under these circumstances, the sign = must be confined to its third meaning, of *equality;* and in order that this may be the case, some one of the symbols must be dependent upon the other for its value: for if all the symbols be considered as arbitrary, and arbitrary values be assigned to them,

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[*] We make use of this expression as a *technical* phrase, the use of which is frequently extremely convenient to express the distinction between quantity considered as continuous and discontinuous: all quantity expressed by numbers is discontinuous, since they designate values which are separated by determinate intervals: whilst quantity expressed by Geometrical lines or general symbols is continuous, since they are capable of comprehending every possible value between given limits.

the equality, which the sign = expresses, cannot *necessarily* take place, since the terms of the expressions on each side of it are neither identical nor capable of being made so.

Object proposed in the solution of equations.

130. It is not our intention to enter at present into the examination of the laws of dependence of one symbol upon the other, in such equations, or in other words, of the theory of the *solution* of equations, an enquiry of great extent and difficulty, and incomparably the most important of all the inverse processes of Algebra: it will be sufficient, in order to explain our meaning, to shew this dependence in a few very simple instances.

Examples.

If $x=a$, the value of one symbol is identical with and therefore dependent upon that of the other: in other words, if one be arbitrary, the other is not so.

If $x=a+b$, and if a and b are arbitrary and assumable at pleasure, the value of x is determined by them, and therefore dependent upon them: again, if x and b be arbitrary, a, which is equal to $x-b$, is dependent upon x and b: if x and a be arbitrary, b, which is equal to $x-a$, is dependent upon x and a.

If $ax+b=cx+d$, the relation existing between the five symbols, x, a, b, c, d, may be exhibited under the five following forms:

$$(1)\qquad x=\frac{d-b}{a-c}.$$

$$(2)\qquad a=\frac{cx+d-b}{x}.$$

$$(3)\qquad b=(c-a)x+d.$$

$$(4)\qquad c=\frac{ax+b-d}{x}.$$

$$(5)\qquad d=(a-c)x+b.$$

It thus appears, that if we consider four of the five symbols in the equation as arbitrary and independent, the fifth is necessarily dependent upon them and its value determined.

Generalization of equivalent forms when the symbols in which they were first discovered were not general.

131. Our great object in the very lengthened discussion which we have just concluded, has been to point out the distinction between the science of Algebra when considered with reference to its own principles, and when considered with reference to its applications, and to shew in what manner and to what extent the assumptions which regulate the combinations of general and arbitrary symbols in Algebra were suggested, and their interpretation limited by other and subordinate sciences: the principles which determine the connection between these sciences being once established, we shall be fully prepared to consider to what extent we can consider equivalent forms suggested or investigated upon the principles of a subordinate science, as equally true when expressed in general symbols.

Thus the principles of Arithmetical Algebra lead to the equation

$$a^n \times a^m = a^{n+m},$$

when n and m were whole numbers: it was the conversion of this conclusion in one science into an assumption in the other, which lead to the same equation,

$$a^n \times a^m = a^{n+m},$$

when n and m were general symbols.

If, however, we had commenced with the *assumption* that there existed some *equivalent* form for $a^n \times a^m$, when n and m were general symbols; and if we had discovered and *proved* that this form in Arithmetical Algebra was a^{n+m}, where n and m were such quantities as Arithmetical Algebra recognizes, then we might infer that such likewise must be the *equivalent* form in Symbolical Algebra: for this form can undergo no change, according to the assumptions which we have made, from any change in the nature of its symbols, and must therefore continue the same when the symbols are numbers: if, therefore, we discover this form in any one case, we discover it for all others.

Law of the permanence of equivalent forms stated

132. Let us again recur to this principle or law of the *permanence of equivalent forms*, and consider it when stated in the form of a *direct* and *converse* proportion.

"*Whatever form is Algebraically equivalent to another, when expressed in general symbols, must be true, whatever those symbols denote.*"

Conversely, if we discover an equivalent form in Arithmetical Algebra or any other subordinate science, when the symbols are general in form though specific in their nature, the same must be an equivalent form, when the symbols are general in their nature as well as in their form.

and proved.

The direct proposition must be true, since the laws of the combinations of symbols by which such equivalent forms are deduced, have been assumed without any reference to their specific nature, and the forms themselves, therefore, are equally independent.

The converse proposition must likewise be true for the following reasons:

If there be an equivalent form when the symbols are general in form and in their nature also, it must coincide with the form discovered and proved in the subordinate science, where the symbols are general in form but specific in their nature: for in passing from the first to the second, no change in its form can take place by the first proposition.

Secondly, we may assume such an equivalent form in general symbols, since the laws of the combinations of symbols are assumed in such a manner as to coincide strictly with the corresponding laws in subordinate sciences, such as Arithmetical Algebra: the conclusions, therefore, so far as their form is concerned, are necessarily the same in both; and the Algebraical *equivalence* which exists in one case must exist likewise in the other.

Its great importance.

133. The principle which is expressed in these propositions, which we have named *the law of the permanence of algebraical forms*, is one of the greatest importance, and merits the most profound and careful consideration: it points out the proper object of those demonstrations in Algebra, which have reference to the research of equivalent forms, and shews why they may be safely generalized, even though they may be obtained by the aid of specific values of the symbols: if a general equivalent form be assumed to exist, it is clearly sufficient if we can discover it in any one of its states of existence, corresponding to the different specific values of its general symbols: and if we commence by detecting its existence for specific values of the symbols, we may generalize the symbols, since the same form continues to be equivalent for all algebraical operations.

An example.

134. Thus, we may assume the existence of an equivalent form for $(1+x)^n$, when n is a general symbol, and we may discover it when n is a whole number: or we may commence by the discovery of the equivalent form of $(1+x)^n$, when n is a whole number, and subsequently assume the existence of it, when n is a general symbol: in the first case, assuming its existence, it is necessarily the form discovered: in the second case, the law we have mentioned and the reasoning by which it is established, would shew that the form discovered is algebraically equivalent, when n is a general symbol.

Not to be superseded by other methods of investigation.

135. It is no objection to the use of this principle, that in consequence of its extreme abstraction and generality, it requires a great and painful effort of the mind to apprehend thoroughly the evidence of its truth, unless

it can be shewn that there are other and more easily intelligible, though less general, methods of arriving at similar conclusions; but a very little consideration will shew, that the province of demonstration is extremely limited when general symbols are employed, since we must reject altogether the aid derived from the specific values of the symbols: we are thus confined to the assumed laws of their combination and incorporation with each other, and consequently demonstration can only extend to cases which those laws comprehend: it is for this reason that there can be no demonstration, independently of additional assumptions, of the existence of an equivalent form for $(1+x)^n$, when n is a general symbol.

Principle of symmetrical combinations.

136. Amongst the most important aids for the discovery and determination of equivalent forms, whether their general existence can be assumed or proved, may be mentioned the principle of *symmetrical combinations*, of which use has been made in many examples in the last chapter: it may be considered as such a modification of the principle of *sufficient reason* as is applicable to the science of Algebra, and in its most general form may be stated as follows:

"*When several events are equally likely to happen, if one can be shewn to happen, the others must: and if one of them can be shewn not to happen, neither can the others happen.*"

When we speak of events as equally likely to happen, we mean that they are similarly circumstanced; and when we speak of one event as necessarily determining the others, we merely say that the same premises must necessarily lead to the same conclusion.

Example.

137. Thus, in our determination of the product or equivalent form of $(x+a)$ $(x+b)$ $(x+c)$, we observed that the general symbols a, b, c were equally involved

in each of the factors $x+a$, $x+b$ and $x+c$: and we concluded from the general principle above-mentioned, that if ax^2 was a term of the product, bx^2 and cx^2 must be equally so: and that if abx was a term of the product, acx and bcx must be equally so: for the same reasons which determined the existence of ax and abx, must determine that of bx^2 and cx^2 in one case, and that of acx and bcx in the other: if we deny these consequences, there is an end of all certainty in our reasoning from the same causes to the same effects.

Different species of Induction.

138. Amongst other processes of reasoning which are employed in the discovery of equivalent forms, we may mention Induction, of which there are several species, presenting very different degrees of evidence, from what is merely probable to what may be considered as demonstrative.

Induction from insulated facts.

139. The first and most humble of these species of Induction, is that by which we draw a general conclusion from the observation of particular facts.

In physical science, such inductions are useful as a means of classifying facts, and as furnishing indications of general truths, which other methods may establish. In Algebra, or rather in Arithmetical Algebra, it may suggest the existence of an equivalent form, but no dependence can be placed upon the generalization of conclusions thus deduced, unless other methods can be invented for their demonstration.*

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* As an example of the distrust with which such generalizations should be viewed, may be mentioned the formula

$$x^2+x+41$$

discovered by Euler, which represents *prime* numbers for all values of x, from 0 to 39: it would be a false induction, however, to infer from so many concurrent facts, that such would generally be the case, for if $x=40$ or 41, the number expressed by the formula is clearly divisible by 41. Another and a more remarkable example occurs in the formula 2^n+1, which Fermat asserted was always a prime number, but which Euler shewed not to be so, when n was 32. In the theory of numbers, we meet with numerous examples of a similar nature.

We advance considerably upon the scale of evidence, when the facts upon which our inductions are founded are connected with each other: thus, if a formula be found to be true for several successive numbers, and there is a discoverable dependence of the several cases upon each other, when taken in the order of their succession, we may assume it to be true for any number; the demonstration, however, is not complete, unless we can shew, that *assuming* its truth for any given number, it *must* be true for the number immediately succeeding. We can then pass upwards from the known cases, by means of this proposition, to any extent we please.

An example of this species of Induction is given in Ex. 12, Art. 39, in ascertaining the law of formation of the product of any number of binomial factors $x+a$, $x+b$, $x+c$, &c.; in order to complete the demonstration of the proposition there deduced, it would be requisite to shew that if true for $(n-1)$ factors, it must necessarily be true for n factors; a step in the process of reasoning which it is not very difficult to supply.

Analogy: to what extent it may be trusted.

140. General conclusions drawn from *analogy*, are founded upon probable evidence only, and require, like those deduced by Induction, the confirmation of demonstrations drawn from other sources: the operations and resulting forms in Arithmetic and Geometry, expressed in symbols, bear a strict analogy to operations bearing the same names, and to the forms resulting similarly from them in Algebra, where the symbols are perfectly general: but it is by the *law of the permanence of equivalent forms*, and not by analogy, that we are enabled to pass from one to the other: it is only so far, therefore, as analogy may be considered as a modified expression of this law, that we are legitimately enabled to generalize conclusions deduced by means of it. We will explain our meaning more fully by an example.

Example.

If we replace $\overline{1+x}|^2$, $\overline{1+x}|^3$, $\overline{1+x}|^4$, $\overline{1+x}|^5$, &c. by equivalent forms, the second terms of these forms will be severally $2x$, $3x$, $4x$, $5x$, &c. where the coefficient of x is the index of the binomial $1+x$: if we sought for equivalent forms of $\overline{1\div x}|^{\frac{5}{2}}$, $\overline{1+x}|^{\frac{7}{2}}$, $\overline{1+x}|^{\frac{9}{2}}$, &c., we should *conclude from analogy*, that their several second terms would be $\frac{5}{2}x$, $\frac{7}{2}x$, $\frac{9}{2}x$, &c.: such a conclusion, however, would not be correct or capable of generalization, unless the second term of the equivalent form of $\overline{1+x}|^{n}$ was nx, whatever n might be: in other words, all such deductions from analogy, where algebraical forms are concerned, depend upon the *law of the permanence of equivalent forms*, and derive their authority from it.

Analogous rules and operations.

141. A different case from the one just considered, is where the analogy which exists between the operations in different sciences, leads to analogous rules in corresponding processes, which differences of notation and other circumstances may make very different from each other; thus, in the course of the next two chapters we shall have to consider the rules for finding the greatest common measure of two numbers, and the highest (in dimensions) common factor of two compound algebraical expressions, which bear the most intimate analogy to each other, both in form and expression, though the process of reasoning by which they are severally deduced, is essentially distinguished in the two cases, in consequence of the essential difference of numerical and algebraical notation.

What is essential to a demonstration.

142. Before we conclude the very long and miscellaneous discussion of the different subjects contained in this chapter, it may be proper to add a few remarks in answer to the question of "What constitutes a demonstration?" or in other words, under what circumstances the proof of a proposition may be considered as complete in point of form as well as in point of evidence: for this purpose, it is necessary to state distinctly in the first place what we mean by a demonstration.

A demonstration is the establishment of a NECESSARY *connection between the proposition to be demonstrated, and one or more other propositions, one of which has been already demonstrated to be true, or may be assumed to be so.*

No demonstrations without definitions.

143. In the first place, *definitions* are essential to demonstrations: for we must perfectly understand what are the objects of the science, as well as the nature of the operations to which they are subject, before we can deduce any certain or *necessary* conclusions respecting them: in Geometry, where lines and the figures formed by them are the objects of the science, we have no concern with such as are actually traced upon paper, but with such only as possess the properties which the definitions assign to them: physical lines and physical triangles do not possess the properties of geometrical lines and geometrical triangles, but are essentially variable, and incapable, therefore, of accurate description, and consequently of definition: there can be no certainty or *necessary* truth, therefore, in general conclusions concerning them: in other words, demonstration in this and in all other cases, can have no existence when the objects of demonstration are incapable of being defined.

Demonstrations terminate in axioms.

144. Demonstrations terminate in demonstrated or admitted truths, and consequently always in the latter, if pursued completely to their conclusion: such admitted truths or axioms, as they are generally called, are propositions not resolvable into others of a more simple nature, and consequently not admitting of formal demonstration: they are propositions, in short, of such a nature, that no reasoning can add to the force of their evidence, and therefore, if they are denied, there must be an end of all demonstration, since at the utmost we can only succeed in establishing a necessary connection and dependence between two controverted propositions, and it is only by the concession of one that we can be said to have demonstrated the other: it is in this sense that demonstrations may be said to depend upon axioms.

Axioms depend upon definitions.

145. Axioms, however, depend upon definitions equally with other propositions: thus, equality in Geometry is defined to be determined by the coincidence of quantities in all their parts: in Arithmetic, it is the identity of the numbers; and in Algebra, the identity of the symbols, by which quantities are represented, or in terms of which they are expressed: without such a definition, the proposition that "things which are equal to the same are equal to one another," could no longer be considered as axiomatic, inasmuch as we should be at a loss for the principle or principles by which the equality was to be ascertained*.

Demonstrations complete without axioms.

146. If demonstrations, therefore, depend upon axioms, they depend upon definitions, and may, therefore, be said to be entirely founded upon them: it remains to be con-

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* The definitions and axioms which appear in Euclid, present many examples of the confusion of one with the other: thus the definition of equality and the axiom above-mentioned, which is founded upon it, are equally referred to the list of axioms: a straight line is defined to be "a line which lies *evenly* between its extreme points," whilst his essential definition of such lines is found amongst the axioms, when it is asserted as a self-evident truth, that "two straight lines cannot enclose a space." Again, parallel lines are defined to be those, "which in the same plane never meet, if produced ever so far;" and they are therefore determined to be so by a test which is incapable of application, unless through the medium of another proposition: we are therefore obliged, in order to complete the theory of such lines, to make use of the 12th axiom, which is not a self-evident consequence of their definition: if however, we should make the 12th axiom itself the definition of parallel lines: or if we should replace it by the following, which is more simple and immediate in its applications; "that lines in the same plane, which make equal angles towards the same parts with any line which meets them, are said to be parallel to each other," we should experience no difficulty in proving not merely the 12th axiom, but likewise every other proposition concerning them which is required for the purposes of Geometry: the proposition, which was before the definition, that parallel lines never meet, if produced ever so far, will now follow as a consequence: its converse, that lines in the same plane which never meet, however far they may be produced, is totally useless in a system of Geometry, inasmuch as it can only present itself as a consequence of other conditions: it is true however that this definition would require some modification of the definition of a straight line, by which it might be inferred, that whatever conclusion is true of one straight line and of any portion of it, must be true likewise of every portion of the same and of every other straight line.

sidered, whether the formal statement of axioms is essential to the form or to the evidence of demonstrations.

In the first place, the preliminary statement of an axiom can add nothing to the evidence of a demonstration, inasmuch as its truth is an equally manifest consequence of the definitions, whether it appears for the first time among a series of dependent propositions or not: and in the second place, it is of little importance as far as the form of the demonstration is concerned, inasmuch as it requires a similar act of the mind to perceive the necessary dependence of one proposition upon another, that is required to perceive the necessary dependence of an axiom upon a definition or definitions.

Use of axioms in Geometry.

147. In systems of Geometry, lists of axioms are given, which, if properly selected, should include all those propositions which are necessary and self-evident consequences of the definitions, and which are not susceptible of formal demonstration: we should exclude, therefore, from such lists, propositions like the following, namely, "That all right angles are equal," and "that all circles are equal, whose radii are equal," which, though obvious consequences of the definition of a right angle and a circle, require and admit of an intermediate process of reasoning to connect them with the definition of equality. In such a science, where the relations of the quantities with respect to magnitude are simple and definite, the use of axioms adds something to the concinnity and completeness of demonstration, by empowering us in all cases to refer to a definition, or to a demonstrated or acknowledged proposition, and in consequence enables us to limit controversy concerning such demonstrations, by shewing that they all terminate when pursued to their conclusion, in a small and determinate number of self-evident propositions.

No axioms in Symbolical Algebra.

148. The case, however, is very different in a science of general symbols, which by themselves express no relations of magnitude: in such a science there are, properly

speaking, no axioms, since the propositions, *immediately* deducible from the definitions and assumptions, must be considered rather as the necessary and immediate consequences of defined operations, than the necessary and self-evident results of reasoning.

Axioms not required to be formally stated in sciences subordinate to Algebra.

149. It is only, therefore, in the sciences subordinate to Algebra, such as Arithmetic and Arithmetical Algebra and Geometry, that we must look for the application of axioms: if stated formally, therefore, they must be specially adapted to those sciences and the series of them which correspond to one science, would only correspond to another so far as the definitions from which they were deduced were the same: under such circumstances, it seems unnecessary to encumber our demonstrations by references to them, as they may in all cases be supplied by the understanding of the reader: a deficiency in form, which will be much less observable in demonstrations in most of the sciences subordinate to Algebra, than in that of Geometry, inasmuch as they are less complete in the mutual dependence of all their parts upon each other, in consequence of the more miscellaneous nature of the quantities which are considered, and the algebraical form under which they are exhibited.

◆

CHAP. IV.

◆

ON THE APPLICATION OF ALGEBRA TO THE THEORY OF NUMERICAL FRACTIONS.

Reasons for the separate consideration of numerical and algebraical fractions.

150. THERE are some advantages attending the separate consideration of numerical and algebraic fractions: for in the first place, the processes for the reduction of numerical and compound algebraical fractions to their most simple terms, are merely connected by analogy, and are not founded upon the same principles: in the second place, there are some propositions which are intimately connected with the theory of numerical fractions, which have no application with respect to those which are algebraical: in the third and last place, the demonstrations of the rules of numerical fractions, merely require the aid of that Arithmetical Algebra, the nature and use of which we have very fully considered in the preceding chapter.

The fraction $\frac{a}{b}$ may be reduced when a and b have a common measure.

151. The fraction $\dfrac{a}{b}$, where a and b are whole numbers, may be reduced to an equivalent fraction in lower terms, when a and b have a common divisor: and the process for finding the greatest common divisor or greatest *common measure*, as it is called, of the numerator and denominator of a fraction, or of two numbers generally, one of the most common and most useful in the arithmetic of fractions, is expressed in the following rule.

Rule for finding the greatest common measure of two numbers.

152. *Divide the greater of the two numbers by the less, and the last divisor by the last remainder, repeating the process until there is no remainder: the last divisor is the greatest common measure required.*

Form of the operation.

153. The form of the operation, expressed in symbols, may be exhibited as follows:

$$\begin{array}{l}
b)\ a\ (p \\
\quad \underline{pb} \\
\qquad c)\ b\ (q \\
\qquad\quad \underline{qc} \\
\qquad\qquad d)\ c\ (r \\
\qquad\qquad\quad \underline{rd} \\
\qquad\qquad\quad\ \cdot\ \cdot
\end{array}$$

Its interpretation.

The interpretation of this form of the process is very easy: b is contained in a, p times, with a remainder c; c is contained in b, q times, with a remainder d; d is contained in c, r times and there is no remainder.

We have supposed the process to terminate after three divisions: the demonstration which follows would be equally applicable, if it had proceeded to a greater number of them: for this purpose, it is convenient for us to premise the two following Lemmas.

Lemma.

154. LEMMA I. If a number measure another, it will measure any multiple of that number.

If $a=cx$, then $ma=mcx$: or x is contained c times in a, and mc times in ma.

Lemma.

155. LEMMA II. If a number measure each of two others, it will measure their sum and difference.

For if $a=cx$ and $b=dx$, we have

$$a+b=cx+dx=(c+d)x;$$

$$\text{and } a-b=cx-dx=(c-d)x:$$

and since c and d are whole numbers, $c+d$ and $c-d$ are whole numbers; and therefore x, which measures a and b, measures $a+b$ and $a-b$.

Proof that d is a measure of a and b.

156. In the first place, we shall prove that d is a measure of a and b.

Since d measures c by the units in r, it measures qc (Art. 154): it measures $qc+d$ or b, since it measures qc and d (Art. 155): it measures b, and therefore pb: it measures pb and c, and therefore $pb+c$ or a: it appears, therefore, that d is a measure both of a and b.

Every measure of a and b is a measure of d.

157. In the second place, we shall prove that every measure of a and b is a measure of d.

For if a number measures a and b, it measures a and pb, and therefore their difference (Art. 155) $a-pb$ or c: it measures b and c, and therefore b and qc, and consequently $b-qc$ or d.

Proof that d is the greatest measure of a and b.

158. Since every number which measures a and b, measures d, the greatest number which measures a and b measures d: therefore d, which measures a and b, is their greatest common measure; for no number greater than d can measure d.

Greatest common measure of three or more numbers.

159. If the greatest common measure of three numbers a, b and c be required, we must find d the greatest common measure of a and b, and then the greatest common measure of d and c is the greatest common measure of a, b and c.

For every common measure of a and b is a measure of d (Art. 157), and therefore the greatest common measure of d and c is the greatest common measure of a, b and c.

The same principle may easily be extended to find the greatest common measure of four or more numbers.

If a be prime to b, there is no other fraction equal to $\frac{a}{b}$, whose

160. If a and b have no common measure except unity, they are said to be *prime* to each other, and the fraction $\frac{a}{b}$ is in its lowest terms: for in that case there

is no other fraction equal to $\frac{a}{b}$, whose terms are not equimultiples of a and b.

terms are not equimultiples of a and b.

In order to prove this proposition, let us suppose $\frac{a}{b}=\frac{a'}{b'}$: then by the process for finding the greatest common measure of a and b, and of a' and b' respectively, we get

$$\begin{array}{lll} b)\,a\,(p & \qquad & b')\,a'\,(p' \\ \overline{\quad} & & \overline{\quad} \\ \quad c)\,b\,(q & & \quad c')\,b'\,(q' \\ \quad \overline{\quad} & & \quad \overline{\quad} \\ \qquad d)\,c\,(r & & \qquad d')\,c'\,(r' \\ \qquad \overline{\quad} & & \qquad \overline{\quad} \\ \qquad \cdot & & \qquad \cdot \end{array}$$

Since $a=pb+c$, and $a'=p'b'+c'$, we have $\frac{a}{b}=p+\frac{c}{b}$ and $\frac{a'}{b'}=p'+\frac{c'}{b'}$: and since $\frac{a}{b}=\frac{a'}{b'}$, we have, therefore, $p+\frac{c}{b}=p'+\frac{c'}{b'}$, and consequently $p=p'$ and $\frac{c}{b}=\frac{c'}{b'}$, since p and p' are whole numbers, and $\frac{c}{b}$ and $\frac{c'}{b'}$ proper fractions.

Since $\frac{c}{b}=\frac{c'}{b'}$, it follows that their reciprocals $\frac{b}{c}$ and $\frac{b'}{c'}$ are also equal; for multiplying them both by bb', we find $\frac{c}{b}\times bb'=\frac{c'}{b'}\times bb'$ or $cb'=c'b$: again, dividing by cc', we get $\frac{cb'}{cc'}=\frac{c'b}{cc'}$ or $\frac{b'}{c'}=\frac{b}{c}$.

Since $b=qc+d$ and $b'=q'c'+d'$, it follows as before, since $\frac{b}{c}=\frac{b'}{c'}$, that $q=q'$, $\frac{d}{c}=\frac{d'}{c'}$, and therefore also $\frac{c}{d}=\frac{c'}{d'}$.

If a be prime to b, the last divisor d must be equal to unity (Art. 152), and therefore $c=r$: and since $\frac{c}{d}=\frac{c'}{d'}$, we have $r=r'$, and there can, under these circumstances, be no remainder or $\frac{c'}{d'}=r'$: it consequently follows that d' is a common measure of a' and b'.

$$\begin{aligned}
\text{Also } c'=cd'&: \text{ for } c'=r'd'=rd'=cd':\\
b'=bd'&: \text{ for } b'=q'c'+d'=qcd'+d'\\
&=(qc+1)\,d'=bd', \quad \text{since } b=qc+1:\\
a'=ad'&: \text{ for } a'=p'b'+c'=pbd'+cd'\\
&=(pb+c)\,d'=ad', \quad \text{since } a=pb+c.
\end{aligned}$$

We thus get $\frac{a'}{b'}=\frac{ad'}{bd'}$: or in other words, a' and b' are either equal to a and b respectively, in which case $d'=1$, or they are equimultiples of them.

Least common multiple of two numbers.

161. In the reduction of several fractions to a common denominator, as well as upon many other occasions, it is important to find the *least common multiple* of two or more numbers; that is, to find the *least* number which is severally divisible by those numbers, without a remainder: we shall first prove the common rule for this purpose for two numbers, and then shew in what manner it may be extended to any number of them.

Let m be the least common multiple of two numbers a and b, whose greatest common measure is x: since m is a multiple of both a and b, we may suppose $m=pa=qb$, where p and q are whole numbers: also, since $pa=qb$, we have (by dividing them both by pb) $\frac{a}{b}=\frac{q}{p}$: and since m is the least possible, q and p are the least possible, and consequently $\frac{q}{p}$ is the fraction $\frac{a}{b}$ in its lowest terms:

for if not, let any other fraction $\frac{q'}{p'}$, be the fraction $\frac{a}{b}$ in its lowest terms: then, since $\frac{q'}{p'}=\frac{a}{b}$, we have $q'b'=p'a$, or there are multiples of a and b, which are less than pa and qb, which is impossible: it follows, therefore, that $\frac{q}{p}$ is the fraction $\frac{a}{b}$ in its lowest terms, and therefore we have (Art. 160) $q=\frac{a}{x}$, and consequently $m=qb=\frac{ab}{x}$: a result, from which we conclude, *that the least common multiple of* a *and* b *is their product divided by their greatest common measure.*

Every other multiple of two numbers is a multiple of their least common multiple.

162. Every other multiple of a and b is a multiple of m: for let M be any other multiple, and suppose $M=Pa=Qb$; we consequently have $\frac{a}{b}=\frac{Q}{P}=\frac{q}{p}$: and since $\frac{q}{p}$ is in its lowest terms, it follows (Art. 160) that $Q=nq$ and $P=np$, when n is a whole number: we thus get $M=Qb=nqb=nm$, since $m=qb$.

Least common multiple of three or more numbers.

163. If the least common multiple of three numbers a, b and c be required, we must find m the least common multiple of a and b, and then m' the least common multiple of m and c, is the least common multiple of a, b and c.

For every common multiple of a and b is a multiple of m, and therefore the least common multiple of m and c is the least common multiple of a, b and c.

In a similar manner we may find the least common multiple of four or more numbers.

Reduction of fractions to the same denominator.

164. In adding fractions together, or in subtracting them from each other, it is necessary to reduce them to

equivalent fractions with the same denominator: this is done by the following rule (Art. 54).

Multiply the numerator and denominator of each fraction into all the denominators of the remaining fractions: the resulting fractions are equivalent to the former and have the same denominator.

Thus $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ are equivalent to $\frac{adf}{bdf}, \frac{cbf}{dbf}, \frac{ebd}{fbd}$, the numerator and denominator of each fraction being multiplied by the same quantity.

Reduction of fractions to the least common denominator.

165. When the denominators of the fractions admit of a common multiple which is less than their continued product, their least common multiple may be made the common denominator, and equivalent fractions found under a more simple form than by the common rule: thus the fractions $\frac{a}{px}, \frac{c}{qx}$ are equivalent to $\frac{aq}{pqx}$ and $\frac{cp}{pqx}$, x being the greatest common measure of the denominators, and pqx their least common multiple: again, the fractions $\frac{a}{px}, \frac{b}{qx}, \frac{c}{ry}, \frac{d}{sy}$, are equivalent to $\frac{aqrsy}{pqrsxy}, \frac{bprsy}{pqrsxy}, \frac{cpysx}{pqrsxy}, \frac{dpqrx}{pqrsxy}$, where $pqrsxy$ is the least common multiple of the denominators. It is always convenient and frequently of essential importance to attend to this reduction of fractions to their most simple forms, with a common denominator.

166. We have on a former occasion (Art. 55, 56) explained the principles of the rules for the addition, subtraction, multiplication and division of numerical fractions: and as those principles would derive no additional evidence or generality from the use of symbolical language, it is not necessary to repeat them here: we shall therefore

confine ourselves to a statement of the rules, with a few examples.

(*α*) Reduce the fractions to a common denominator, add the numerators together and subscribe the common denominator. **Rule for the addition of fractions.**

Ex. 1. $$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

Ex. 2. $$\frac{a}{b}+\frac{c}{d}+\frac{e}{f}=\frac{adf}{bdf}+\frac{bcf}{bdf}+\frac{bde}{bdf}$$
$$=\frac{adf+bcf+bde}{bdf}.$$

Ex. 3. $$\frac{a}{bx}+\frac{c}{dx}=\frac{ad}{bdx}+\frac{bc}{bdx}=\frac{ad+bc}{bdx}\ \text{(Art. 165)}.$$

Ex. 4. $$\frac{a}{x}+\frac{b}{cx^2}=\frac{acx}{cx^2}+\frac{b}{cx^2}=\frac{ax+b}{cx^2}\ \text{(Art. 165)}.$$

(*β*) Reduce the fractions to a common denominator, subtract the numerator or sum of the numerators of the fractions to be subtracted, from the numerator or sum of the numerators of the others and subscribe the common denominator. **Rule for the subtraction of fractions.**

Ex. 1. $$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}.$$

Ex. 2. $$\frac{a}{b}-\frac{c}{d}-\frac{e}{f}=\frac{adf}{bdf}-\frac{bcf}{bdf}-\frac{bde}{bdf}$$
$$=\frac{adf-bcf-bde}{bdf}.$$

Ex. 3. $$\frac{a}{bx}-\frac{c}{dx^2}+\frac{e}{fx^3}=\frac{adfx^2-bcfx+bde}{bdfx^3}$$
(Art. 165).

Rule for the multiplication of fractions.

(γ) Multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Ex. 1. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Ex. 2. $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$.

Ex. 3. $\frac{a}{bx} \times \frac{cx}{d} = \frac{acx}{bdx} = \frac{ac}{bd}$.

Ex. 4. $\frac{ax}{by} \times \frac{c}{d} \times \frac{ey}{fx^2} = \frac{acexy}{bdfx^2y} = \frac{ace}{bdfx}$.

Rule for the division of fractions.

(δ) Invert the divisor and then proceed as in Multiplication.

Ex. 1. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

Ex. 2. $\frac{ax}{by} \div \frac{cx}{dy} = \frac{ax}{by} \times \frac{dy}{cx} = \frac{adxy}{bcxy} = \frac{ad}{bc}$.

Ex. 3. $\frac{ax^3}{bxy} \div \frac{cx^2}{dy^2} = \frac{ax^3}{bxy} \times \frac{dy^2}{cx^2} = \frac{adx^3y^2}{bcx^3y} = \frac{ady}{bc}$.

CHAP. V.

ON THE REDUCTION OF ALGEBRAICAL EXPRESSIONS TO EQUIVALENT AND MORE SIMPLE FORMS.

Reduction of algebraic fractions, whose terms have a common divisor.

167. In the division of algebraical quantities, where the quotient would not be complete and finite, it is commonly most convenient to leave the dividend and divisor under a fractional form: but in many cases such a form would not be the most simple which the expression is capable of receiving, without altering its value and signification: inasmuch as the numerator and denominator may be divided by any factor, whether simple or compound, which is common to them both.

A common factor, which is a single term.

168. If this factor or common divisor be a simple Algebraical term, it is discoverable by inspection, and the reduction required may be effected at once: thus, a is a divisor of every term of the numerator and denominator of $\frac{a^2 - ax}{a^2 + ax}$, which is therefore equivalent to $\frac{a - x}{a + x}$: ax is a divisor of every term of

$$\frac{a^3x - 2a^2x^2 + ax^3}{a^4x + a^3x^2 + a^2x^3 + ax^4},$$

which is at once reducible therefore to

$$\frac{a^2 - 2ax + x^2}{a^3 + a^3x + ax^2 + x^3}:$$

and other instances will present themselves amongst the examples which follow.

A numerical factor.

Again, if all the terms of the numerator and denominator have numerical coefficients, a common measure may exist amongst them, which may be detected either by inspection or by means of the rule (Art. 152), the proof of which has been given in the preceding Chapter.

A compound factor.

169. But the reduction of an algebraical fraction which is effected by these means, is frequently not the only one of which it admits: for in many cases the numerator and denominator may have a common factor consisting of more than one term, which is not discoverable by inspection: for as the factors of a numerical product are lost sight of in the result, so likewise an algebraical factor consisting of more than one term, after its incorporation with other quantities by multiplication, may leave no discoverable traces of its separate existence: it becomes an enquiry, therefore, the importance of which is not limited to the reduction of algebraic fractions, to find the factor of the highest dimensions which is common to two algebraical expressions: we say factors of the highest dimensions and not the greatest, for the terms *greater* and *less* can have no application when we are reasoning with general symbols, to which no specific values are given.

The process for finding compound factors.

170. The process for finding such compound factors is analogous to that for finding the greatest common measure of two numbers.

Let A and B represent two compound algebraical expressions: arrange them as much as possible according to the powers of some one letter, and make that quantity the divisor whose dimensions are not the highest: let the remainder after division be Cc, where C is a compound quantity, and where c is a quantity whether simple or compound, which is obviously not a divisor both of A and B: make C alone the new divisor and the last divisor the new dividend, and let the remainder, if any, after division, be Dd, where d is not a factor of A and B: make D a new divisor and C a new dividend: if there is no remainder after this division, then D is the required compound factor of A and B: if there is a remainder, we must proceed as before until a divisor is found which leaves no remainder: if no such divisor can be found, A and B have no common factor, or this process fails to discover it.

The following scheme may help to make the process more intelligible:

$$\begin{array}{l}
B)\,A\,(P \\
\quad\;\underline{PB} \\
\quad\;Cc \\
\qquad C)\,B\,(Q \\
\qquad\quad\;\underline{QB} \\
\qquad\quad\;Dd \\
\qquad\qquad D)\,C\,(R \\
\qquad\qquad\quad\;\underline{RD} \\
\qquad\qquad\quad\;\cdot\ \cdot
\end{array}$$

In order to prove the truth of this process, we must premise the following Lemma.

Lemma.

171. LEMMA. If D is the highest common divisor of A and B, it is likewise the highest common divisor of Aa and Bb, if a and b have no common factors. For the factors which are common to A and B, are the same which are common to Aa and Bb, no new common factor having been introduced by multiplying a into A and b into B.

It follows from this Lemma, that we may divide or multiply either A or B, by a factor which is not common to both, or which has no divisor common to both, without affecting the dimensions or form of their highest common factor.

Proof of the rule.

172. That D, the last divisor found by this process, is a factor of A and B, may be proved as follows.

$$A = PB + Cc.$$
$$B = QC + Dd.$$
$$C = RD.$$

D is a divisor of C, and therefore of QC and Dd, and therefore of $QC + Dd$ or B: and since it is a divisor of B and C, it is a divisor of PB and Cc, and therefore of $PB + Cc$ or A.

Again, every compound divisor of A and B is a divisor of D.

For every compound factor of A and B is a factor of A and PB, and therefore of $A - PB$ or Cc, and therefore of C alone, since c has no factor common to A and B: it consequently divides B and QC, and therefore $B - QC$ or Dd, and therefore D alone, since d has no factor common to A and B.

It follows, therefore, that D is the highest compound divisor of A and B.

Examples. 173. (1) To reduce the fraction $\frac{x^3 - a^3}{x^2 - a^2}$ to its lowest terms.

$$\begin{array}{lllll}
 & x^2 - a^2) & x^3 - a^3\ (x & & \\
 & & \underline{x^3 - a^2 x} & & \\
\text{(Dividing by } a^2\text{)} & & a^2 x - a^3 & & \\
 & & x - a) & x^2 - a^2\ (x + a & \\
 & & & \underline{x^2 - ax} & \\
 & & & \quad ax - a^2 & \\
 & & & \quad \underline{ax - a^2} & \\
 & & & \quad \cdot \quad \cdot &
\end{array}$$

The highest common divisor, therefore, of $x^3 - a^3$ and $x^2 - a^2$, is $x - a$, and the fraction reduced is $\frac{x^2 + ax + a^2}{x + a}$

In this example we have stopped the first division after one operation: if we had proceeded further, the successive terms in the quotient would have been $\frac{a^2}{x}$, $\frac{-a^3}{x^2}$, $\frac{a^4}{x^3}$, &c. and the successive remainders would have been $-a^3+\frac{a^4}{x}$, $\frac{a^4}{x}-\frac{a^5}{x^2}$, $-\frac{a^5}{x^2}+\frac{a^6}{x^3}$, &c. which, when severally divided by $-\frac{a^3}{x}$, $\frac{a^4}{x^2}$, $-\frac{a^5}{x^3}$, &c. would have given the same quantity $x-a$ for the new divisor: in order to avoid such unnecessary divisions, it is *convenient* to adopt this general rule: *continue the division as far as possible in each case without introducing a term into the quotient under a fractional form, and no further.*

Rule for avoiding unnecessary divisions.

(2) Reduce $\frac{x^2+(a-b)x-ab}{x^2+(a+b)x+ab}$ to its lowest terms.

When the first terms of each expression, arranged according to the powers of some one letter, are identical, write one of them beneath the other and subtract.

$$\begin{array}{rr}
 & x^2+(a-b)x-ab \\
 & x^2+(a+b)x+ab \\
\hline
(\text{Dividing by } -2b) & -2bx-2ab
\end{array}$$

$$\begin{array}{r}
x+a)\ x^2+(a+b)x+ab\ (x+b \\
\underline{x^2+ax\qquad\qquad} \\
bx+ab \\
\underline{bx+ab} \\
\cdot\quad\cdot
\end{array}$$

The common factor is $x+a$, and the fraction reduced is $\frac{x-b}{x+b}$.

(3) Reduce $\dfrac{x^3-39x+70}{x^2-\ 3x-70}$ to its lowest terms.

$$\begin{array}{lr}
 & x^2-3x-70)\ x^3-39x+70\ (x+3 \\
 & \underline{x^3-\ 3x^2-70x} \\
 & 3x^2+31x+\ 70 \\
 & \underline{3x^2-\ 9x-210} \\
\text{(Dividing by 40)} & 40x+280 \\
 & x+7)\ x^2-3x-70\ (x-10 \\
 & \underline{x^2+7x} \\
 & -10x-70 \\
 & \underline{-10x-70} \\
 & \cdot\ \ \cdot
\end{array}$$

The fraction reduced is $\dfrac{x^2-7x+10}{x-10}$.

In this case we proceed as far as two terms in the quotient of the first division, no fractional quantity being thus introduced: we then divide the remainder, $40x+280$, by 40, which is a measure of both its terms.

(4) Reduce $\dfrac{a^3+(1+a)ax+x^2}{a^4-x^2}$ to its lowest terms.

$$\begin{array}{lr}
 & a^3+(1+a)ax+x^2)\ a^4-x^2\qquad (a \\
 & \underline{a^4+(1+a)a^2x+ax^2} \\
\text{[Dividing by }-(1+a)x] & +(1+a)a^2x-(1+a)x^2 \\
 & a^2+x)\ a^3+(1+a)ax+x^2\ (a+x \\
 & \underline{a^3+ax} \\
 & a^2x+x^2 \\
 & \underline{a^2x+x^2} \\
 & \cdot\ \ \cdot
\end{array}$$

The fraction reduced is $\dfrac{a+x}{a^2-x}$.

The first remainder in this case involves a^3, and the division might have been continued without introducing a term under a fractional form into the quotient: it is more convenient, however, to avail ourselves at once of the obvious reduction which arises from dividing by $-(1+a)x$.

(5) Reduce $\dfrac{abc+9bc-5c^2}{12adf+18bdf-10cdf}$ to its lowest terms.

It is obvious that c divides every term of the numerator, and $2df$ every term of the denominator: if we separate these factors, the fraction assumes the form

$$\frac{c}{2df} \times \frac{ab+9b-5c}{a+9b-5c} = \frac{c}{2df}.$$

(6) Reduce $\dfrac{acx^2+(ad+bc)x+bd}{a^2x^2-b^2}$ to its lowest terms.

If we should begin with dividing the numerator by the denominator, the first term in the quotient would be $\dfrac{c}{a}$, a quantity under a fractional form: in order to avoid this, we multiply the numerator by a, as follows:

$$\begin{array}{r|l}
 & acx^2+(ad+bc)x+bd \\
 & a \\ \hline
a^2x^2-b^2) & a^2cx^2+(ad+bc)ax+abd\ (c \\
 & a^2cx^2-b^2c \\ \hline
\end{array}$$

Dividing by $(ad+bc)$, $(ad+bc)ax+abd+b^2c$

$$\begin{array}{r|l}
ax+b) & a^2x^2-b^2\ (ax-b \\
 & a^2x^2+abx \\ \hline
 & -abx-b^2 \\
 & -abx-b^2 \\ \hline
\end{array}$$

The fraction reduced is $\dfrac{cx+d}{ax-b}$.

(7) Reduce $\dfrac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$ to its lowest terms.

In order that the first term of the quotient may not be fractional, we multiply the numerator by 4.

$$\begin{array}{rl}
 & 3a^3-3a^2b+ab^2-b^3 \\
 & 4 \\ \hline
4a^2-5ab+b^2) & 12a^3-12a^2b+4ab^2-4b^3\ (3a \\
 & 12a^3-15a^2b+3ab^2 \\ \hline
 & \qquad 3a^2b+ab^2-4b^3 \\
 & \qquad 4
\end{array}$$

$$\begin{array}{rl}
4a^2-5ab+b^2) & 12a^2b+\ 4ab^2-16b^3\ (3b \\
 & 12a^2b-15ab^2+\ 3b^3 \\ \hline
\text{(Dividing by } 19b^2\text{)} & \qquad 19ab^2-19b^3
\end{array}$$

$$\begin{array}{rl}
a-b) & 4a^2-5ab+b^2\ (4a-b \\
 & 4a^2-4ab \\ \hline
 & \qquad -ab+b^2 \\
 & \qquad -ab+b^2 \\ \hline
\end{array}$$

The fraction reduced is $\dfrac{3a^2+b^2}{4a-b}$.

We might have made the first remainder the new divisor, after dividing it by b: it is more convenient, however, to multiply it by 4, and to retain the old divisor.

(8) Reduce the fraction $\dfrac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}$ to its lowest terms.

$$\begin{array}{lr}
 & 27x^4+\ 63x^3-12x^2-\ 28x \\
 & 5\qquad\qquad\qquad\qquad\qquad \\ \hline
15x^3+35x^2+3x+7) & 135x^4+315x^3-60x^2-140x\ (9x \\
 & 135x^4+315x^3+27x^2+\ 63x \\ \hline
\text{(Dividing by } -29x) & -87x^2-203x
\end{array}$$

$$\begin{array}{lr}
3x+7) & 15x^3+35x^2+3x+7\ (5x+1 \\
 & 15x^3+35x^2\qquad\qquad \\ \hline
 & 3x+7\qquad \\
 & 3x+7\qquad \\ \hline
 & \cdot\quad\cdot\qquad
\end{array}$$

The fraction reduced is $\dfrac{5x^2+1}{9x^3-4x}$.

In this case 5×27 or 135 is the least common multiple of 15 and 27.

(9) Reduce $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}$ to its lowest terms.

$$\begin{array}{r}
a^2+b^2+c^2+2ab+2ac+2bc \\
a^2-b^2-c^2-2bc\qquad\qquad \\ \hline
2ab+2ac+2b^2+4bc+2c^2 \\
\text{or }\ 2(b+c)a+2b^2+4bc+2c^2
\end{array}$$

$$\begin{array}{lr}
\text{Dividing by } 2(b+c)\quad a+b+c) & a^2-b^2-2bc-c^2\ (a-b-c \\
 & a^2+ab+ac\qquad\qquad\qquad \\ \hline
 & -ab-ac-b^2-2bc-c^2 \\
 & -ab-b^2-bc\qquad\qquad \\ \hline
 & -ac-bc-c^2 \\
 & -ac-bc-c^2 \\ \hline
 & \cdot\quad\cdot\quad\cdot
\end{array}$$

The fraction reduced is $\dfrac{a+b+c}{a-b-c}$.

Case where simple powers only of each symbol are involved.

(10) Reduce $\dfrac{ac+ad+bc+bd}{ae+af+be+bf}$ to its lowest terms.

In cases of this kind, where simple powers only of each symbol are involved, the rule for finding common factors, which has been employed above, no longer can furnish us with any aid.

A little consideration, however, of the nature of the factors which produce such products, will direct us to the mode of detecting those which are in common.

Such products can only arise from the multiplication of compound factors, all whose terms are different from each other.

Such common factors must exist, therefore, as coefficients of more than one of the symbols involved.

Form, therefore, successively the coefficients of each symbol, until such a one is found as divides both the numerator and denominator: if no such factor is found, they can have no common divisor.

In the present instance, the coefficient of a is $c+d$, of b the same, of c, $a+b$, and of d the same: it will be found that $a+b$ alone divides both numerator and denominator, and the reduced fraction becomes $\dfrac{c+d}{e+f}$.

It is obvious that this method will enable us, in all cases, to resolve algebraical products into their component factors, whenever *different powers* of the same symbol are not involved.

Thus, the coefficient of a in

$$ae-be-bf-ad+af+ce-cd+cf+bd$$

is $e+f-d$: and it will be found by trial to be one of the factors of this product.

The coefficient of a in

$$ace + acf + ade + adf + bce + bcf + bde + bdf$$

is $ce + cf + de + df$, which is a factor of the product: and again, the coefficient of c in $ce + cf + de + df$ is $e + f$, which is likewise one of its factors: we thus resolve the original product into the simple factors $a + b$, $c + d$ and $e + f$.

(11) Reduce the fraction

$$\frac{4a^3cx - 4a^3dx + 24a^2bcx - 24a^2bdx + 36ab^2cx - 36ab^2dx}{7abcx^3 - 7abdx^3 + 7ac^2x^3 - 7acdx^3 + 21b^2dx^3 + 21b^2cx^3 + 21bc^2x^3 + 21bd^2x^3}$$

to its lowest terms.

In the first place, we discover by inspection that $4ax$ is a factor of the numerator, and $7x^3$ a factor of the denominator: dividing the fraction by $\frac{4ax}{7x^3}$ or $\frac{4a}{7x^2}$, in order to simplify the remaining part of the operation, and arranging the result according to powers of a, we get

$$\frac{(c - d)a^2 + 6(bc - bd)a + 9(b^2c - b^2d)}{(bc - bd + c^2 - cd)a + 3(b^2c + bc^2 + b^2d + bd^2)}.$$

In the next place, we find by trial that $c - d$, the coefficient of a^2, is a factor of the numerator and denominator: the fraction reduced, in consequence, becomes

$$\frac{a^2 + 6ab + 9b^2}{(b + c)a + 3(b^2 + bc)}.$$

We find by trial, that $b + c$ is a factor of the denominator, and the other factor which results is a divisor of the numerator: we thus reduce the fraction to the form

$$\frac{a + 3b}{b + c}:$$

if we multiply this fraction by $\frac{4a}{7x^2}$, by which the original fraction was divided, we get

$$\frac{4a(a + 3b)}{7x^2(b + c)}$$

for the most simple equivalent form to which it can be reduced.

The common factors of two algebraical expressions under a rational form may always be found.

174. This method of finding the common factors of two algebraical expressions, though in many cases one of successive trials, yet will always succeed in detecting such factors whenever they exist.

For in the first place, when the same power of each letter only exists, the expressions are separately resolvible into their component factors.

In the second place, when different powers of the same letter exist, the factors which do not involve that symbol must exist in the coefficients of every term when the expressions are arranged according to the powers of that letter.

In the third place, common factors involving that letter, may always be found by the common rule: for by successive divisions, according to that rule, we must arrive at a remainder in which that letter does not appear: if this remainder be zero, or identically equal to zero, the last divisor is a common factor: if not, there is no common factor in which that symbol is involved.

Under these circumstances, it will only remain to examine the common factors of all the coefficients of that symbol: and if those coefficients involve more than one power of some one or more letters, they will only require the application of the preceding theory to detect the common factors, if any, which they involve.

Other examples of reductions.

175. We shall subjoin a series of examples of the combination and incorporation of algebraical expressions, whether fractional or not, by addition, subtraction, multiplication and division, reducing them in every instance to their most simple forms: we shall add occasionally such remarks as may be requisite to guide the student,

either in effecting such reductions or in shortening the process for that purpose.

(1) $\frac{a+b}{2}+\frac{a-b}{2}=a.$

The sum of the semi-sum and semi-difference of two quantities is equal to the greater.

(2) $\frac{a+b}{2}-\frac{a-b}{2}=b.$

The difference of the semi-sum and semi-difference of two quantities is equal to the less.

(3) $\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{a^2+2ab+b^2}{a^2-b^2}+\frac{a^2-2ab+b^2}{a^2-b^2}$

$=\frac{2a^2+2b^2}{a^2-b^2}.$

The fractions in this and all similar cases are reduced to a common denominator.

(4) $\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{4ab}{a^2-b^2}.$

(5) $\frac{1}{1+x}+\frac{1}{1-x}=\frac{2}{1-x^2}.$

(6) $\frac{1}{1+x}-\frac{1}{1-x}=-\frac{2x}{1-x^2}.$

(7) $\frac{1}{1-x}-\frac{2}{1-x^2}=\frac{1+x}{1-x^2}-\frac{2}{1-x^2}=\frac{x-1}{1-x^2}$

$=-\frac{1}{1+x}$: reducing the fraction $\frac{x-1}{1-x^2}$,

or $-\frac{1-x}{1-x^2}$ to its lowest terms.

(8) $\dfrac{1}{x-2} - \dfrac{1}{x-5} = -\dfrac{3}{x^2-7x+10}.$

(9) $\dfrac{1}{x-6} - \dfrac{1}{x-5} = \dfrac{1}{x^2-11x+30}.$

(10) $\dfrac{7a-6b}{10} - \dfrac{7a-2b}{11} = \dfrac{7a-46b}{110}.$

(11) $\dfrac{a-3x}{4} + \dfrac{3a-5x}{5} + \dfrac{3a-5x}{20} = a-2x.$

The fractions must be reduced to their least common denominator 20.

(12) $\dfrac{13a-5b}{4} - \dfrac{7a-2b}{6} - \dfrac{3a}{5} = \dfrac{89a-55b}{60}.$

The number 60 is the least common multiple of 4, 6 and 5.

(13) $\dfrac{3a-4b}{7} - \dfrac{2a-b-c}{3} + \dfrac{15a-4c}{12} = \dfrac{85a-20b}{84}.$

(14) $\dfrac{7x-10}{5} - \dfrac{3x-7}{6} - \dfrac{27x-30}{30} = \dfrac{1}{6}.$

(15) $\dfrac{x}{1-x} + \dfrac{x^2}{(1-x)^2} = \dfrac{x(1-x)+x^2}{(1-x)^2} = \dfrac{x}{(1-x)^2}.$

(16) $1+x+\dfrac{x^2}{1-x} = \dfrac{1}{1-x}.$

(17) $1+x+x^2+\dfrac{x^3}{1-x} = \dfrac{1}{1-x}.$

(18) $\dfrac{a}{c} - \dfrac{(ad-bc)x}{c(c+dx)} = \dfrac{a+bx}{c+dx}.$

(19) $$\frac{b}{d}+\frac{ad-bc}{d(c+dx)}=\frac{a+bx}{c+dx}.$$

(20) $$1-\frac{2x^2}{a^2}+\frac{2x^4}{a^2(a^2+x^2)}=\frac{a^2-x^2}{a^2+x^2}.$$

(21) $$1-2x+4x^2-\frac{6x^3-4x^4}{1+x-x^2}=\frac{1-x+x^2}{1+x-x^2}.$$

(22) $$\frac{1}{3(1+x)}+\frac{2-x}{3(1-x+x^2)}=\frac{1}{1+x^3}.$$

(23) $$\frac{1}{3(1-x)}+\frac{2+x}{3(1+x+x^2)}=\frac{1}{1-x^3}.$$

(24) $$\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1+x^2)}=\frac{1}{1-x^4}.$$

(25) $$\frac{1}{8(x-1)}-\frac{1}{4(x-3)}+\frac{1}{8(x-5)}$$
$$=\frac{1}{x^3-9x^2+23x-15}.$$

(26) $$\frac{-1}{x+2}+\frac{1}{2(x+1)}+\frac{1}{2(x+3)}=\frac{1}{x^3+6x^2+11x+6}.$$

(27) $$\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$$
$$=\frac{1+x+x^2}{1-x-x^4+x^5}.$$

(28) $$\frac{4x+13}{25(x+2)^2}-\frac{4x-3}{25(x^2+1)}=\frac{1}{x^4+4x^3+5x^2+4x+4}.$$

(29) $$\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}=\frac{x^2}{x^3+6x^2+11x+6}.$$

(30) $$\frac{a^2}{(b-a)(c-a)}\cdot\frac{1}{x+a}+\frac{b^2}{(a-b)(c-b)}\cdot\frac{1}{x+b}$$
$$+\frac{c^2}{(a-c)(b-c)}\cdot\frac{1}{x+c}=\frac{x^2}{(x+a)(x+b)(x+c)}.$$

(31) $$\frac{1}{3(x+1)}-\frac{12}{(x+2)}+\frac{135}{2(x+3)}-\frac{352}{3(x+4)}$$
$$+\frac{125}{2(x+5)}=\frac{x^4-7x^3}{x^5+15x^4+85x^3+225x^2+274x+120}.$$

(32) $$\frac{a}{b}+\frac{a^2-3ab}{cd}+\frac{a^3-a^2b-ab^2}{bcd}=\frac{a^3-4ab^2+acd}{bcd}.$$

(33) $$\frac{a}{b}-\frac{(a^2-b^2)x}{b^2}+\frac{a(a^2-b^2)x^2}{b^2(b+ax)}=\frac{a+bx}{b+ax}.$$

(34) $$\frac{a}{10}+\frac{b}{10^2}+\frac{c}{10^3}=\frac{100a+10b+c}{10^3}.$$

(35) $$\frac{a}{x^5}+\frac{b}{x^3}+\frac{c}{x}=\frac{a+bx^2+cx^4}{x^5}.$$

(36) $$\frac{a^3}{(a+b)^3}-\frac{ab}{(a+b)^2}+\frac{b}{a+b}=\frac{a^3+ab^2+b^3}{(a+b)^3}.$$

(37) $$\frac{a}{a-x}-\frac{a^2}{(a-x)^2}+\frac{a^3}{(a-x)^3}-\frac{a^4}{(a-x)^4}$$
$$=-\frac{2a^3x-2a^2x^2+ax^3}{(a-x)^4}.$$

(38) $$\frac{1+2x}{(3-x)(1+x)}+\frac{7}{(2+x)(1-3x)}+\frac{x}{(1+x)(2+x)}$$
$$=\frac{23+16x-30x^2-3x^3}{(3-x)(1+x)(2+x)(1-3x)}.$$

(39) $$\frac{1-x}{1+x}+\frac{(1-x)(1-x^2)}{(1+x)(1+x^2)}+\frac{(1-x)(1-x^2)(1-x^3)}{(1+x)(1+x^2)(1+x^3)}$$
$$=\frac{3-3x-x^2+x^3-x^4+x^5-x^6}{(1+x)(1+x^2)(1+x^3)}.$$

(40) $$\frac{1-x}{1+x}+\frac{1-x-x^2}{1+x+x^2}+\frac{1-x-x^2-x^3}{1+x+x^2+x^3}$$
$$=\frac{3+3x-x^2-6x^3-9x^4-7x^5-3x^6}{1+3x+5x^2+6x^3+5x^4+3x^5+x^6}.$$

(41) $\frac{a+b}{2}\times\frac{a-b}{2}=\frac{a^2-b^2}{4}$, multiplying the numerators together for a new numerator, and the denominators together for a new denominator.

(42) $$\frac{a+b}{2}\div\frac{a-b}{2}\text{ or }\frac{\frac{a+b}{2}}{\frac{a-b}{2}}=\frac{2(a+b)}{2(a-b)}=\frac{a+b}{a-b}:$$

multiplying the numerator of the dividend into the denominator of the divisor for the numerator of the quotient, and the denominator of the dividend into the numerator of the divisor for the denominator of the quotient.

(43) $$\frac{a+b}{a-b}\times\frac{a-b}{a+b}=\frac{a^2-b^2}{a^2-b^2}=1.$$

(44) $$\frac{a+b}{a-b}\div\frac{a-b}{a+b}\text{ or }\frac{\frac{a+b}{a-b}}{\frac{a-b}{a+b}}=\frac{a^2+2ab+b^2}{a^2-2ab+b^2}.$$

(45) $$\frac{a^2+b^2}{a^2-b^2}\times\frac{a-b}{a+b}=\frac{a^3-a^2b+ab^2-b^3}{a^3+a^2b-ab^2-b^3}=\frac{a^2+b^2}{a^2+2ab+b^2}:$$

reducing the fraction to its most simple terms.

(46) $$\frac{a^2+b^2}{a^2-b^2} \div \frac{a-b}{a+b} = \frac{a^3+a^2b+ab^2+b^3}{a^3-a^2b-ab^2+b^3} = \frac{a^2+b^2}{a^2-2ab+b^2}.$$

(47) $$\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3} \times \frac{a^2-ax+x^2}{a+x} = \frac{a^4+a^2x^2+x^4}{a^4-x^4}.$$

(48) $$\frac{x^2-9x+20}{x^2-6x} \times \frac{x^2-13x+42}{x^2-5x}$$

$$= \frac{x^4-22x^3+179x^2-638x+840}{x^4-11x^3+30x^2}$$

$$= \frac{x^2-11x+28}{x^2},$$

when reduced to its lowest terms.

(49) $$\frac{x^2+3x+2}{x^2+2x+1} \times \frac{x^2+5x+4}{x^2+7x+12} = \frac{x+2}{x+3},$$

the first product being reduced to its lowest terms.

(50) $$\frac{x^2+3x+2}{x^2+2x+1} \div \frac{x^2+5x+4}{x^2+7x+12} = \frac{x^2+5x+6}{x^2+2x+1}.$$

(51) $$\frac{ac+(ab+bc)x+b^2x^2}{a-bx} \times \frac{ae+(af-be)x-bfx^2}{a+bx}$$

$$= ce+(cf+be)x+bfx^2.$$

(52) $$\frac{a^2+(2ac-b^2)x^2+c^2x^4}{a^2+2abx+(2ac+b^2)x^2+2bcx^3+c^2x^4}$$

$$\times \frac{a^2+(ac-b^2)x^2-bcx^3}{a^2+(ac-b)x^2+bcx^3} = \frac{a-bx}{a+bx},$$

when reduced to its most simple form.

(53) $$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{e}{f}+\frac{g}{h}}=\frac{\frac{ad+bc}{bd}}{\frac{eh+fg}{fh}}=\frac{(ad+bc)fh}{(eh+fg)bd}:$$

first adding together the fractions in the numerator and denominator.

(54) $$\frac{\frac{a}{b}+\frac{c}{d}+\frac{e}{f}}{\frac{a'}{b'}+\frac{c'}{d'}+\frac{e'}{f'}}=\frac{(adf+bcf+bde)b'd'f'}{(a'd'f'+b'c'f'+b'd'e')bdf}.$$

(55) $$\frac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a}{a-b}-\frac{b}{a+b}}=\frac{\frac{a^2+b^2}{a^2-b^2}}{\frac{a^2+b^2}{a^2-b^2}}=1.$$

(56) $$\frac{\frac{1}{1+x}+\frac{x}{1-x}}{\frac{1}{1-x}-\frac{x}{1+x}}=1.$$

(57) $$\frac{\frac{1}{1+x}}{1-\frac{1}{1+x}}=\frac{1}{x}.$$

(58) $$\frac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}=\frac{a^2+x^2}{2ax}=\frac{1}{2}\left\{\frac{a}{x}+\frac{x}{a}\right\}.$$

(59) $$\frac{\frac{a+bx}{a-bx}+\frac{b+ax}{b-ax}}{\frac{a+bx}{a-bx}-\frac{b+ax}{b-ax}}=\frac{ab(1-x^2)}{(b^2-a^2)x}.$$

(60) $$\frac{\dfrac{x}{1+x}+\dfrac{1-x}{x}}{\dfrac{x}{1+x}-\dfrac{1-x}{x}}=\frac{1}{2x^2-1}.$$

(61) $$\frac{\dfrac{1}{1+x}+\dfrac{1}{1-x}}{\dfrac{1}{1-x}-\dfrac{1}{1+x}}=\frac{1}{x}.$$

(62) $$\frac{1}{1+\dfrac{1}{x}}=\frac{x}{1+x}.$$

(63) $$\frac{1}{1+\dfrac{1}{x+\dfrac{1}{x}}}=\frac{x+\dfrac{1}{x}}{1+x+\dfrac{1}{x}}=\frac{1+x^2}{1+x+x^2}:$$

putting $x+\frac{1}{x}$ in the place of x in Ex. 62.

(64) $$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}=\frac{1}{1+\dfrac{x}{1+x}}=\frac{1+x}{1+2x}.$$

(65) $$\frac{a}{b+\dfrac{c}{d}}=\frac{ad}{bd+c}.$$

(66) $$\frac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}=\frac{a\left(d+\dfrac{e}{f}\right)}{b\left(d+\dfrac{e}{f}\right)+c}=\frac{adf+ae}{(bd+c)f+be}.$$

putting $d+\frac{e}{f}$ in place of d in Ex. 65.

(67) $$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\cfrac{g}{h}}}} = \frac{ad\left(f+\frac{g}{h}\right)+ae}{(bd+c)\left(f+\frac{g}{h}\right)+be}$$

$$= \frac{(adf+ae)h+adg}{\{(bd+c)f+be\}h+(bd+c)g}.$$

(68) $$\frac{1}{\dfrac{1}{x+a}+\dfrac{1}{x+b}} = \frac{x^2+(a+b)x+ab}{2x+a+b}.$$

(69) $$\frac{1}{\dfrac{1}{x+3}+\dfrac{1}{x+5}+\dfrac{1}{x-8}} = \frac{x^3-49x-120}{3x^2-49}.$$

(70) $$\cfrac{x+1}{x-1+\cfrac{x+2}{x-2+\cfrac{x+3}{x-3}}} = \cfrac{x+1}{x-1+\cfrac{x^2-x-6}{x^2-4x+9}}$$

$$= \frac{x^3-3x^2+5x+9}{x^3-4x^2+12x-15}.$$

(71) $$\cfrac{x}{1+\cfrac{x}{1+x+\cfrac{x}{1+x+x^2+\cfrac{x}{1+x+x^2+x^3}}}}$$

$$= \frac{x+5x^2+7x^3+7x^4+6x^5+3x^6+x^7}{1+6x+10x^2+10x^3+9x^4+5x^5+2x^6}.$$

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CHAP. VI.

FURTHER DEVELOPEMENT OF THE THEORY OF INDICES.

Restatement of the principle of indices.

176. OUR first assumption of indices was made for the purpose of abbreviating the expression of the continued product of a symbol (whether arithmetical or not) into itself, the index denoting the number of times that symbol appeared as a factor: thus a^2 was taken to denote aa, a^3 to denote aaa, and a^n to denote the product of a into itself, when repeated a number of times equal to n: under these circumstances, our interpretation of the meaning of the expression a^n was confined to those values of n, which were positive whole numbers: and it was deduced as a necessary consequence of this notation, that whenever n and m denoted whole numbers, $a^n \times a^m$ was identical with a^{n+m}. (Art. 11, 12.)

Taking this conclusion as a guide for our assumptions, we next assumed the existence of expressions such as a^n, where n was perfectly arbitrary, like all other symbols in Algebra: but in order that the interpretation of the meaning of such expressions should not be arbitrary likewise, when particular values of the index were assigned, we assumed as a general principle that in all cases the product of a^n and a^m, whatever n and m might be, should be equivalent to a^{n+m}.

It was in conformity with this principle that we shewed that $a^{\frac{1}{2}}$ was equivalent to $\sqrt{a}$, $a^{\frac{1}{3}}$ equivalent to $\sqrt[3]{a}$, $a^{\frac{2}{3}}$ equivalent to $\sqrt[3]{a^2}$ or to the cube root of a^2, and $a^{\frac{n}{m}}$ equivalent to $\sqrt[m]{a^n}$, or to the m^{th} root of a^n, where m and n are whole numbers: by this means a consistent meaning and interpretation was given to all indices, which were positive fractions. (Art. 13.)

It is our present object to examine the other consequences of the same principle.

Negative indices: a^{-n} equivalent to $\frac{1}{a^n}$.

177. Since the index of a^m is perfectly arbitrary, it may be negative as well as positive: it remains to ascertain the algebraical meaning of such expressions as a^{-m}.

The general principle of indices, gives $a^n \times a^{-m} = a^{n-m}$, since $n-m$ is the algebraical *sum* of n and $-m$. Again, if we multiply a^{n-m} into a^m, the result by the same principle is $a^{n-m+m} = a^n$: and if we multiply $\frac{a^n}{a^m}$ into a^m, the result is likewise a^n, since multiplication and division are inverse operations: it follows, therefore, that a^{n-m} is equivalent to $\frac{a^n}{a^m}$, since they both produce the same result when multiplied into the same quantity: and consequently, it is indifferent whether we multiply a^n or any other quantity by a^{-m} or divide it by a^m, the two results being equivalent to each other.

Again, since a^n multiplied into $\frac{1}{a^m}$ is equivalent to a^n divided by a^m (Art. 42. Ex. 5.); and since a^n multiplied into a^{-m} is likewise equivalent to a^n divided by a^m; it follows therefore that a^{-m} and $\frac{1}{a^m}$ are equivalent to each other.

Quantities may be transferred from the numerator to the denominator of a fraction, or conversely by changing the signs of their indices.

178. This is a most important conclusion, and is altogether independent of the particular value of the index: it shews that any quantity may be transferred from the numerator to the denominator of a fraction by merely changing the sign of its index and conversely.

Thus, $a^{-1} = \frac{1}{a}$: $a^{-2} = \frac{1}{a^2}$: $a^{-3} = \frac{1}{a^3}$:

$$a=\frac{1}{a^{-1}}:\quad a^2=\frac{1}{a^{-2}}:\quad a^3=\frac{1}{a^{-3}}:$$

$$\frac{a}{b}=\frac{1}{a^{-1}b}=\frac{b^{-1}}{a^{-1}}=ab^{-1}:$$

$$\frac{a^2}{b^3c^4}=\frac{1}{a^{-2}b^3c^4}=\frac{b^{-3}}{a^{-2}c^4}=\frac{b^{-3}c^{-4}}{a^{-2}},$$

$$=a^2b^{-3}c^4=\frac{a^2b^{-3}}{c^4}=\frac{a^2c^{-4}}{b^3}.$$

Proof that $a^0=1$.

179. Since $\frac{a^n}{a^n}=1=a^{n-n}=a^0$, it follows that a^0 is in all cases equivalent to *unity*, being the representative of a quantity divided by itself: this result is a necessary consequence of the notation.

Proof that $(a^m)^n=a^{mn}$.

180. The continued product of a^m into itself repeated n times, would be denoted by $(a^m)^n$, according to the general principle of indices: and it may likewise be shewn by the same principle, that the same product may be denoted by a^{mn}, or by a with an index equal to n times the index of the simple factor a^m: for $a^m \times a^m = a^{m+m} = a^{2m}$: $a^m \times a^m \times a^m = a^{3m}$: and if a^m appear four times in the product, the result is a^{4m}, if five times, it is a^{5m}, and consequently, if it appears n times, (where n is a whole number) it is denoted by a^{mn}.

We have thus obtained an equivalent form for $(a^m)^n$, where the symbols m and n are, one or both of them, particular in their representation, though general in their form; it follows therefore, from the *law of the permanence of equivalent forms* (Art. 131. 132), that a^{mn} is equally equivalent to $(a^m)^n$, where m and n are any quantities whatsoever.*

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* It may be useful to restate, with reference to this case, the reasoning made use of in the establishment of that law: if there be an equivalent form for $(a^m)^n$, it must necessarily be a^{mn}, inasmuch as it must coincide with the result obtained in arithmetical Algebra, in conformity with the principle of indices,

181. We subjoin a series of examples which will explain more fully the consequences of these general conclusions, and will serve to shew the great variety of equivalent forms which we are consequently enabled to give to algebraical expressions. Examples.

$$(1)\quad a^m \times a^n = a^{m+n} = \frac{a^m}{a^{-n}} = \frac{a^n}{a^{-m}} = \frac{1}{a^{-(m+n)}}.$$

$$(2)\quad a^m \times a^{-n} = a^{m-n} = \frac{a^m}{a^n} = \frac{a^{-n}}{a^{-m}} = \frac{1}{a^{-(m-n)}}.$$

$$(3)\quad a^{-m} \times a^n = a^{-(m-n)} = \frac{a^{-m}}{a^{-n}} = \frac{a^n}{a^m} = \frac{1}{a^{m-n}}.$$

$$(4)\quad a^{-m} \times a^{-n} = a^{-(m+n)} = \frac{a^{-m}}{a^n} = \frac{a^{-n}}{a^m} = \frac{1}{a^{m+n}}.$$

$$(5)\quad 3a^{-2} \times -4a^{-1} \times -7a^3 = 84a^0 = 84, \text{ since } a^0 = 1.$$

$$(6)\quad \frac{\frac{7}{6}a^3}{\frac{2}{5}a^5} = \frac{35}{12}a^{-2} = \frac{35}{12a^2}.$$

$$(7)\quad a^{\frac{3}{4}} \times a^{-\frac{4}{5}} = a^{\frac{3}{4}-\frac{4}{5}} = a^{-\frac{1}{20}} = \frac{1}{a^{\frac{1}{20}}}.$$

$$(8)\quad \frac{a^{\frac{2}{3}} \times a^{\frac{7}{9}}}{a^3} = \frac{a^{\frac{13}{9}}}{a^3} = a^{\frac{13}{9}-3} = a^{-\frac{14}{9}} = \frac{1}{a^{\frac{14}{9}}}.$$

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indices, when m and n are whole numbers; otherwise the form in question is not a general equivalent form for all values of m and n: and secondly, we may safely assume such an equivalent form of $(a^m)^n$, inasmuch as it coincides with the general form in the subordinate science, with whose rules and operations those of Algebra have been assumed to coincide, when the symbols coincide in signification; and it can never therefore lead to results inconsistent with any other results, which those rules and operations may give.

(9) $ab^{-1}c^{-2} \times a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}} = a^{\frac{3}{2}}b^{-\frac{2}{3}}c^{-\frac{7}{4}} = \dfrac{a^{\frac{3}{2}}}{b^{\frac{2}{3}}c^{\frac{7}{4}}}$.

(10) $a^{\frac{7}{3}}\left(\dfrac{1}{a} - \dfrac{3b}{a^2} + \dfrac{4c^2}{a^3}\right) = a^{\frac{4}{3}} - 3a^{\frac{1}{3}}b + \dfrac{4c^2}{a^{\frac{2}{3}}}$.

(11) $a^{\frac{3}{4}}x^{-\frac{1}{2}}\left(1 - \dfrac{a^{\frac{4}{3}}}{x^{\frac{3}{4}}} + a^{\frac{1}{3}}x^{\frac{1}{4}} - \dfrac{x^{\frac{5}{4}}}{a^{\frac{2}{3}}}\right)$

$$= \frac{a^{\frac{3}{4}}}{x^{\frac{1}{2}}} - \frac{a^{\frac{25}{12}}}{x^{\frac{5}{4}}} + \frac{a^{\frac{13}{12}}}{x^{\frac{1}{4}}} - a^{\frac{1}{12}}x^{\frac{3}{4}}.$$

(12) $\sqrt{a} \times \sqrt[3]{a} = a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{5}{6}}$. (Art. 13.)

(13) $\sqrt{a} \times \sqrt{a} \times \sqrt[4]{a} = a^{\frac{1}{2}} \times a^{\frac{1}{3}} \times a^{\frac{1}{4}} = a^{\frac{13}{12}}$.

(14) $\sqrt{\sqrt[3]{a}} = \sqrt{a^{\frac{1}{3}}} = (a^{\frac{1}{3}})^{\frac{1}{2}} = a^{\frac{1}{6}}$.

(15) $\sqrt{\sqrt[3]{\sqrt[4]{a}}} = \sqrt{\sqrt[3]{a^{\frac{1}{4}}}} = \sqrt{(a^{\frac{1}{4}})^{\frac{1}{3}}} = \sqrt{a^{\frac{1}{12}}} = a^{\frac{1}{24}}$.

This notation is equivalent to $\{(a^{\frac{1}{4}})^{\frac{1}{3}}\}^{\frac{1}{2}}$.

(16) $\sqrt{\sqrt[3]{\sqrt[4]{\sqrt[5]{a}}}} = \sqrt{\sqrt[3]{\sqrt[4]{a^{\frac{1}{5}}}}} = \sqrt{\sqrt[3]{a^{\frac{1}{20}}}}$

$$= \sqrt{a^{\frac{1}{60}}} = a^{\frac{1}{120}}.$$

This notation is equivalent to $\{\{(a^{\frac{1}{5}})^{\frac{1}{4}}\}^{\frac{1}{3}}\}^{\frac{1}{2}}$.

(17) $(a^{-1})^{-2} = a^2$.

(18) $\{(a^{-1})^{-2}\}^{-3} = (a^2)^{-3} = a^{-6} = \dfrac{1}{a^6}$.

(19) $\{(a^{-\frac{1}{2}})^{-\frac{1}{3}}\}^{-\frac{1}{4}} = (a^{\frac{1}{6}})^{-\frac{1}{4}} = a^{-\frac{1}{24}} = \dfrac{1}{a^{\frac{1}{24}}}$.

(20) $(a^{\frac{3}{4}}\,b^{\frac{2}{3}})^{\frac{1}{3}} = a^{\frac{1}{4}}\,b^{\frac{2}{9}} = a^{\frac{9}{36}}b^{\frac{8}{36}} = (a^9b^8)^{\frac{1}{36}} = \sqrt[36]{(a^9b^8.)}$

In this case, the indices are reduced to a common denominator.

(21) $(a^2b^{-\frac{1}{2}}\,c^{\frac{2}{5}})^{-\frac{1}{4}} = a^{-\frac{1}{2}}\,b^{\frac{1}{8}}\,c^{-\frac{1}{10}} = \dfrac{b^{\frac{5}{40}}}{a^{\frac{20}{40}}\,c^{\frac{4}{40}}}$

$$= \left(\frac{b^5}{a^{20}c^4}\right)^{\frac{1}{40}} = \sqrt[40]{\left(\frac{b^5}{a^{20}c^4}\right)}.$$

(22) $(ab^2\sqrt{(ab^3)}\,\sqrt[3]{ab^4}\,\sqrt[4]{ab^5})^{\frac{1}{5}}$

$$= \{ab^2\,(ab^3)^{\frac{1}{2}}\,(ab^4)^{\frac{1}{3}}\,(ab^5)^{\frac{1}{4}}\}^{\frac{1}{5}}$$

$$= \{ab^2 \times a^{\frac{1}{2}}b^{\frac{3}{2}} \times a^{\frac{1}{3}}b^{\frac{4}{3}} \times a^{\frac{1}{4}}b^{\frac{5}{4}}\}^{\frac{1}{5}}$$

$$= \{a^{\frac{25}{12}}b^{\frac{73}{12}}\}^{\frac{1}{5}} = a^{\frac{5}{12}}b^{\frac{73}{60}}.$$

(23) $\sqrt[6]{(a^3\,b\sqrt[5]{a^3bc})^5} = \{a^3b \times a^{\frac{3}{5}}b^{\frac{1}{5}}c^{\frac{1}{5}}\}^{\frac{5}{6}}$

$$= (a^{\frac{18}{5}}\,b^{\frac{6}{5}}\,c^{\frac{1}{5}})^{\frac{5}{6}} = a^3bc^{\frac{1}{6}} = a^3b\sqrt[6]{c}.$$

(24) $\left\{\dfrac{a^2x}{(a+x)^{\frac{5}{4}}}\right\}^{-\frac{1}{5}} = \dfrac{a^{-\frac{2}{5}}x^{-\frac{1}{5}}}{(a+x)^{-\frac{1}{4}}} = \dfrac{(a+x)^{\frac{1}{4}}}{a^{\frac{2}{5}}\,x^{\frac{1}{5}}}$

$$= \frac{\sqrt[4]{(a+x)}}{\sqrt[5]{(a^2x)}}.$$

(25) $\left\{(a+x)\sqrt{\left(\dfrac{b^3}{(a+x)^{\frac{1}{2}}}\right)}\right\}^{\frac{1}{3}} = \left\{\dfrac{(a+x)\times b^{\frac{3}{2}}}{(a+x)^{\frac{1}{4}}}\right\}^{\frac{1}{3}}$

$$= \{b^{\frac{3}{2}}\,(a+x)^{\frac{3}{4}}\}^{\frac{1}{3}}$$

$$= b^{\frac{1}{2}}\,(a+x)^{\frac{1}{4}}$$

$$= \sqrt{b} \times \sqrt[4]{(a+x)}.$$

Amongst the preceding examples, the indication of roots is sometimes effected by signs, and sometimes by indices: before the reduction, however, of the indices is effected, it is always convenient and frequently necessary to replace the signs of the different roots by their corresponding indices: it is usual, indeed, in nearly all cases, to replace radical signs beyond the square root by indices, and it would conduce very greatly both to the clearness and uniformity of notation, if the use of the former was abandoned altogether.

The transformation of binomial and other expressions consisting of more than one term, as in the following ten examples, to equivalent forms, where the first term of the binomial or polynomial is 1, will be found extremely useful in simplifying their developement by the binomial theorem or by other methods.

$$(26)\quad (a^2+x^2)^{\frac{1}{2}}=\left\{a^2\left(1+\frac{x^2}{a^2}\right)\right\}^{\frac{1}{2}}=a\left(1+\frac{x^2}{a^2}\right)^{\frac{1}{2}}.$$

$$(27)\quad (a^2+ax)^{\frac{1}{3}}=\left\{a^2\left(1+\frac{x}{a}\right)\right\}^{\frac{1}{3}}=a^{\frac{2}{3}}\left(1+\frac{x}{a}\right)^{\frac{1}{3}}.$$

$$(28)\quad (\sqrt{a}-\sqrt{x})^{\frac{1}{4}}=\left\{\sqrt{a}\left(1-\frac{\sqrt{x}}{\sqrt{a}}\right)\right\}^{\frac{1}{4}}$$

$$=a^{\frac{1}{8}}\left(1-\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)^{\frac{1}{4}}.$$

$$(29)\quad (a^2x-ax^2)^{-\frac{2}{3}}=\left\{a^2x\left(1-\frac{x}{a}\right)\right\}^{-\frac{2}{3}}$$

$$=a^{-\frac{4}{3}}x^{-\frac{2}{3}}\left(1-\frac{x}{a}\right)^{-\frac{2}{3}}.$$

$$(30)\quad a^{-\frac{4}{5}}(a^5-a^4x)^{-\frac{1}{25}}=a^{-\frac{4}{5}}\left\{a^5\left(1-\frac{x}{a}\right)\right\}^{-\frac{1}{25}}$$

$$=a^{-\frac{4}{5}}\times a^{-\frac{1}{5}}\left(1-\frac{x}{a}\right)^{-\frac{1}{25}}=a^{-1}\left(1-\frac{x}{a}\right)^{-\frac{1}{25}}.$$

(31) $$a^{\frac{1}{2}}x^{\frac{2}{3}}(a^3x^2-a^2x^3)^{\frac{1}{6}}=a^{\frac{1}{2}}x^{\frac{2}{3}}\left\{a^3x^2\left(1-\frac{x}{a}\right)\right\}^{\frac{1}{6}}$$

$$=a^{\frac{1}{2}}x^{\frac{2}{3}}\times a^{\frac{1}{2}}x^{\frac{1}{3}}\left(1-\frac{x}{a}\right)^{\frac{1}{6}}=ax\left(1-\frac{x}{a}\right)^{\frac{1}{6}}.$$

(32) $$(a^2-ax+x^2)^{\frac{1}{7}}=\left\{a^2\left(1-\frac{x}{a}+\frac{x^2}{a^2}\right)\right\}^{\frac{1}{7}}$$

$$=a^{\frac{2}{7}}\left\{1-\frac{x}{a}+\frac{x^2}{a^2}\right\}^{\frac{1}{7}}.$$

(33) $$a^{\frac{7}{2}}x^{\frac{1}{2}}(a^3x^5+a^4x^4+a^5x^3)^{\frac{3}{2}}$$

$$=a^{\frac{7}{2}}x^{\frac{1}{2}}\left\{a^3x^5\left(1+\frac{a}{x}+\frac{a^2}{x^2}\right)\right\}^{\frac{3}{2}}$$

$$=a^8x^8\left(1+\frac{a}{x}+\frac{a^2}{x^2}\right)^{\frac{3}{2}}.$$

(34) $$\left\{\sqrt[3]{\frac{27a^5x}{2b}}-\sqrt[3]{\frac{a^2x^4}{2b}}+\sqrt[3]{\frac{125x^7}{2ab}}\right\}^{\frac{1}{3}}$$

$$=\left\{3a\sqrt[3]{\frac{a^2x}{2b}}-x\sqrt[3]{\frac{a^2x}{2b}}+\frac{5x^2}{a}\sqrt[3]{\frac{a^2x}{2b}}\right\}^{\frac{1}{3}}$$

$$=\left\{\frac{a^2x}{2b}\right\}^{\frac{1}{9}}\left\{3a-x+\frac{5x^2}{a}\right\}^{\frac{1}{3}}$$

$$=\left\{\frac{a^2x}{2b}\right\}^{\frac{1}{9}}(3a)^{\frac{1}{3}}\left\{1-\frac{x}{3a}+\frac{5x^2}{3a^2}\right\}^{\frac{1}{3}}$$

$$=\left\{\frac{27a^5x}{2b}\right\}^{\frac{1}{9}}\left\{1-\frac{x}{3a}+\frac{5x^2}{3a^2}\right\}^{\frac{1}{3}}.$$

(35) $$\left\{\frac{\sqrt{x+y}}{x^2}+\frac{2(x+y)^{\frac{3}{2}}}{x^2y}+\frac{(x+y)^{\frac{5}{2}}}{x^2y^2}\right\}^{\frac{1}{\sqrt{2}}}$$

$$=\left\{\frac{(x+y)^{\frac{1}{2}}}{x^2}\left(1+\frac{2(x+y)}{y}+\frac{(x+y)^2}{y^2}\right)\right\}^{\frac{1}{\sqrt{2}}}$$

$$= \frac{(x+y)^{\frac{1}{2\sqrt{2}}}}{x^{\sqrt{2}}} \left\{1 + \frac{x+y}{y}\right\}^{\frac{2}{\sqrt{2}}}$$

$$= \frac{(x+y)^{\frac{1}{2\sqrt{2}}}}{x^{\sqrt{2}}} \left\{2 + \frac{x}{y}\right\}^{\sqrt{2}}$$

$$= \frac{(x+y)^{\frac{1}{2\sqrt{2}}}}{x^{\sqrt{2}}} \left\{2\left(1 + \frac{x}{2y}\right)\right\}^{\sqrt{2}}$$

$$= \frac{2^{\sqrt{2}}(x+y)^{\frac{1}{2\sqrt{2}}}}{x^{\sqrt{2}}} \left(1 + \frac{x}{2y}\right)^{\sqrt{2}}$$

$$= \left\{\frac{4\sqrt{(x+y)}}{x^2}\right\}^{\frac{1}{\sqrt{2}}} \left(1 + \frac{x}{2y}\right)^{\sqrt{2}}.$$

The following are examples of miscellaneous reductions of compound expressions involving radical quantities and indices, to other and equivalent forms.

$$(36) \quad \frac{\sqrt{(1-x)} + \dfrac{1}{\sqrt{(1+x)}}}{1 + \dfrac{1}{\sqrt{(1-x^2)}}} = \sqrt{(1-x)}.$$

$$(37) \quad \frac{\sqrt{\left(\dfrac{2x}{1+x}\right)} + \dfrac{\sqrt{(1+x)}}{\sqrt{2}}}{1 + \dfrac{1+x}{2\sqrt{x}}} = \sqrt{\left\{\frac{2x}{1+x}\right\}}.$$

$$(38) \quad \frac{\sqrt{(1-x^2)} + \dfrac{x^2}{\sqrt{(1-x^2)}}}{1-x^2} = \frac{1}{(1-x^2)^{\frac{3}{2}}}.$$

$$(39) \quad \frac{1 + \dfrac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}}{\sqrt{(a^2+x^2)} + \sqrt{(a^2-x^2)}} = \frac{1}{\sqrt{(a^2+x^2)}}.$$

(40) $$\frac{\dfrac{\sqrt{(1+x)}}{\sqrt{(1-x)}}-\dfrac{\sqrt{(1-x)}}{\sqrt{(1+x)}}}{\dfrac{\sqrt{(1+x)}}{\sqrt{(1-x)}}+\dfrac{\sqrt{(1-x)}}{\sqrt{(1+x)}}}=x.$$

(41) $$\frac{\sqrt{(1+2x)}-\dfrac{1}{\sqrt{(1+2x)}}}{\sqrt{(1+2x)}+\dfrac{1}{\sqrt{(1+2x)}}}=\frac{x}{1+x}.$$

(42) $$\frac{\dfrac{1+x}{\sqrt{(1+x^2)}}-\dfrac{\sqrt{(1+x^2)}}{1+x}}{\dfrac{1+x}{\sqrt{(1+x^2)}}+\dfrac{\sqrt{(1+x^2)}}{1+x}}=\frac{x}{1+x+x^2}.$$

(43) $$\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$$
$$=\frac{\{\sqrt{(1+x)}+\sqrt{(1-x)}\}^2}{(1+x)-(1-x)},$$
$$=\frac{1+\sqrt{(1-x^2)}}{x}=\frac{x}{1-\sqrt{(1-x^2)}},$$

multiplying the numerator and denominator by $\sqrt{(1+x)}+\sqrt{(1-x)}$ in one case, and by $\sqrt{(1+x)}-\sqrt{(1-x)}$ in the other.

(44) $$\frac{\sqrt{(x^2+1)}-1}{\sqrt{(x^2+1)}+1}=\frac{x^2}{\{\sqrt{(x^2+1)}+1\}^2}$$
$$=\frac{\{\sqrt{(x^2+1)}-1\}^2}{x^2},$$

multiplying numerator and denominator in the first instance by $\sqrt{(x^2+1)}+1$, and in the second by $\sqrt{(x^2+1)}-1$.

$$(45)\quad \frac{\dfrac{\sqrt{(1-x)}}{2\sqrt{(1+x)}}+\dfrac{\sqrt{(1+x)}}{2\sqrt{(1-x)}}}{1-x}=\frac{1}{(1-x)\sqrt{(1-x^2)}}$$

$$=\frac{1}{(1-x)^{\frac{3}{2}}\sqrt{(1+x)}}$$

since $\sqrt{(1-x^2)}=\sqrt{(1-x)}\sqrt{(1-x)}$.

$$(46)\quad \frac{\sqrt{(a+b)}+\sqrt{(a-b)x}}{\sqrt{(a+b)}-\sqrt{(a-b)x}}$$

$$=\frac{a+b+(a-b)x+2\sqrt{(a^2-b^2)x}}{a+b-(a-b)x}$$

$$=\frac{a(1+x)+b(1-x)+2\sqrt{(a^2-b^2)x}}{a(1-x)+b(1+x)}$$

$$=\frac{a(1-x)+b(1+x)}{a(1+x)+b(1-x)-2\sqrt{(a^2-b^2)x}},$$

multiplying the numerator and denominator of the fraction. in the first instance, by its numerator, and in the second by its denominator.

$$(47)\quad \sqrt{(a+\sqrt{b})}=\frac{\sqrt{(a^2-b)}}{\sqrt{(a-\sqrt{b})}},$$

multiplying and dividing by $\sqrt{(a-\sqrt{b})}$.

$$(48)\quad \sqrt{(a-\sqrt{b})}=\frac{\sqrt{(a^2-b)}}{\sqrt{(a+\sqrt{b})}},$$

multiplying and dividing by $\sqrt{(a+\sqrt{b})}$.

$$(49)\quad \sqrt[n]{(a+\sqrt{b})}=(a+\sqrt{b})^{\frac{1}{n}}=\frac{(a^2-b)^{\frac{1}{n}}}{(a-\sqrt{b})^{\frac{1}{n}}},$$

multiplying and dividing by $(a-\sqrt{b})^{\frac{1}{n}}$: for $(a+\sqrt{b})^{\frac{1}{n}}\times(a-\sqrt{b})^{\frac{1}{n}}=\{(a+\sqrt{b})(a-\sqrt{b})\}^{\frac{1}{n}}=\{a^2-b\}^{\frac{1}{n}}$.

(50) $$\sqrt{(a-\sqrt{b})}=\frac{(a^2-b)^{\frac{1}{2}}}{(a+\sqrt{b})^{\frac{1}{2}}}.$$

(51) $$\sqrt{\left\{\frac{a+\sqrt{b}}{a-\sqrt{b}}\right\}}=\frac{\sqrt{(a^2-b)}}{a-\sqrt{b}},$$

multiplying and dividing by $\sqrt{(a-\sqrt{b})}$.

(52) $$\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}=\frac{\sqrt{(ab)}-\sqrt{(ac)}}{b-c},$$

multiplying and dividing by $\sqrt{b}-\sqrt{c}$.

(53) $$\frac{\sqrt{\{\sqrt{(ab)}+\sqrt{(ac)}\}}}{\sqrt{a}}=\sqrt{\left\{\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{2}}c^{\frac{1}{2}}}{a}\right\}}$$

$$=\sqrt{\left\{\frac{b^{\frac{1}{2}}+c^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right\}}=\sqrt{\left\{\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right\}}.$$

(54) $$\frac{c\sqrt{a}+d\sqrt{b}}{e\sqrt{a}+f\sqrt{b}}=\frac{ace-bdf+(de-cf)\sqrt{ab}}{ae^2-bf^2}$$

$$=\frac{ac^2-bd^2}{ace-bdf-(de-cf)\sqrt{ab}},$$

multiplying the numerator and denominator of the fraction by $e\sqrt{a}-f\sqrt{b}$ in the first instance, and by $c\sqrt{a}-d\sqrt{b}$ in the second.

(55) $$\frac{2\sqrt{x}+3\sqrt{y}}{3\sqrt{x}+4\sqrt{y}}$$

$$=\frac{6x-12y+\sqrt{(xy)}}{9x-16y}=\frac{4x-9y}{6x-12y-\sqrt{(xy)}}.$$

This is a particular case of the preceding example.

(56) $$\frac{a-b}{\sqrt{a}-\sqrt{b}}=\sqrt{a}+\sqrt{b}.$$

(57) $$\frac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}=a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}.$$

(58) $$\frac{a-b}{a^{\frac{1}{4}}-b^{\frac{1}{4}}}=a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}.$$

(59) $$\frac{\sqrt{a}-\sqrt{b}}{\sqrt[4]{a}-\sqrt[4]{b}}=\frac{a^{\frac{1}{2}}-b^{\frac{1}{2}}}{a^{\frac{1}{4}}-b^{\frac{1}{4}}}=a^{\frac{1}{4}}+b^{\frac{1}{4}}.$$

The four last results are obtained by the ordinary process of division.

(60) $$\frac{1}{\sqrt[4]{a}+\sqrt[4]{b}}=\frac{a^{\frac{3}{4}}-a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}-b^{\frac{3}{4}}}{a-b},$$

multiplying the numerator and denominator by the quotient of $a-b$ divided by $\sqrt[4]{a}+\sqrt[4]{b}$.

(61) $$\frac{\sqrt[4]{a}+\sqrt[4]{b}}{\sqrt[4]{a}-\sqrt[4]{b}}=\frac{a+b+2a^{\frac{1}{2}}b^{\frac{1}{2}}+2a^{\frac{3}{4}}b^{\frac{1}{4}}+2a^{\frac{1}{4}}b^{\frac{3}{4}}}{a-b},$$

the numerator being the product of $\sqrt[4]{a}+\sqrt[4]{b}$ into the quotient $a-b$ divided by $\sqrt[4]{a}-\sqrt[4]{b}$.

(62) $$\begin{aligned}&\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}\\&=\sqrt{\{\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}\}^2}\\&=\sqrt{\{2a+2\sqrt{(a^2-b)}\}}.\end{aligned}$$

(63) $$\begin{aligned}&\sqrt{(a+\sqrt{b})}-\sqrt{(a-\sqrt{b})}\\&=\sqrt{\{\sqrt{(a+\sqrt{b})}-\sqrt{(a-\sqrt{b})}\}^2}\\&=\sqrt{\{2a-2\sqrt{(a^2-b)}\}}.\end{aligned}$$

(64) $\sqrt[4]{(a+\sqrt{b})}+\sqrt[4]{(a-\sqrt{b})}$

$$=\sqrt{\{\sqrt[4]{(a+\sqrt{b})}+\sqrt[4]{(a-\sqrt{b})}\}^2}$$

$$=\sqrt{\{\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}+2\sqrt[4]{(a^2-b)}\}}$$

$$=\sqrt{\{\sqrt{\{2a+2\sqrt{(a^2-b)}\}}+2\sqrt[4]{(a^2-b)}\}}$$

replacing $\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}$ by its equivalent form given in Ex. 61.

If the radical signs be replaced by indices, the last result will appear as follows:

$$[\{2a+2\,(a^2-b)^{\frac{1}{2}}\}^{\frac{1}{2}}+2\,(a^2-b)^{\frac{1}{4}}]^{\frac{1}{2}}.$$

(65) $\sqrt[8]{(a+\sqrt{b})}+\sqrt[8]{(a-\sqrt{b})}$

$$=\sqrt{\{\sqrt[8]{(a+\sqrt{b})}+\sqrt[8]{(a-\sqrt{b})}\}^2}$$

$$=\sqrt{\{\sqrt[4]{(a+\sqrt{b})}+\sqrt[4]{(a-\sqrt{b})}+2\sqrt[8]{(a^2-b)}\}}$$

$$=\sqrt{\{\sqrt{\{\sqrt{\{2a+2\sqrt{(a^2-b)}\}}+2\sqrt[4]{(a^2-b)}\}}}$$
$$+2\sqrt[8]{(a^2-b)}\}.$$

The last result may be exhibited under the following form:

$$[[\{2a+2\,(a^2-b)^{\frac{1}{2}}\}^{\frac{1}{2}}+2\,(a^2-b)^{\frac{1}{4}}]^{\frac{1}{2}}+2\,(a^2-b)^{\frac{1}{8}}]^{\frac{1}{2}}.$$

(66) $\sqrt{ax}+\dfrac{ax}{a-\sqrt{ax}}=\dfrac{a\sqrt{x}}{\sqrt{a}-\sqrt{x}}.$

(67) $\dfrac{ax}{\sqrt{(a+x)}}-\dfrac{2ax^2}{(a+x)^{\frac{3}{2}}}+\dfrac{ax^3}{(a+x)^{\frac{5}{2}}}=\dfrac{a^3x}{(a+x)^{\frac{5}{2}}}.$

reducing the several fractions to their lowest common denominator.

(68) $$\frac{1}{a^2x(a+bx^2)^{\frac{1}{2}}}+\frac{2bx}{a^2(a+bx^2)^{\frac{3}{2}}}-\frac{4bx}{a(a+bx^2)^{\frac{5}{2}}}$$
$$-\frac{3b^2x^3}{a^2(a+bx^2)^{\frac{5}{2}}}=\frac{1}{x(a+bx^2)^{\frac{5}{2}}}.$$

(69) $$\frac{2c}{\sqrt{(a+bx+cx^2)}}-\frac{(2cx+b)^2}{2(a+bx+cx^2)^{\frac{3}{2}}}$$
$$=\frac{4ac-b^2}{2(a+bx+cx^2)^{\frac{3}{2}}}.$$

(70) $$\frac{5}{2\sqrt{(2x-x^2)}}-\left(\frac{x^2}{3}+\frac{5x}{6}+\frac{5}{2}\right)\frac{1-x}{\sqrt{(2x-x^2)}}$$
$$-\left(\frac{2x}{3}+\frac{5}{6}\right)\sqrt{(2x-x^2)}=\frac{x^3}{\sqrt{(2x-x^2)}}.$$

(71) $$\frac{1}{a^3x\sqrt{(a+bx+cx^2)}}-\frac{b+cx}{a^3(a+bx+x^2)^{\frac{3}{2}}}$$
$$-\frac{b+cx}{a^2(a+bx+cx^2)^{\frac{5}{2}}}-\frac{b+cx}{a(a+bx+cx^2)^{\frac{7}{2}}}$$
$$=\frac{1}{x(a+bx+cx^2)^{\frac{7}{2}}}.$$

(72) $$\frac{1+x^{\frac{1}{2}}+x+x^{\frac{3}{2}}}{2x+2x^{\frac{3}{2}}+3x^2+3x^{\frac{5}{2}}}=\frac{1+x}{2x+3x^2}.$$

When simple radicals, like $\sqrt{x}$ or $x^{\frac{1}{2}}$, are involved in a fraction, it may be reduced to its most simple terms, in the same manner as if all its terms had been rational.

(73) $$\frac{1-x+2x^{\frac{3}{2}}+x^3}{1-2\sqrt{x}+x+2x^{\frac{3}{2}}-2x^2+x^3}=\frac{1+x^{\frac{1}{2}}+x^{\frac{3}{2}}}{1-x^{\frac{1}{2}}+x^{\frac{3}{2}}},$$

the common factor of the numerator and denominator being $1-x^{\frac{1}{2}}+x^{\frac{3}{2}}$.

$$(74)\qquad \frac{a-x}{a^{\frac{5}{2}}-a^2\sqrt{x}+a^{\frac{3}{2}}x-ax^{\frac{3}{2}}+a^{\frac{1}{2}}x^2-x^{\frac{5}{2}}}=\frac{\sqrt{a}+\sqrt{x}}{a^2+ax+x^2},$$

the common factor of the numerator and denominator, being $\sqrt{a}-\sqrt{x}$.

$$(75)\qquad \frac{x^2+2x^{\frac{3}{2}}y^{\frac{1}{2}}+3xy+2x^{\frac{1}{2}}y^{\frac{3}{2}}+y^2}{x^2+xy+y^2}$$

$$=\frac{x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y}{x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y},$$

the common factor of the numerator and denominator, being $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

$$(76)\qquad \frac{x^2+x-1}{x^3-3x^2+3x+(3x^2-x+1)\sqrt{1-x}}.$$

In a case like this, where a compound radical is involved, along with powers of the same letter which appears in that radical, the rule for detecting the common factors of the numerator and denominator, must obviously fail, inasmuch as we can no longer arrange the dividend and divisor according to the powers of that letter, upon which the application of the rule depends.

In this instance, we know from other sources that $x+\sqrt{1-x}$ is a factor of the numerator and denominator, and that consequently the fraction is reducible to

$$\frac{x-\sqrt{1-x}}{x^2-x+1+2x\sqrt{1-x}}.$$

◆

CHAP. VII.

ON THE THEORY OF DECIMAL FRACTIONS.

Numbers may be expressed in an algebraical form.

182. ANY number may be expressed in an algebraical form, by multiplying each digit into a power of 10 equal to the number of digits which succeed it, and connecting the results together with the sign +: thus the number

$$31245$$

expressed in this manner becomes

$$3\times 10^4+1\times 10^3+2\times 10^2+4\times 10+5,$$

the truth of which conversion will be manifest, if we write the number in words at full length and keep in mind that 10^2, 10^3 and 10^4, are 100, 1000 and 10000 respectively.

Decimals: their meaning.

183. If we divide this number by 10^3, the result expressed algebraically, would be

$$3\times 10+1+2\times 10^{-1}+4\times 10^{-2}+5\times 10^{-3}$$

or, otherwise

$$3\times 10+1+\frac{2}{10}+\frac{4}{10^2}+\frac{5}{10^3},$$

which, expressed arithmetically, is

$$31.245,$$

the digits after the mark (.), which is called the decimal point, being divided by powers of 10, whose indices express their distance from the place of units.

If we divide the same number by 10^5, the result expressed algebraically is

$$\frac{3}{10}+\frac{1}{10^2}+\frac{2}{10^3}+\frac{4}{10^4}+\frac{5}{10^5},$$

which is expressed arithmetically by

$$.31245$$

where all the digits are decimals.

If we had divided this number by 10^7 instead of 10^5, the algebraical result would have been

$$\frac{3}{10^3}+\frac{1}{10^4}+\frac{2}{10^5}+\frac{4}{10^6}+\frac{5}{10^7},$$

and the arithmetical form of it

$$.0031245$$

where the first significant digit is in the *third* place from the place of units, 3 being the index of 10 in the denominator of the fraction $\frac{3}{10^3}$.

It follows from the notation just explained, that the digits of all numbers are to be multiplied or divided by powers of 10, whose indices express their distances to the left or to the right of the place of units.

Thus 300 means 3×10^2 and .03 means $\frac{3}{10^2}$: 70000 means 7×10^4 and .0007 means $\frac{7}{10^4}$: 3250 means $3 \times 10^3 + 2 \times 10^2 + 5 \times 10$ and .00325 means $\frac{3}{10^3}+\frac{2}{10^4}+\frac{5}{10^5}$ and similarly in all other cases.

Conversion of a decimal into an equivalent fraction.

184. We are thus enabled to express any decimal number, by a series of equivalent fractions connected by

the sign + : if we add those fractions together, reducing them to the *lowest* common denominator, which is a *power* of 10, we shall obtain a single fraction which is equivalent to the decimal.

Thus,

$$.314 = \frac{3}{10} + \frac{1}{10^2} + \frac{4}{10^3}$$

$$= \frac{3 \times 10^2 + 1 \times 10 + 4}{10^3} = \frac{314}{10^3}$$

$$.07598 = \frac{7}{10^2} + \frac{5}{10^3} + \frac{9}{10^4} + \frac{8}{10^5}$$

$$= \frac{7 \times 10^3 + 5 \times 10^2 + 9 \times 10 + 8}{10^5}$$

$$= \frac{7598}{10^5}.$$

Inasmuch as the reduction of these partial fractions to the lowest common denominator which is a power of 10, leads to the multiplication of the several digits in their numerators, by powers of 10 equal to their respective distances from the last decimal place, it follows that the sum of these numerators (of the fractions with a common denominator) is equal to the integral number which arises from omitting the decimal point: we thus get the following rule for the conversion of a decimal into an equivalent single fraction.

Rule. *Omit the decimal point and divide the resulting integral number, by a power of* 10, *whose index is the number of decimal places.*

Thus,

$$754 = \frac{754}{10^3}$$

$$.00419 = \frac{419}{10^5}$$

$$.000073 = \frac{73}{10^6}.$$

If the original quantity be partly integral and partly decimal, the same rule applies: for the numerator of the improper fraction which is formed by adding together the integral number and the fraction equivalent to the decimal part, is equal to the integral number, which arises from the omission of the decimal point in the first instance: thus,

$$7.35 = 7 + \frac{35}{10^2} = \frac{700 + 35}{10^2} = \frac{735}{10^2}$$

$$3.045 = 3 + \frac{45}{10^3} = \frac{3045}{10^3}$$

$$73.0126 = 73 + \frac{126}{10^4} = \frac{730126}{10^4}:$$

for in this and all similar cases, the integral part being multiplied by a power of 10, which is equal to the number of decimal places, its last digit must be removed from the place of units by the same number of places: the numerator of the fraction equivalent to the decimal part is therefore added to it by writing its digits consecutively in their proper positions.

Conversion of a fraction whose denominator is a power of 10 into an equivalent decimal.

185. Any fraction, whose denominator is a power of 10, may be converted into an equivalent decimal, by omitting its denominator and making a number of decimal places in the numerator equal to the index of 10. This is the converse of the proposition just demonstrated, and is a necessary consequence of it.

Thus,

$$\frac{375}{10^2} = 3.75$$

$$\frac{4191}{10^5} = .04191$$

$$\frac{31}{10^6} = .000031.$$

Proof of the rule for the multiplication of decimals.

186. The knowledge of the proposition and its converse, which are demonstrated in two preceding Articles, will enable us very easily to demonstrate the common rules for the multiplication and division of decimals.

Let d and d' be two decimals, with n and n' decimal places respectively, and let N and N' be the integral numbers, which arise from omitting the decimal point in each: then by what we have just shewn, we have

$$d = \frac{N}{10^n} \text{ and } d' = \frac{N'}{10^{n'}}:$$

consequently their product is

$$dd' = \frac{N}{10^n} \times \frac{N'}{10^{n'}} = \frac{NN'}{10^{n+n'}};$$

or is equal to the decimal, which arises from striking off from NN', the product of the decimals considered as whole numbers, a number of decimal places equal to $n + n'$, or to the number of decimal places in the multiplicand and multiplier together: this is the ordinary rule.

Thus,

$$.576 \times .3854 = \frac{576}{10^3} \times \frac{3854}{10^4}$$

$$= \frac{2219904}{10^7} = .2219904:$$

$$113.5 \times .072 = \frac{1135}{10} \times \frac{72}{10^3} = \frac{81720}{10^4}$$

$$= 8.172$$

$$.00005 \times .017 = \frac{5}{10^5} \times \frac{17}{10^3} = \frac{85}{10^8}$$

$$= .00000085.$$

Proof of the rule for the division of decimals.

187. Again, if we divide d by d', we get

$$\frac{d}{d'} = \frac{N}{10^n} \div \frac{N'}{10^{n'}} = \frac{N}{N'} \times \frac{10^{n'}}{10^n}:$$

we must here consider three cases, according as n' is greater, equal to or less than n.

Case 1.

Let n' be greater than n, and therefore

$$\frac{d}{d'} = \frac{N}{N'} \times 10^{n'-n}:$$

or, the quotient of the decimals, is equal to the quotient of the decimals considered as integral numbers, with a number of *zeros* attached, equal to $n' - n$, or to the excess of the number of decimal places in the divisor above the number of decimal places in the dividend.

Thus,

$$400 \div .25 = \frac{400}{25} \times 10^2 = 16 \times 10^2 = 1600:$$

$$10287.36 \div .00036 = \frac{1028736}{36} \times 10^3$$

$$= 28576 \times 10^3 = 28576000$$

$$.01 \div .0002 = \frac{1}{2} \times 10^2 = \frac{10}{2} \times 10 = 50.$$

Case 2.

Let $n' = n$, and therefore

$$\frac{d}{d'} = \frac{N}{N'} \times 10^0 = \frac{N}{N'}:$$

or the quotient of the decimals is the same as if they were considered as whole numbers.

Thus,

$$145.817 \div .563 = \frac{145817}{563} = 259:$$

$$56.40 \div .15 = \frac{5640}{15} = 376.$$

Case 3.

Let n' be less than n, and therefore

$$\frac{d}{d'} = \frac{N}{N'} \times \frac{1}{10^{n-n'}},$$

or the quotient of the decimals will be found by striking off from the quotient of the decimals considered as whole numbers, a number of decimal places, equal to $n-n'$, or to the excess of the number of decimal places in the dividend above their number in the divisor.

Thus,

$$2.53944 \div 7.2 = \frac{253944}{72} \times \frac{1}{10^4}$$

$$= \frac{3527}{10^4} = .3527:$$

$$.00048 \div 12 = \frac{48}{12} \times \frac{1}{10^5} = \frac{4}{10^5} = .00004.$$

Any number of zeros may be added after the significant figures.

188. The ordinary rule for division of decimals reduces the three cases to one, by directing us to subjoin zeros to the dividend when necessary, so that the number of decimal places in it may be at least equal to the number of decimal places in the divisor: for a very little consideration will shew, that the value and signification of a decimal is not affected by such additions: thus 3.78 is equal to 3.7800 or to 3.7800000, which, if converted into equivalent fractions according to the rule, severally become

$$\frac{378}{10^2}, \quad \frac{37800}{10^4}, \quad \frac{37800000}{10^7},$$

the numerator and denominator of the first fraction, being in each case multiplied by the same powers of 10.

189. We may therefore continue the division of two numbers, whether decimal or not, as long as there is any remainder, by subjoining zeros to the dividend as long as may be necessary. Conversion of fractions generally into decimals.

Thus,

$$\frac{7.5}{16} = \frac{7.50000}{16} = .46875$$

$$\frac{3}{4} = \frac{3.00}{4} = .75$$

$$\frac{1}{25} = \frac{1.00}{25} = .04$$

$$\frac{1}{3} = \frac{1.000\ldots}{3} = .333\ldots\ldots$$

190. In this last case, the division never terminates, there being the same digit for a remainder after each division: these remainders, however, are continually decreasing in value, being the digits of a decimal quantity whose distances from the place of units are always increasing: the quotient therefore continually approximates to the value of the true quotient, though not arithmetically equal to it, unless the division be supposed to be continued without limit. Circulating decimals.

191. $\frac{3}{7} = .428571428571\ldots\ldots\ldots$

In this case, there is the same digit for the remainder, after every six divisions, which is the same with the original numerator: the same six digits 428571 must therefore appear in the quotient periodically, since they arise from the recurrence of the same dividend. Recurring period.

Their theory.

It is evident, that in every case of this kind, where the division never terminates, the digits in the resulting decimal must recur in periods, the number in each being always less than the original divisor: for when it is necessary to introduce zeros in the dividend, it is very manifest that there cannot be a series of consecutive remainders different from each other, and also from 0 and the divisor, which can be equal to the divisor: the same remainder must therefore recur, and if so, the digits in the decimal quotient must recur likewise.

It is not often that the number of places in the repeating period will reach this extreme limit: we have an instance in all fractions in their lowest terms, whose denominators are 17: thus

$$\frac{1}{17} = .058823529411764\dot{7}\ldots\ldots$$

and the repeating period consists of 16 places: there is of course an equal number of places in the periods of the decimals resulting from fractions which are multiples of $\frac{1}{17}$ by numbers which are prime to 17: thus,

$$\frac{16}{17} = .882352941176470\dot{5}\ldots\ldots$$

a period of 16 places, where the digits are the same as in the former period, removing the two first from the beginning to the end of it.

Lemma I. If a number is prime to each of two others, it is prime to their product.

192. It becomes a question of some interest, to ascertain in what cases the division will terminate and in what not: preparatory to its discussion, we must premise the following Lemmas, which are of great importance in the theory of numbers.

LEMMA 1. If a number c be prime to each of two others a and b, it is prime to their product ab.

For if not, let c and ab have a common measure x: and let $ab = xp$, and therefore $\frac{a}{x} = \frac{p}{b}$: but since a is prime to x, (for a is prime to c) the fraction $\frac{a}{x}$ is in its lowest terms, and therefore b is either equal to x, or to a multiple of x; (Art. 160.) in neither case therefore, can it be prime to c, which is contrary to the hypothesis: it follows, therefore, that ab and c have no common measure.

Lemma 2. If a number be prime to another, it is prime to any power of it.

193. LEMMA 2. If c be prime to a, it is prime to a^n.

For by the same process as in the last Lemma, it would appear, that if c be prime to a, it is prime to $a \times a$ or a^2: and therefore if prime to a and a^2, it is prime to $a \times a^2$ or a^3: if prime to a and a^3, it is prime to $a \times a^3$ or a^4: and in a similar manner it may be shewn to be prime to any integral power of a.

What fractions produce finite decimals.

194. The fraction $\frac{a}{b}$ will terminate in a finite decimal, whenever it can be reduced to an equivalent fraction, whose denominator is a power of 10, and in no other case: for whenever the corresponding decimal is finite, it may be converted into such a fraction: (Art. 184.) we must have, therefore, $\frac{a}{b} = \frac{a \times 10^n}{b \times 10^n} = \frac{N}{10^n}$, where N is a whole number: now, if b or any factor of b be prime to a and to 10, b or that factor of b is prime to a and 10^n, and therefore to $a \times 10^n$: (Art. 193.) the quantity $\frac{a \times 10^n}{b}$ can never, therefore, become a whole number, and the decimal corresponding to $\frac{a}{b}$ can never terminate: but if b contain no factor prime to 10, or in other words, if b be composed entirely of powers of 2 and

of 5, the division will terminate after a number of operations equal to the index of the highest of those powers: for if $b = 2^p 5^q$, and if p be the greatest the two numbers p and q, then $\frac{a}{b} = \frac{a \times 10^p}{b \times 10^p} = \frac{a \times 2^p \times 5^p}{2^p \times 5^q \times 10^p}$ $= \frac{a \times 5^{p-q}}{10^p}$: and if q be greater than p, then $\frac{a}{b}$ $= \frac{a \times 10^q}{b \times 10^q} = \frac{a \times 2^q \times 5^q}{2^p \times 5^q \times 10^q} = \frac{a \times 2^{q-p}}{10^q}$.

$$\text{Thus,}\quad \frac{3}{4} = \frac{3}{2^2} = \frac{3 \times 10^2}{2^2 \times 10^2} = \frac{3 \times 2^2 \times 5^2}{2^2 \times 10^2}$$

$$= \frac{3 \times 5^2}{10^2} = \frac{75}{10^2} = .75:$$

$$\frac{7}{125} = \frac{7}{5^3} = \frac{7 \times 10^3}{5^3 \times 10^3} = \frac{7 \times 2^3 \times 5^3}{5^3 \times 10^3}$$

$$= \frac{7 \times 2^3}{10^3} = \frac{56}{10^3} = .056:$$

$$\frac{1}{400} = \frac{1}{2^4 \times 5^2} = \frac{10^4}{2^4 \times 5^2 \times 10^4} = \frac{2^4 \times 5^4}{2^4 \times 5^2 \times 10^4}$$

$$= \frac{5^2}{10^4} = .0025.$$

Circulating decimals: their conversion into equivalent fractions.

195. We have already shewn that a finite decimal may be at once converted into an equivalent decimal whose denominator is a power of 10; and it is obviously necessary merely to reduce that fraction to its lowest terms, in order to get the most simple fraction from which the decimal originated, or to which it is equivalent: it remains to consider in what manner we may convert an *indefinite* but *circulating* decimal into an equivalent fraction, or in other words, in what manner we can discover the fraction from which such a decimal arose: for this purpose it is convenient to premise the following Lemma.

Lemma. The decimal corresponding to the fraction $\frac{1}{10^n - 1}$, is a circulating decimal, whose repeating period consists of n places, with zero in all of them, except the last, in which is 1.

Lemma.

For if we subtract $10^n - 1$ from 10^n, the remainder is 1: or in other words, if we divide 10^n by $10^n - 1$, the first remainder is 1; to which, if n zeros be subjoined, we get the original dividend: and so on, however long the division may be continued.

In a similar manner, if we divide 1 by $10^n - 1$, the first remainder is 1, and the first significant figure in the quotient is 1 in the n^{th} decimal place: the continuation of the process will give a second 1 in the $2n^{\text{th}}$ decimal place, a third 1 in the $3n^{\text{th}}$ decimal place, and so on for ever.

Fractions which produce any assigned circulating decimals.

196. It follows from hence, that if p be any number consisting of not more than n places, then $\frac{p}{10^n - 1}$ will be convertible into a circulating decimal, whose repeating period is p: for if the circulating decimal corresponding to $\frac{1}{10^n - 1}$, be multiplied by p, each of its circulating periods will be changed from 1 into p.

Conversely, therefore, it follows, that all circulating decimals, whose repeating period is p, and consisting of n places, commencing immediately from the decimal point, originate from the fraction $\frac{p}{10^n - 1}$.

If, however, the repeating period commences after the m^{th} decimal place, and not before, the fraction from which the circulating part of the decimal originates is $\frac{p}{10^m \times (10^n - 1)}$: for if we divide the first and all subsequent repeating periods by 10^m, they are expressed by the same series of digits removed m places further from the decimal point.

General formula.

If in those m decimal places after the decimal point, or in any places before it, there exist significant digits which, when the decimal point is removed, form the integral number q, then the whole decimal, including its non-repeating as well as its repeating parts, is expressed by

$$\frac{q}{10^m} + \frac{p}{10^m(10^n-1)} = \frac{q \times 10^n - q + p}{10^m(10^n-1)}.$$

Rule for converting a circulating decimal into an equivalent fraction.

197. We from hence derive the following general rule for the conversion of a circulating decimal into an equivalent fraction.

"From the repeating period subtract the non-repeating part of the decimal, both considered as integral numbers, and add the remainder to the non-repeating part followed by as many zeros as there are places in the repeating period: this is the numerator of the fraction, whose denominator is formed by the digit 9 repeated as often as there are places in each repeating period, followed by as many zeros as there are decimal places before the repeating period begins."

"If the repeating period be less than the non-repeating part, both considered as integral numbers, we must subtract the first from the second, and afterwards subtract the remainder from the non-repeating part followed by as many zeros as there are places in each repeating period, in order to determine the numerator of the equivalent fraction."

This equivalent fraction must be reduced to its lowest terms, in order to get the most simple form of the fraction from which the circulating decimal originated.

198. The following Examples will serve to illustrate the application of this rule. Examples.

(1) $.333\ldots.$

The repeating period is 3, commencing from the decimal point. The equivalent fraction is, therefore,

$$\frac{3}{10-1}=\frac{3}{9}=\frac{1}{3}.$$

(2) $.12\dot{5}\ 12\dot{5}\ 12\dot{5}\ldots.$

The repeating period is 125, commencing from the decimal point; and the corresponding fraction is

$$\frac{125}{10^3-1}=\frac{125}{999},$$

which is not reducible to lower terms.

(3) $.0243\dot{9}\ 0243\dot{9}\ldots.$

The repeating period is .02439, and the fraction corresponding is $\frac{2439}{99999}=\frac{1}{41}$, when reduced to its lowest terms.

(4) $.02325581395348837209\dot{3}0232558\ldots.$

The repeating period consists of 21 places, commencing from the decimal point; and the fraction corresponding is

$$\frac{23255813953488372093}{999999999999999999999}=\frac{1}{43},$$

when reduced to its lowest terms.

(5) $.63\ 6\dot{4}\ 6\dot{4}\ 6\dot{4}\ldots.$

The repeating period is 64, and the non-repeating part 63: their difference is 1: and the fraction corresponding to it is, therefore, $\frac{6300+1}{9900}=\frac{6301}{9900}$.

(6) $1.14285\dot{7}\ 14285\dot{7}\ldots.$

The non-repeating part is 1, and the repeating period is 142857: the fraction corresponding is, therefore,

$$\frac{1142857-1}{999999}=\frac{1142856}{999999}=\frac{8}{7},$$

when reduced to its lowest terms.

(7) $4.53\dot{7}77\ldots.$

The corresponding fraction is $\frac{1021}{225}$.

(8) $.1394\ 23076\dot{9}\ 23076\dot{9}\ldots.$

The fraction corresponding is $\frac{29}{208}$.

(9) $.337\dot{1}8\dot{1}8$

The fraction corresponding is $\frac{3709}{11000}$.

(10) $21.676190\dot{4}76190\dot{4}$

The fraction corresponding is $\frac{2276}{105}$.

◆

CHAP. VIII.

ON INVERSE OPERATIONS IN ALGEBRA, AND ON THE EXTRACTION OF THE SQUARE AND OTHER ROOTS OF ALGEBRAICAL AND NUMERICAL QUANTITIES.

Difficulty of detecting the component factors of a product.

199. IF the product of two or more algebraical quantities be required, the process for finding it is general and certain; but if the product alone is given, and it is required to find the factors, the question becomes more difficult, and in many cases it admits not of solution, and as far as the signs of the factors are concerned, it is always to a certain extent ambiguous.

It is not our intention at present to enter upon the general question of the resolution of an algebraical product into its component factors, as we shall have occasion to discuss it when we come to the general theory of equations: we shall merely mention in the first instance, a few cases, where such a resolution may be effected, and afterwards proceed to the specific consideration of the extraction of the square, cube and other roots.

It must be kept in mind that our attention is confined to the determination of such factors as present themselves under a rational and possible form.

Factors of any expression may be found which is homogeneous and symmetrical.

200. The factors, if any exist, of any homogeneous expression may be found, where the symbols are equally involved.

For the factors must be symmetrical expressions of the symbols, and the dimensions of one of them at least must not exceed one half of the dimensions of the original product.

It is merely requisite, therefore, to form a series of symmetrical functions of the symbols, and so find by trial those which will succeed.

Examples. 201. To find the factors, if any, of the expression

$$a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 + 3abc.$$

Since the symbols a, b, c are equally involved, it follows that $a + b + c$ must be a factor in this case, if any exist, and it is found by trial to succeed.

The two factors are $a + b + c$ and $ab + ac + bc$.

(2) To find the factors, if any, of

$$a^4 + b^4 + c^4 + a^2b^2 + a^2c^2 + b^2c^2 - 2a^2bc - 2b^2ac - 2c^2ab.$$

The factors of which trial may be made are

$$a + b + c$$
$$a^2 + b^2 + c^2$$
$$ab + ac + bc$$
$$a^2 + b^2 + c^2 + ab + ac + bc$$
$$a^2 + b^2 + c^2 - ab - ac - bc.$$

The two last of which succeed.

It is not necessary to make trial of every symmetrical function which may present itself, whose dimensions are within the required limits, as there may be some which clearly cannot succeed: of this kind is $ab + ac + bc$; for in case it was a factor, there could be no such terms as a^4, b^4, c^4.

(3) To find the factors, if any, of

$$2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4.$$

The first factor, $a + b + c$, of which trial is made, is found to succeed, and the result of the division by it, gives us

$$a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2 - a^3 - b^3 - c^3 - 2abc.$$

If this quantity be resolvible into other factors, and if one of them be $b+c-a$, there must be other two, which are $a+b-c$ and $a+c-b$; otherwise the three symbols a, b, c, would not be equally involved, and the expression would not be symmetrical with respect to them: they will be found by trial to succeed.

(4) To find the factors, if any exist, of

$$a^3-b^3+c^3-2a^2b+2ab^2+2a^2c+2ac^2+2b^2c$$
$$-2bc^2-3abc.$$

A very slight examination of this expression would shew that it would be symmetrical with respect to a, b and c, if the sign of b was changed: in other words, it is symmetrical, with respect to a, $-b$ and c, and the factors, if any, must be symmetrical functions or combinations of $a, -b$ and c.

The only such combination of one dimension is $a-b+c$, which is found to succeed: the other factor is $a^2+b^2+c^2-ab+ac-bc$.

202. We have already considered the resolution of expressions into their component factors, in which one power only of each symbol appears (Art. 173. Ex. 10): in this case, the same symbol can present itself in one factor only, and therefore the product of all the others must be a factor of its coefficient: the following are examples:

Expressions where one power only of the same symbol is found.

Examples.

(1) To find the factors of

$$60xyz+72xy+75xz+80yz+90x+96y+100z+120.$$

The coefficient of x is $60yz+72y+75z+90$, and if the whole expression be divided by it, we get a finite quotient $x+\frac{4}{3}$: it follows, therefore, that $x+\frac{4}{3}$ or $3x+4$ (since there is no fraction in the product) is one of the factors required.

The coefficient of y in $60yz + 72y + 75z + 90$ or rather in $20yz + 24y + 25z + 30$ (the quotient of the original expression divided by $3x + 4$) is $20z + 24$: by which, if this expression be divided, we get a finite quotient $y + \frac{5}{4}$: therefore $y + \frac{5}{4}$ or rather $4y + 5$, is a factor of $20yz + 24y + 25z + 30$, and therefore of the original product.

The third and remaining factor is $5z + 6$.

(2) To find the factors of

$4xz + 8xu + 6yz + 12yu + 6x + 9y + 4u + 2z + 3$.

The coefficient of x is $4z + 8u + 6$, and if the expression be divided by it, we get a finite quotient $x + \frac{3y}{2} + \frac{1}{2}$: it follows, therefore, that $2x + 3y + 1$ is one factor of this product, and $2z + 4u + 3$ the other.

(3) To find the factors, if any, involving two or one only, of the quantities x, y, z, of $x^2y^2z^2 + 4x^2y^2 - 4x^2z^2 - 16x^2 + 3xy^2z^2 + 12xy^2 - 12xz^2 - 48x + 4y^2z^2 + 16y^2 - 16z^2 - 64$.

In this case $y^2z^2 + 4y^2 - 4z^2 - 16$ is a common measure of the coefficient of x^2 and x, and it is also a divisor of the whole expression, producing $x^2 + 3x + 4$ for its quotient: the other two factors detected by this method are $y^2 - 4$ and $z^2 + 4$.

203. When the expressions are not homogeneous with respect to all the symbols involved, and when factors are required involving a symbol of which different powers present themselves, these and similar methods admit not of application, and we must resort to the processes, which will be afterwards considered, for the resolution of equations: we must except, however, the case where the expression is to be resolved into two or more equal factors, the discussion of which will occupy the remainder of this Chapter.

Extraction of the square root.

204. The process for extracting the square root of any algebraical expression, is founded upon the direct process for the formation of the square, merely reversing the steps: it is of course equivalent to resolving an expression into two equal and identical factors: when such factors exist, the expression is a complete square, and its root is determined in a finite number of terms: but when the expression is not a complete square, and there exists no such finite root, a root will be found *by the rule* in the form of an interminable series, in a similar manner and upon the same principle with the series which represent incomplete quotients in division (Art. 46. Ex. 20...)

Investigation of the rule.

205. In order to define the rule for finding this root, let us examine the law of formation of the squares.

The square of a is a^2.

The square of $(a+b)$ is $a^2+2ab+b^2$, which may be put under the form

$$a^2+(2a+b)b.$$

The square of $(a+b+c)$ is $(a+b)^2+2(a+b)c+c^2$, considering $a+b$ as one quantity; which may be put under the form

$$(a+b)^2+(2a+2b+c)c:$$

and if it be written at full length, and its terms arranged in alphabetical order, it becomes

$$a^2+2ab+2ac+b^2+2bc+c^2.$$

In the two last cases, the reversed process would stand as follows:

(1)
$$\begin{array}{l}
a^2+2ab+b^2\ (a+b \\
a^2 \\
\hline
2a+b)\ 2ab+b^2 \\
\qquad\qquad 2ab+b^2=(2a+b)b \\
\hline
\qquad\qquad \cdot \quad \cdot
\end{array}$$

(2)
$$\begin{array}{rl}
 & a^2+2ab+2ac+b^2+2bc+c^2\ (a+b+c \\
 & a^2 \\ \hline
2a+b) & 2ab+2ac+b^2+2bc+c^2 \\
 & \quad 2ab+b^2 \\ \hline
2a+2b+c) & 2ac+2bc+c^2 \\
 & 2ac+2bc+c^2 \\ \hline
\end{array}$$

The examination of the direct and inverse process in these cases, will lead to the following rule.

Rule. 206. *Arrange the terms in alphabetical order: find the square root* (a) *of the first term* (a^2)*: subtract its square* (a^2)*: double the root* (a) *already found, and divide the first term* ($2ab$) *of the remainder by it: the quotient* (b)*, is the second term of the root: to double the first term of the root, add the second: the sum* ($2a+b$) *is called the* DIVISOR*: from the remainder* ($2ab+b^2$) *subtract the product of the divisor* ($2a+b$) *and of the second term* (b) *of the root: if there is no remainder, the terms in the root constitute the square root required: if not, we must consider the terms of the root,* ($a+b$)*, as one term, and repeat the same process again: and so on, as often as may be required.*

Its principle. The principle of this rule is, that after every operation, we have subtracted from the original expression, the complete square of the terms in the root: thus, in the first case, by the two operations, we take away first a^2 and then $(2a+b)b$, which are together equal to the square of $a+b$. In the second case, we first take away (by two operations) the square of $a+b$, and by the third operation, we also take away $(2a+2b+c)c$, which added to $(a+b)^2$, makes the complete square of $a+b+c$.

Examples.

207. (1) Extract the square root of $a^2-ab+\frac{b^2}{4}$.

$$\begin{array}{rl}
 & a^2-ab+\frac{b^2}{4}\ \Big(a-\frac{b}{2}\\
 & a^2\\ \cline{2-2}
2a-\frac{b}{2}\Big) & -ab+\frac{b^2}{4}\\
 & -ab+\frac{b^2}{4}\\ \cline{2-2}
 & \quad\cdot\quad\cdot
\end{array}$$

In this case, we divide $-ab$ by $2a$, which gives $-\frac{b}{2}$, for the second term in the root.

(2) Extract the square root of $\frac{a^2}{b^2}+\frac{b^2}{a^2}-2$.

$$\begin{array}{rl}
 & \frac{a^2}{b^2}-2+\frac{b^2}{a^2}\ \Big(\frac{a}{b}-\frac{b}{a}\\
 & \frac{a^2}{b^2}\\ \cline{2-2}
\frac{2a}{b}-\frac{b}{a}\Big) & -2+\frac{b^2}{a^2}\\
 & -2+\frac{b^2}{a^2}\\ \cline{2-2}
 & \quad\cdot\quad\cdot
\end{array}$$

In arranging the terms, according to the powers of the letter a, $\frac{b^2}{a^2}$ must be placed after -2, inasmuch as it involves a negative power of a: the neglect of this arrangement might make the process interminable.

(3) Extract the square root of $\frac{9}{4}+6x-17x^2-28x^3+49x^4$.

$$\begin{array}{rl}
 & \frac{9}{4}+6x-17x^2-28x^3+49x^4\ \left(\frac{3}{2}+2x-7x^2\right. \\
 & \frac{9}{4} \\
\hline
3+2x) & 6x-17x^2 \\
 & 6x+4x^2 \\
\hline
3+4x-7x^2) & -21x^2-28x^3+49x^4 \\
 & -21x^2-28x^3+49x^4 \\
\hline
\end{array}$$

If we had arranged the terms of the expression in the contrary order, or as follows,

$$49x^4-28x^3-17x^2+6x+\frac{9}{4}$$

we should find $7x^2-2x-\frac{3}{2}$ for the root, a result differing from the former or $\frac{3}{2}+2x-7x^2$, in the sign of the whole quantity: but it must be kept in mind that $\left(7x^2-2x-\frac{3}{2}\right)$ and $-\left(7x^2-2x-\frac{3}{2}\right)$ or $\frac{3}{2}+2x-7x^2$, are equally roots of $\left(7x^2-2x-\frac{3}{2}\right)^2$: and in every instance, in passing from the square to the square root, there are two results, which equally answer the conditions, differing merely in their signs. Thus the square root of a^2 is equally $+a$ and $-a$: the square root of $a^2+2ab+b^2$ is equally $(a+b)$ and $-(a+b)$.

The square root of $\frac{a^2}{b^2}-2+\frac{b^2}{a^2}$ is equally $\frac{a}{b}-\frac{b}{a}$ and

$$-\left(\frac{a}{b}-\frac{b}{a}\right) \text{ or } \frac{b}{a}-\frac{a}{b}.$$

And similarly in all other cases.

(4)
$$\begin{array}{l}
9a^2-6ab+30ac+6ad+b^2-10bc-2bd+25c^2+10cd+d^2\,(3a-b+5c+d \\
9a^2 \\
\hline
6a-b)\,-6ab+b^2 \\
\qquad\quad -6ab+b^2 \\
\hline
6a-2b+5c)\,30ac-10bc+25c^2 \\
\qquad\qquad\qquad 30ac-10bc+25c^2 \\
\hline
6a-2b+10c+d)\,6ad-2bd+10cd+d^2 \\
\qquad\qquad\qquad\qquad 6ad-2bd+10cd+d^2 \\
\hline
\end{array}$$

The other root is $-3a+b-5c-d$.

(5)
$$\begin{array}{l}
ac^2+2cd\sqrt{ab}+bd^2\,(c\sqrt{a}+d\sqrt{b} \\
ac^2 \\
\hline
2c\sqrt{a}+d\sqrt{b})\;2cd\sqrt{ab}+bd^2 \\
\qquad\qquad\qquad 2cd\sqrt{ab}+bd^2 \\
\hline
\end{array}$$

(6) $\sqrt{\{a^{2m}-4a^{m+n}+4a^{2n}\}}=a^m-2a^n$:

(7) $\sqrt{\left\{\frac{a^2-2a+1}{a^4-2a^3+3a^2-2a+1}\right\}}=\frac{a-1}{a^2-a+1}$:

The square roots of fractions are found by extracting the square roots of the numerator and denominator successively.

$$\begin{array}{rl} (8) \qquad & 7+4\sqrt{3}\,(2+\sqrt{3} \\ & 4 \\ \cline{2-2} 4+\sqrt{3}) & 4\sqrt{3}+3 \\ & 4\sqrt{3}+3 \\ \cline{2-2} & \cdot \quad \cdot \end{array}$$

Roots of binomial surds.

In binomial expressions of this kind, one term may be considered as the sum of the squares of the roots, and the other as their product; when the root exists, it may always be found by this method, though the process is tentative only, an incorporation of the terms having taken place which obliterates some of those traces of the root upon which the rule is founded: the question will be shewn hereafter to resolve itself into the solution of a quadratic equation.

The same remark may be extended to algebraical expressions such as

$$\frac{a^2}{4} + \frac{x}{2}\sqrt{(a^2-x^2)},$$

whose root is

$$\frac{x+\sqrt{(a^2-x^2)}}{2},$$

which are more easily and generally treated by other methods.

Extraction of roots where the process does not terminate.

208. In the preceding examples, the process has always terminated: but it is evident that the process may be continued as long as there is any remainder, and therefore indefinitely, whenever the quantity subjected to its operation does not admit of resolution into two finite identical factors.

(1) Let it be required to extract the square root of $1-x$.

$$
\begin{array}{rl}
 & 1-x\left(1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\ldots\right. \\
 & 1 \\ \cline{2-2}
2-\frac{x}{2}\Big) & -x \\
 & -x+\frac{x^2}{4} \\ \cline{2-2}
2-x-\frac{x^2}{8}\Big) & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{64} \\ \cline{2-2}
2-x-\frac{x^2}{4}-\frac{x^3}{16}\Big) & -\frac{x^3}{8}-\frac{x^4}{64} \\
 & -\frac{x^3}{8}+\frac{x^4}{16}+\frac{x^5}{64}+\frac{x^6}{256} \\ \cline{2-2}
2-x-\frac{x^2}{4}-\frac{x^3}{8}-\frac{5x^4}{128}\Big) & -\frac{5x^4}{64}-\frac{x^5}{64}-\frac{x^6}{256} \\
 & -\frac{5x^4}{64}+\frac{5x^5}{128}+\frac{5x^6}{512}+\frac{5x^7}{1024}+\frac{25x^8}{16384} \\ \cline{2-2}
 & -\frac{7x^5}{128}-\frac{7x^6}{512}-\frac{5x^7}{1024}-\frac{25x^8}{16384}
\end{array}
$$

In this case, the operation can never terminate, inasmuch as there are always two more terms in each subtrahend, than in the remainder from which it must be taken: in other words, the remainder can never disappear.

The series $1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-$ &c., is the result of the operation of extracting the square root in conformity with the general rule: when we write the equation

$$\sqrt{(1-x)}=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\text{\&c.},$$

we must keep in mind, that the sign = does not necessarily indicate that the two members of the equation are arithmetically equal to each other, but merely that one member is the result of an operation, which in the other is indicated and not performed. (Art. 22 and 117.)

The arithmetical equality of the two members of this equation can exist, in those cases only, where x is less than 1: and they can only be considered as algebraically equivalent to each other, upon the principle which we have explained before in Art. 120.

$$(2)\quad \sqrt{(1+x)} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c.$$

This series may be deduced from the former, by changing the sign of x.

The resulting series for the square root of $x + 1$, considering x as the first term, is essentially different from the one which we have just found for the square root of $1 + x$: the operation stands as follows:

$$\begin{array}{rl}
 & x + 1 \left(x^{\frac{1}{2}} + \dfrac{1}{2x^{\frac{1}{2}}} - \dfrac{1}{8x^{\frac{3}{2}}} + \&c. \right. \\
 & x \\
 \hline
 \left. 2x^{\frac{1}{2}} + \dfrac{1}{2x^{\frac{1}{2}}} \right) & 1 \\
 & 1 + \dfrac{1}{4x} \\
 \hline
 \left. 2x^{\frac{1}{2}} + \dfrac{1}{x^{\frac{1}{2}}} - \dfrac{1}{8x^{\frac{3}{2}}} \right) & - \dfrac{1}{4x} \\
 & - \dfrac{1}{4x} - \dfrac{1}{8x^2} + \dfrac{1}{64x^3}.
\end{array}$$

If x be less than unity, it is the first, and if x be greater than unity, it is the second, series, which is arithmetically equal to $\sqrt{(1+x)}$.

(3) $$\sqrt{(a+x)}=\sqrt{a}\sqrt{\left(1+\frac{x}{a}\right)}$$

$$=\sqrt{a}\left\{1+\frac{x}{2a}-\frac{x^2}{8a^2}+\frac{x^3}{16a^3}-\&c.\right\}$$

$$=a^{\frac{1}{2}}+\frac{x}{2a^{\frac{1}{2}}}-\frac{x^2}{8a^{\frac{3}{2}}}+\frac{x^3}{16a^{\frac{5}{2}}}-\&c.$$

Or under another form,

$$\sqrt{(x+a)}=\sqrt{x}\sqrt{\left(1+\frac{a}{x}\right)}$$

$$=x^{\frac{1}{2}}\left\{1+\frac{a}{2x}-\frac{a^2}{8x^2}+\frac{a^3}{16x^3}-\&c.\right\}$$

$$=x^{\frac{1}{2}}+\frac{a}{2x^{\frac{1}{2}}}-\frac{a^2}{8x^{\frac{3}{2}}}+\frac{a^3}{16x^{\frac{5}{2}}}-\&c.$$

(4) $$\sqrt{(a^2+ax+x^2)}=a+\frac{1}{2}x+\frac{3}{8}\cdot\frac{x^2}{a}-\frac{3}{16}\cdot\frac{x^3}{a^2}-\&c.$$

If we put the powers of x in the first place, we shall find

$$\sqrt{(x^2+ax+a^2)}=x+\frac{1}{2}a+\frac{3}{8}\cdot\frac{a^2}{x}-\frac{3}{16}\cdot\frac{a^3}{x}-\&c.$$

(5) $$\sqrt{\left\{\frac{a+x}{a-x}\right\}}=\sqrt{\left\{1+\frac{2x}{a}+\frac{2x^2}{a^2}+\frac{2x^3}{a^3}+\&c.\right\}}$$

$$=1+\frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{2a^3}+\frac{3x^4}{2a^4}+\&c.$$

In this case, we first divide $a+x$ by $a-x$, and then extract the square root of the resulting series.

Rule for the extraction of the square root in numbers.

209. The rule for the extraction of the square root in numbers, is entirely founded upon the rule for the corresponding operation in Algebra, being merely modified in such a manner as may adapt it to arithmetical notation: for the purpose of shewing this connection more clearly, we will first exhibit it under an algebraical, and, secondly, under its common arithmetical form.

Required the square root of 61009.

$$
\begin{array}{lr|l}
 & & 61009\ (\overset{a}{200} + \overset{b}{40} + \overset{c}{7} \text{ or } 247 \\
 & & \underline{40000} = a^2 \\
2a + b = & \underset{(440)}{400 + 40}\ \Big) & 21009 \\
 & & \underline{17600} = (2a + b)\,b \\
2a + 2b + c & \underset{(487)}{480 + 7}\ \Big) & 3409 \\
 & & \underline{3409} = (2a + 2b + c)\,c
\end{array}
$$

If for 200, 40 and 7, we put a, b and c respectively, we shall find the process precisely the same as when exhibited in algebraical symbols.

The arithmetical form of the process is as follows:

$$
\begin{array}{rl}
 & \dot{6}1\dot{0}0\dot{9}\ (247 \\
 & \underline{4} \\
44) & 210 \\
 & \underline{176} \\
487) & 3409 \\
 & \underline{3409}
\end{array}
$$

This form of the process may be considered as the skeleton of the one above given; it rejects all superfluous writing, in conformity with the spirit of all arithmetical processes.

210. We mark off periods of two, commencing with the place of units: for the number of zeros after the significant digits in the squares of 10, 20, 30, &c. is two; in the squares of 100, 200, 300, &c. is four; and so on, being always double of their number in the root: by thus marking or pointing off two places, there is a place in the root corresponding to each period: having thus determined the number of places in the root, it only remains to determine the digits in those places.

Method of pointing.

Its uses.

The digit in the highest place of the root, is the greatest number, whose square is less than the first period: for if a larger number be taken, the square of that number with its proper zeros, must be larger than the original number whose root is required.

First digit in the root.

We subtract the square of this number from the first period and to the remainder, we subjoin the second period only: for the digits in the succeeding periods are not affected by the second operation, since, if the process was exhibited at full length, zeros only would be placed beneath them.

We double the first digit in the root and divide the remainder or *dividend* formed as above (omitting the last digit) by it: the result, taken in defect if necessary, is, or may be, the next digit in the root: we subjoin this digit to the divisor, multiply the resulting quantity by it, and subtract the product from the first remainder. For the first digit in the root has properly one more zero after it than the second, and if we add the numbers which they represent together, they will form consecutive digits in the sum: also the product of the numbers which they represent, has a number of zeros after it, less by unity than the double of the number of zeros after the first digit of the root: we therefore omit the last digit, in dividing the remainder by the double of the first digit of the root in order to determine the second.

Second digit in the root.

We afterwards consider the digits in the root, as one number or digit, and repeat the same process until all the periods are exhausted.

Other digits.

Method of extracting the square root of a decimal.

211. The square root of $\frac{N}{10^{2n}}$ is $\frac{\sqrt{N}}{10^n}$: in extracting therefore the square root of a decimal, if the number of decimal places be even, we may proceed as if it were a whole number, making a number of decimal places in the root equal to one half the number of decimal places in the square: for $d=\frac{N}{10^{2n}}$, if $2n$ be the number of decimal places in d, and N the whole number which arises from removing the decimal point.

Method of pointing in decimals.

We can always make the number of decimal places even, by subjoining a zero if necessary: in pointing, therefore, we may begin always from the place of units, marking to the right as well as to the left: and the decimal places in the root will begin, when the integral periods in the process of extraction are exhausted.

Thus let it be required to extract the square root of

```
          .  .  .  .
         2583.6889 (50.83
         25
         ----
  1008)  8368
         8064
         -----
 10163) 30489
        30489
        -----
          . .
```

In this case, we are obliged to bring down two periods in the second operation, inasmuch as the first divisor 10 is greater than the first remainder 8 (omitting the last digit of 83.)

Indefinite roots in decimals.

212. It is upon this principle, that we may continue the operation of extracting the square root of a number

which is not a complete square as far as we choose: for any number of zeros may be subjoined to the significant digits of a number whether originally a decimal or not, after the decimal point: and as the remainder, when not a whole number, diminishes after each operation, since its digits recede continually from the decimal point, the root obtained approximates continually to the true value, and may be supposed equal to it, if we suppose the operation indefinitely continued, and therefore the *ultimate* remainder arithmetically evanescent.

1. $\sqrt{5} = 2.23606\ldots$
2. $\sqrt{101} = 10.04987\ldots$
3. $\sqrt{10} = 3.16228$
4. $\sqrt{.1} = .316228$
5. $\sqrt{.01} = .1$
6. $\sqrt{.001} = .0316228$
7. $\sqrt{.0001} = .01$
8. $\sqrt{.00000256} = .0016$
9. $\sqrt{\frac{582169}{956484}} = \frac{763}{978}.$

In this case, the numerator and denominator of the fraction are complete squares, and their square roots may be determined separately: in most cases, however, it is most convenient to begin by reducing the fraction to an equivalent decimal, and then to extract its square root.

10. $\sqrt{\frac{7}{4}} = \sqrt{1.75} = 1.32287\ldots$
11. $\sqrt{\frac{1}{17}} = \sqrt{.0588235294\ldots} = .24253\ldots$
12. $\sqrt{3\frac{1}{3}} = \sqrt{3.3333\ldots.}$

$= 1.8257\ldots\ldots$

Rule for extracting the cube root of algebraical quantities.

213. In the same manner, in which we deduced the rule for the square root, by observing the law of formation of its square, we may deduce likewise a rule for finding the cube root, by attending to the law of formation of the cube: thus, since we have

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$
$$(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3,$$

and so on, whatever be the number of terms in the root. The inverse process will stand as follows:

$$\begin{array}{ll} & a^3+3a^2b+3ab^2+b^3\ (a+b \\ & a^3 \\ \hline \text{Divisor} & 3a^2)\ 3a^2b+3ab^2+b^3 \\ & \qquad\ 3a^2b+3ab^2+b^3 \\ \hline & \qquad\ \cdot \quad \cdot \quad \cdot \end{array}$$

Rule.

The terms being arranged according to the powers of some one letter, we find the cube root of the first term: we then take three times its square, divide the first term of the remainder by it and the quotient is the second term: we then take away the remaining part of the cube of the terms in the root; namely three times the square of the first term multiplied into the second, three times the first term into the square of the second, and the cube of the second term: if there is any remainder, we proceed as before, considering the terms already in the root, as constituting one term.

Example.

$$\begin{array}{l} x^3+6x^2+12x+8\ (x+2 \\ x^3 \\ \hline 3x^2)\ 6x^2+12x+8 \\ \qquad\ 6x^2+12x+8=3x^2\times 2+3x\times 2^2+2^3 \\ \hline \qquad\ \cdot \quad \cdot \quad \cdot \end{array}$$

The cube root is $x+2$.

(2) $a^2x^6 - 3a^2bx^5 + 3(ab^2 + a^2c)x^4 - (b^3 + 6abc)x^3 + 3(b^2c + ac^2)x^2 - 3bc^2x + c^3 \left(ax^2 - bx + c. \right.$

a^2x^6

$3a^2x^4) - 3a^2bx^5 + 3(ab^2 + a^2c)x^4 - (b^3 + 6abc)x^3 + \ldots$

$- 3a^2bx^5 + 3ab^2x^4 - b^3x^3$

$3a^2x^4 - 6abx^3 + 3b^2x^2 \Big)\; 3a^2cx^4 - 6abcx^3 + 3(b^2c + ac^2)x^2 - 3bc^2x + c^3$

or $3(ax^2 - bx)^2$. $\;\; 3a^2cx^4 - 6abcx^3 + 3b^2cx^2 + 3ac^2x^2 - 3bc^2x + c^3$

. . . . . .

(3) $\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$: or properly arranged

$x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} \left(x + \frac{1}{x} \right.$

x^3

$3x^2)\; 3x + \frac{3}{x} + \frac{1}{x^3}$

$3x + \frac{3}{x} + \frac{1}{x^3}$

.

(4) $\sqrt[3]{}(a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3$
$+ 3b^2c + 3bc^2 + c^3) = a + b + c.$

(5) $\sqrt[3]{}\left\{\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}} - \frac{3x}{a} + \frac{6x^{\frac{1}{2}}}{a^{\frac{1}{2}}} - 7 + \frac{6a^{\frac{1}{2}}}{x^{\frac{1}{2}}} - \frac{3a}{x} + \frac{a^{\frac{3}{2}}}{x^{\frac{3}{2}}}\right)$

$= \frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}} - 1 + \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}.$

(6) $\sqrt[3]{}\{8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2$
$+ 108c^5x - 27c^6) = 2x^2 + 4cx - 3c^2$

(7) $\sqrt[3]{}\{1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 + 10x^6 - 6x^7 + 3x^8 - x^9)$
$= 1 - x + x^2 - x^3.$

(8) Let it be required to extract the cube root of $1-x$.

$$\begin{array}{rl}
& 1-x\ \left(1-\frac{x}{3}-\frac{x^2}{9}-\frac{5x^3}{81}-\right. \\
& 1 \\
& \overline{} \\
3) & -x \\
& -x+\frac{x^2}{3}-\frac{x^3}{27} \\
& \overline{\phantom{-x+\frac{x^2}{3}-\frac{x^3}{27}}} \\
3-2x+\frac{x^2}{3} & \Big)\ -\frac{x^2}{3}+\frac{x^3}{27} \\
\text{or } 3\left(1-\frac{x}{3}\right)^2 & \Big)\ -\frac{x^2}{3}+\frac{2x^3}{9}-\frac{x^4}{27}+\frac{x^4}{27}-\frac{x^5}{81}-\frac{x^6}{729} \\
& \overline{\phantom{-\frac{x^2}{3}+\frac{2x^3}{9}-\frac{x^4}{27}+\frac{x^4}{27}}} \\
3-2x-\frac{x^2}{3}+\frac{2x^3}{9}+\frac{x^4}{27} & \Big)\ -\frac{5x^3}{27}+\frac{x^5}{81}+\frac{x^6}{729} \\
\text{or } 3\left(1-\frac{x}{3}-\frac{x^2}{9}\right)^2 & \Big)\ -\frac{5x^3}{27}+\frac{10x^4}{81}+\frac{5x^5}{243}+\frac{15x^6}{2187}-\frac{40x^7}{6561}-\frac{25x^8}{19683}-\frac{125x^9}{531441} \\
& \overline{\phantom{-\frac{5x^3}{27}+\frac{10x^4}{81}+\frac{5x^5}{243}+\frac{15x^6}{2187}-\frac{40x^7}{6561}-\frac{25x^8}{19683}}} \\
& -\frac{10x^4}{81}-\&c\ldots\ldots
\end{array}$$

It is obvious that the process can never terminate, inasmuch as there must be a greater number of terms in each subtrahend than in the corresponding remainder.

$$(9)\quad \sqrt[3]{(1+x)}=1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}-\frac{10x^4}{243}+\&c.$$

which differs from the series in the last example, in having its *even* terms positive.

$$(10)\quad \sqrt[3]{(x+1)}=x^{\frac{1}{3}}+\frac{1}{3x^{\frac{2}{3}}}-\frac{1}{9x^{\frac{5}{3}}}+\frac{5}{81x^{\frac{8}{3}}}-\frac{10}{243x^{\frac{11}{3}}}+\&c.$$

The observations which were made in Art. 208. with reference to the series corresponding to $\sqrt{(1+x)}$ and $\sqrt{(x+1)}$, are equally applicable to the series for $\sqrt[3]{(1+x)}$ and $\sqrt[3]{(x+1)}$, and therefore need not be repeated in this place.

Extraction of the cube root in numbers. 214. The rule for the extraction of the cube root in numbers, is founded upon the rule for the corresponding operation in Algebra, as will be manifest if the process be exhibited at full length: let us take for this purpose the number 49836032, whose cube root is required.

$$\begin{array}{rll}
 & 49836032 & (\overset{a}{300}+\overset{b}{60}+\overset{c}{8}=a^3 \\
 & 27000000 & \\
\cline{2-2}
3a^2=270000) & 22836032 & \text{Resolvend} \\
\cline{2-2}
 & 16200000=3a^2b & \\
 & 3240000=3ab^2 & \\
 & 216000=b^3 & \\
\cline{2-2}
 & 19656000 & \text{Subtrahend} \\
\cline{2-2}
3\,(a+b)^2=388800) & 3180032 & \text{Resolvend} \\
\cline{2-2}
 & 3110400=3\,(a+b)^2c & \\
 & 90120=3\,(a+b)c^2 & \\
 & 512=c^3 & \\
\cline{2-2}
 & 3180032 & \text{Subtrahend.} \\
\cline{2-2}
\end{array}$$

The following is the arithmetical form of the same process.

$$\begin{array}{rll}
 & 4\dot{9}83\dot{6}03\dot{2} & (368 \\
 & 27 & \\
\cline{2-2}
27) & 22836 & \text{Resolvend} \\
\cline{2-2}
 & 162 & \\
 & \;\;324 & \\
 & \;\;\;\;216 & \\
\cline{2-2}
 & 19656 & \text{Subtrahend} \\
\cline{2-2}
3888) & 3180032 & \text{Resolvend} \\
\cline{2-2}
 & 31104 & \\
 & \;\;\;\;6912 & \\
 & \;\;\;\;\;\;512 & \\
\cline{2-2}
 & 3180032 & \text{Subtrahend} \\
\cline{2-2}
\end{array}$$

The first and complete form of the process requires no explanation, inasmuch as it is a transcript in numbers of the algebraical process of extracting the cube root of $(a+b+c)^3$; the second or proper arithmetical form of it, is abbreviated from the first by omitting the zeros, introduced by the multiplications, which succeed the significant digits, in every instance: it is very easy to pass from thence to the general arithmetical rule.

Method of pointing.

215. We commence by separating the number whose cube root is required, into periods of three places, commencing from the place of units, for the following reasons: the cube of all digits, or of all numbers less than 10, cannot exceed three places, that of 9 being 729: the cube of 10, and of all multiples of 10 less than 100, is followed by three zeros, and therefore cannot exceed six places: the cube of 100, and of all multiples of 100 less than 1000, is succeeded by six zeros, and therefore cannot exceed nine places: and so on, the number of zeros by which the significant digits are succeeded, increasing by three for every additional zero in the root: by this separation of periods we therefore determine the number of places in the root, and the first digit in the root must be the greatest number whose cube does not exceed the first period.

Rule and its explanation.

The cube of the first digit is subtracted from the first period; and to the remainder, if any, the second period is subjoined: this is called the *resolvend*. It is not necessary to bring down any other period, inasmuch as zeros only would be placed beneath them, if the process was exhibited at full length: we then take three times the square of the first digit in the root, and divide the *resolvend* by it, omitting the last two digits, in order to determine the next digit in the root; the quotient being taken in defect, if necessary: this is done in consequence of the omission of the zero, which should properly follow the first digit in the root already determined, and therefore causing an omission of two zeros in its square:

we then place underneath each other, advancing the place of units successively one place to the right, the product of three times the square of the first digit into the second, of three times the first digit into the square of the second, and the cube of the second: we add them together in their proper positions, and thus determine the *subtrahend*, which must be subtracted from the resolvend: the reason of this successive position of these quantities, is the omission of two zeros in the first product, and of one zero in the second.

To the remainder, after the subtraction is made, we subjoin the third period, if any, and proceed as before, merely considering the whole number already determined in the root, in the same light as the first digit in the root in the former case: we are thus enabled to continue the operation as far as may be necessary.

Cube roots of decimals.

216. When the cube root of a decimal is required, we mark off periods of three places to the right as well as to the left of the place of units, adding as many zeros as may be necessary to make the number of decimal places 3, or a multiple of 3: for

$$\sqrt[3]{\frac{N}{10^3}} = \frac{\sqrt[3]{N}}{10}, \quad \sqrt[3]{\frac{N}{10^6}} = \frac{\sqrt[3]{N}}{10^2}, \quad \sqrt[3]{\frac{N}{10^9}} = \frac{\sqrt[3]{N}}{10^3},$$

and so on: and therefore the process of extraction may proceed as if all the periods were integral, it being merely necessary to mark off one decimal place in the root for every decimal period of three places.

Indefinite cube roots in decimals.

217. Upon the same principle, the extraction of the cube root of a number, whether decimal or not, which is not a complete cube, may be continued indefinitely by adding zeros continually to the integral nnmber, and considering all the places in the root corresponding to them as decimal places: we can thus approximate as near as we please to the root required.

Examples.

218. (1) $\sqrt[3]{(64481201)} = 401.$

(2) $\sqrt[3]{(113028882875)} = 4835.$

(3) $\sqrt[3]{(8108486729)} = 2009.$

(4) $\sqrt[3]{1000} = 10.$

(5) $\sqrt[3]{100} = 4.641\ldots\ldots$

(6) $\sqrt[3]{10} = 2.154\ldots\ldots$

(7) $\sqrt[3]{1} = 1.$

(8) $\sqrt[3]{.1} = .4641\ldots\ldots$

(9) $\sqrt[3]{.01} = .2154\ldots\ldots$

(10) $\sqrt[3]{.001} = .1.$

(11) $\sqrt[3]{.0001} = .04641\ldots\ldots$

(12) $\sqrt[3]{.00001} = .02154\ldots\ldots$

(13) $\sqrt[3]{.000001} = .01\ldots\ldots$

(14) $\sqrt[3]{102.875} = 4.68565\ldots\ldots$

(15) $\sqrt[3]{\frac{2}{3}} = \sqrt[3]{.666\ldots} = .87358\ldots\ldots$

(16) $\sqrt[3]{3\frac{4}{5}} = \sqrt[3]{3.8} = 1.56049\ldots\ldots$

Extraction of roots of higher powers.

219. The extraction of the roots of higher powers, such as the fourth or fifth, &c. is founded upon the formation of the powers themselves: thus

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4;$$

and if we should propose to pass from this fourth power to its root, the process would stand as follows:

$$\begin{array}{rl}
 & a^4+4a^3b+6a^2b^2+4ab^3+b^4\ (a+b \\
 & a^4 \\ \hline
4a^3) & 4a^3b+6a^2b^2+4ab^3+b^4 \\
 & 4a^3b+6a^2b^2+4ab^3+b^4 \\ \hline
 & \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

The first term in the root is the fourth root of the first term: in order to determine the second term in the root, divide the first term of the remainder, after subtracting the first term, by four times the cube of the first term of the root: we afterwards, when two terms in the root are determined, form the several terms of the subtrahend, according to the law of formation, of all the terms after the first, of $(a+b)^4$: if there is any remainder, the process may be continued, considering the two terms in the root already determined as one.

It is not necessary, however, to exemplify this process, or to shew in what manner it may be extended to higher roots: for it is very rarely that such operations are required in Algebra and its applications, when the root is finite; and in all other cases, the binomial or multinomial theorem, which we shall investigate in a subsequent chapter, will furnish a more rapid and certain mode of determining them: the same remark applies, with greater force, to the extraction of the higher roots of numbers, which are effected by means of logarithms, even in the case of the cube root, more rapidly than by any other methods.

◆

CHAP. IX.

THEORY OF PERMUTATIONS AND COMBINATIONS.

Permutations.

220. THE different orders, in which any quantities can be arranged, are called their Permutations.

Thus the permutations of a and b are ab and ba: the permutations of a, b and c are abc, bac, acb, cab, bca and cba; whilst the permutations of the same three letters, taken two and two together, are ab, ba, ac, ca, bc and cb.

Permutations distinguished from Variations.

221. The term *Permutation* is confined by some authors to the different arrangements of the *whole* of any number of things: whilst the term *Variation* is applied to the different arrangements of any number of them less than the whole: it may be convenient generally to adopt this distinction, and to use the term Permutation *absolutely*, inasmuch as the properties of such permutations require more particular attention and examination than those of other classes of variations.

Number of variations of n things.

222. We shall now proceed to determine the expressions for the number of *variations* of n things, taken two and two, three and three, four and four, and generally r and r together, where r is any number less than n.

Let the n things, whose variations are required, be severally represented by a_1, a_2, $a_3 \ldots a_n$, where the numbers subscribed to the same letter a distinguish the things represented from each other, and likewise determine the order of their succession. (Art. 3. and Art. 39. Ex. 22.)

The number of variations, when taken separately or one by one, is clearly n.

223. The number of variations, taken *two* and *two* together, is $n(n-1)$.

Taken two and two together.

For a_1 may be placed successively before a_2, $a_3,\ldots a_n$, and thus form $(n-1)$ variations two and two: a_2 may be placed successively before a_1, a_3, $a_4,\ldots a_n$, and thus form $(n-1)$ variations two and two, which are different from the former: and the same thing may be equally done with $a_3, a_4,\ldots a_n$, there being $(n-1)$ variations corresponding to each letter in the first place, which are different from each other and from all the others (Art. 136): the whole number of such variations is, therefore, n times the number corresponding to each letter in the first place, and is consequently $n(n-1)$.

224. The number of variations, taken *three* and *three* together, is $n(n-1)(n-2)$.

Taken three and three together.

For if we form all the variations possible of $(n-1)$ letters (omitting a_1), taken two and two together, which are $(n-1)(n-2)$ in number (putting $n-1$ in the place of n in the expression last determined Art. 223,) and place a_1 before each of them, we shall have $(n-1)(n-2)$ variations, taken *three* and *three* together, where a_1 occupies the first place: and there must be the same number of variations in which a_2, $a_3\ldots\ldots$ and all the other letters successively occupy the first place in each (Art. 136): the whole number, therefore, of such variations, must be n times the number corresponding to each letter in the first place, and is consequently $n(n-1)(n-2)$.

225. By a similar process of reasoning, we should shew that the number of variations of n things taken *four* and *four* together, would be expressed by

Taken four and four together.

$$n(n-1)(n-2)(n-3),$$

where there are four factors, which are the natural numbers *descending* from n: and the law which is found to prevail in the formation of the expressions for the number of variations taken two and two, three and three, four

Law of formation extended by induction to any number.

and four together, may be easily extended by *induction* to the expression for the number of variations, when any number of things (r) are taken together.

Its proof.

226. In order to *demonstrate* the correctness of this induction, we must shew, that if this law is true for any one class of variations, it must necessarily be true for the class next superior to it (Art. 139).

Assuming, therefore, the expression for the number of variations of n things taken $(r-1)$ and $(r-1)$ together, which is

$$n(n-1)\ldots\ldots(n-r+2) \quad (\alpha)$$

it is required to prove, that the expression for the number of their variations, when they are taken r and r together, is

$$n(n-1)\ldots\ldots(n-r+1) \quad (\beta)^*.$$

Omit the first of the letters a_1, and form all the variations taken $(r-1)$ and $(r-1)$ together, of the $(n-1)$ letters remaining: the expression for the number of them will clearly be found by putting $n-1$ in the place of n

---

[*] This is the usual mode of denoting the expression for the continued product of a series of terms, whose differences are equal: we write the first, second and last terms, merely interposing a series of *dots* (in the place of the deficient terms) between the second and the last: the two first terms give us the common difference of the successive terms, and the last determines the extent to which the series of them is carried.

The last term is determined in the following manner: in the first expression (α), the number of terms is $(r-1)$: the first term is n, the second $n-1$, the third $n-2$, the fourth $n-3$, and so on, the number subtracted from n in each term being *less by unity*, than the number which determines the position of the term in the series: it follows, therefore, that the number which is subtracted from n in the last or $(r-1)^{\text{th}}$ term, is $r-2$: and the last term is, therefore, $n-(r-2)$ or $n-r+2$. The last term in the product of r such factors, as in the second expression (β), is of course less than $n-r+2$ by *unity*, and is, therefore, $n-r+1$.

This determination of any assigned term in a series, whose successive terms have equal differences (an arithmetical series), is frequently required in investigations connected with permutations and combinations, as well as other subjects: thus the r^{th} term of such a series, whose first term is n and second $n-b$, is $n-(r-1)b$: and if the first term be n and the second $n+b$, the r^{th} term is $n+(r-1)b$.

in the formula (α), or, what is the same thing, by subtracting *unity* from each of its terms, when it becomes

$$(n-1)\ (n-2)\ldots\ldots(n-r+1):$$

before each of these variations a_1 may be placed, and there will therefore be

$$(n-1)\ (n-2)\ldots\ldots(n-r+1),$$

variations taken r and r together, in which a_1 occupies the first place: there will clearly be the same number of such variations corresponding to each of the other letters in the first place: and the whole number of variations is, therefore, n times the number corresponding to each letter in the first place, and is, consequently,

$$n\,(n-1)\ldots\ldots(n-r+1)\qquad(\beta).$$

The law of formation, therefore, if true for one class of variations, is necessarily true for the next superior class: it has been shewn to be true for the number of variations, when taken two and two (Art. 223), three and three (Art. 224), four and four together (Art. 224): it is necessarily true, therefore, for the number of these variations when taken five and five together, six and six together, and so on, proceeding step by step from one class to the next superior to it, until we arrive at any assigned number (r).

Fundamental formula.

227. The formula (β) just demonstrated, is the fundamental formula in the theory of Permutations and Combinations; and we have given its investigation with great detail, partly on account of its great importance, and partly on account of its presenting a perfectly complete and legitimate example of that species of *Demonstrative Induction* which we have noticed in a former chapter, (Art. 139.) and which we shall have frequent occasion to employ. It may be proper to add a few remarks upon the course pursued in this mode of investigation.

Observations upon this mode of investigation.

228. We commence by the direct investigation of the formula for the number of variations of the first, second, third and fourth class, in order to discover the law of

its formation for those cases, and their connection with each other: this constitutes the foundation of our first *induction*, by which we *divine* the formula or the law of the formation of the expression for the number of variations of any class whatever.

We next assume the truth of this law for any assigned class, and shew that it is necessarily true for the class next superior to it.

We lastly recommence from the cases demonstrated, and pass from them, by means of the proposition connecting the truth of the formula for one class with the one which succeeds it, to any class which may be required to be expressed by a general symbol (r): we thus exhibit the necessary connection of the extreme propositions with those intermediate to them, and consequently with each other (Art. 142).

229. We shall now proceed to examine some of the consequences of the preceding formula.

Number of permutations.

If $r=n$, or if the things be taken *all* together, then the last term or $n-r+1=n-n+1=1$, and the formula becomes

$$n\,(n-1)\ldots\ldots\ldots 2.1,$$

or writing it in the contrary order,

$$1.2\ldots\ldots.(n-1)n,$$

which is the product of the natural numbers continued as far as n: this is the expression for the number of *permutations* of n things, conformably to the meaning which we have assigned to this term: thus the number of permutations of two things $(a, b)=1.2$ or 2: the number of permutations of three things $(a, b, c)=1.2.3$ or 6: of four things $(a, b, c, d)=1.2.3.4=24$: of five things $(a, b, c, d, e)=1.2.3.4.5=120$: of six things $(a, b, c, d, e, f)=1.2.3.4.5.6=720$: and so on for greater numbers.

230. A case which frequently presents itself for consideration, is the determination of the number of permutations of n things, when any assigned number of them become identical with each other, or when different classes of them become so. Thus, suppose it was required to find an expression for the number of permutations of n things, r of which are identical with each other.

Number of permutations when different classes of letters become identical.

The expression for the number of permutations, supposing them all different from each other is

$$n\,(n-1)\ldots.2.1:$$

if r of these quantities become identical, the permutations which arise from their interchange with each other, or from their *particular* permutations, which are $1.2\ldots r$ in number, for any assigned position of the other quantities, are reduced to one: the number of permutations, therefore, when all of them are different is $1.2\ldots.r$ times as great as when r of them become identical: or in other words,

$$\frac{n(n-1)\ldots.2.1}{1.2\ \ldots\ r},$$

is the expression for the number of permutations under these circumstances.

If, in addition to r quantities which become identical, there are s others, which though different from the former, are still identical with each other, then there are $1.2\ldots s$ permutations corresponding to their interchange with each other, which are reduced to one, for any given position of the other quantities: the expression for the number of permutations under these circumstances becomes

$$\frac{n\,(n-1)\ldots2.1}{1.2\ldots r\times1.2\ldots s}.$$

In a similar manner, it may be shewn, that if, of n quantities, a_1 are of one kind, a_2 of another, a_3 of a third, and so on, as far as a_m of the m^{th} class: then the whole number of their permutations is

$$\frac{n(n-1)\ldots\ldots 2.1}{1.2\ldots a_1 \times 1.2\ldots . a_2 \times 1.2\ldots . a_3 \times \ldots . \times 1.2\ldots a_m}.$$

231. It may be useful to illustrate these formulæ by a few examples.

Examples. (1) To find the number of permutations (p) of the letters in the word *Algebra*.

In this case $n=7$, and the letter a appears twice: consequently

$$p = \frac{7.6.5.4.3.2.1}{1.2} = 2520.$$

(2) To find the number of permutations of the letters in the word *perseverance*.

In this case $n=12$, and the letter e appears four times, and r twice: therefore

$$p = \frac{12.11.10.9.8.7.6.5.4.3.2.1}{1.2.3.4 \times 1.2} = 9979200.$$

(3) To find the number of permutations of the product $a^3 b^5 c^2$ written at full length.

In this case $n=10$, and the letter a appears *three* times, b *five* times, and c *twice*; therefore the number of permutations (p)

$$= \frac{10.9.8.7.6.5.4.3.1}{1.2.3 \times 1.2.3.4.5 \times 1.2} = 2520.$$

(4) To find the number of permutations of the letters in the expressions $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, and $a^{m-r}b^r$.

$$(p) \text{ in } a^{m-1}b = \frac{m\,(m-1)\ldots . 2.1}{1.2\ldots\ldots (m-1)} = m.$$

$$(p) \text{ in } a^{m-2}b^2 = \frac{m\,(m-1)\,(m-2)\ldots . 2.1}{1.2 \times 1.2\ldots . .(m-2)} = \frac{m\,(m-1)}{1.2},$$

striking out the $(m-2)$ last factors, from the numerator and denominator, which are severally identical with each other:

$$(p) \text{ in } a^{m-3}b^3 = \frac{m\,(m-1)\,(m-2)\,(m-3)\ldots.2.1}{1.2.3 \times 1.2\ldots.(m-3)}$$

$$= \frac{m\,(m-1)\,(m-2)}{1.2.3}.$$

$$(p) \text{ in } a^{m-r}b^r = \frac{m\,(m-1)\ldots.(m-r+1)\,(m-r)\ldots2.1}{1.2\ldots..r \times 1.2\ldots(m-r)}$$

$$= \frac{m\,(m-1)\ldots.(m-r+1)}{1.2\ldots.r}.$$

striking out the $(m-r)$ last factors of the numerator and denominator.

These expressions are remarkable, inasmuch as they will be found to be severally the expressions for the number of *combinations* of n things, taken *one* and *one*, *two* and *two*, *three* and *three*, and r and r together.

Combinations: their meaning.

232. By the *combinations* of different quantities, we mean the different collections which can be made of any assigned number of them, without reference to the order of their arrangement.

Thus, *ab*, *ac* and *bc* are different combinations of the three letters *a*, *b*, *c*, taken 2 and 2 together: there is only one combination of the same three letters, taken *all* together, though they form six different permutations.

Number of combinations.

233. We shall now proceed to determine the number of combinations of n things, taken r and r together, where r is less than n.

The number of combinations of n things, taken separately or one and one together, is clearly n.

Taken two and two together.

The number of combinations of n things, taken two and two together, is $\frac{n\,(n-1)}{1.2}$.

For the number of variations of n things, taken two and two together, is $n(n-1)$; and there are two permutations (ab, ba) corresponding to one combination: the number of combinations is found, therefore, by dividing the number of variations by 2 or by 1.2.

Three and three together.

234. The number of combinations of n things, taken three and three together, is $\frac{n(n-1)(n-2)}{1.2.3}$.

For the number of variations of n things, taken three and three together, is $n(n-1)(n-2)$, and there are $1.2.3$ permutations for one combination of three things: the number of combinations is therefore found by dividing the number of variations by $1.2.3$.

Taken r and r together.

235. The number of combinations of n things, taken r and r together, is $\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}$.

For the number of variations of n things, taken r and r together, is $n(n-1)\dots(n-r+1)$; and there are $1.2\dots r$ permutations corresponding to each combination of r things: the number of combinations of n things, therefore, taken r and r together, is found by dividing the number of their corresponding variations by $1.2\dots r$.

236. There are some properties of these expressions which are attended with very important consequences, and which we shall now proceed to point out.

The number of combinations of n things taken r and r together, the same as when taken $n-r$ and $n-r$ together.

In the first place, the number of combinations of n things, taken r and r together, is the same as the number of combinations of n things, taken $n-r$ and $n-r$ together.

For the number of combinations of n things, taken r and r together, is

$$\frac{n(n-1)\dots\dots(n-r+1)}{1.2\dots\dots r} \quad (a).$$

The number of combinations of n things, taken $n-r$ and $n-r$ together, will be expressed by putting $n-r$ in the place of r in the preceding expression (α), when the last term of its numerator becomes $n-(n-r)+1$ or $r+1$, the last term of the denominator $n-r$, and the expression itself becomes

$$\frac{n(n-1)\ldots\ldots(r+1)}{1.2\quad\ldots\ldots(n-r)}\ (\beta).$$

The same expression (β), somewhat differently written, though identical in signification, is

$$\frac{n(n-1)\ldots.(n-r+1)\ \ (n-r)\ldots.(r+1)}{1.2\ \ \ldots.\ \ \ r\ \ .\ \ (r+1)\ldots.(n-r)};$$

where the last terms of the numerator and denominator of (α), and the first and last additional terms introduced into the numerator and denominator of (α) in order to obtain (β), are written down: we thus see that all those additional terms are common both to the numerator and denominator, and that consequently the first expression (α) is equal to the second (β): or, in other words, the number of combinations of n things, taken r and r together, is equal to the number of combinations of n things, taken $(n-r)$ and $(n-r)$ together.

237. The same conclusion may be otherwise and more simply obtained as follows: if we take the n quantities $a_1, a_2, \ldots. a_n$, and form any one combination of r of these quantities, those which remain will form a corresponding and *supplementary* combination of $(n-r)$ quantities: it follows, therefore, that no combination can exist without its *supplement*, and that consequently they must be equal in number.

Second proof.

Supplementary combinations.

238. If we further examine the expression (α) for the number of combinations and its successive values, it will obviously continue to increase, until the additional term introduced into the numerator becomes equal to or less than

Greatest number of combinations when n is odd.

the corresponding additional term in the denominator: now the r^{th} term of the numerator is $n-r+1$, and the corresponding term in the denominator is r: if these be equal to each other, we have $n-r+1=r$, and consequently $r=\frac{n+1}{2}$: and since r is necessarily a whole number, n must, in this case, be necessarily an odd number: there is, therefore, in this case, the same number, and also the greatest number, of combinations, when they are taken $\frac{n-1}{2}$ together and $\frac{n+1}{2}$ together, such combinations being *supplementary* to each other.

When n is even.

If n be an even number, the greatest number of combinations takes place when they are taken $\frac{n}{2}$ together: for in this case, $n-r+1$ becomes $\frac{n}{2}+1$, which is greater than the corresponding value of r or $\frac{n}{2}$; whilst the next succeeding value of $n-r+1$, which is $\frac{n}{2}$, is less than the corresponding value of r, or of the last term of the denominator, which is $\frac{n}{2}+1$.

Law of formation of the product of binomial factors.

239. We shall now proceed to consider the formation of the product of n binomial factors, x_1+a_1, x_2+a_2, x_3+a_3, ... x_n+a_n, in connection with the theory of combinations, from which the most important consequences will be found to follow.

In the first place, the terms in this product are homogeneous, and each of them involves n of the quantities x_1, a_1, x_2, a_2, &c.

For there is an additional dimension given to every term already existing or formed, by every additional factor: and as the terms in the first and every other factor are of one dimension, the terms in the product of two factors are, therefore, of two dimensions, in the product of three factors, of three dimensions, and so on, increasing by unity for every additional factor: those in the product of n factors are, therefore, of n dimensions, and homogeneous.

Terms are homogeneous,

In the second place, the combinations of letters in each of the terms are different from each other.

For the same letter occurring but once in the same or different factors, can only be once employed as a multiplier: if the terms already existing or formed, be, therefore, different from each other, those formed by the introduction of new and successive factors must continue different from each other.

and all different from each other.

In the third place, the letters which appear in the same factor, can never appear together in the same term.

For those letters are employed only once as multipliers; and when so employed, they form terms which are all of them different from each other.

In the fourth place, the terms in the product may be arranged in classes, where the combinations of x's and a's, or of the first and last letters of the several factors, are *complementary* to each other.

Complementary combinations.

For in whatever term, any combination of r of the first letters of r factors appear, in the same must be found the combination of $(n-r)$ second letters of the $(n-r)$ other factors: such combinations may be termed *complementary* to each other, to distinguish them from *supplementary* combinations, which involve the same series of letters only.

The first term of the product is the continued product of $x_1, x_2, \ldots x_n$, or of the first terms of the several binomial factors.

The first class of *complementary* combinations comprehends $(n-1)$ x's and one a, the number of which is n.

The second class of *complementary* combinations comprehends $(n-2)$ x's and two a's: the number of them is, therefore, $\dfrac{n(n-1)}{1.2}$.

The fourth class of *complementary* combinations comprehends $(n-3)$ x's and three a's: the number of them is, therefore, $\dfrac{n(n-1)(n-2)}{1.2\ .\ 3}$.

The r^{th} class of *complementary* combinations comprehend $(n-r)$ x's and r a's: the number of them is, therefore,

$$\frac{n(n-1)\ldots.(n-r+1)}{1.2\ \ldots.\ \ r}.$$

The last term is the continued product of the a's, or of $a_1, a_2, \ldots a_n$.

Number of terms.

240. The whole number of terms, or the whole number of their combinations, is 2^n: for the number of terms is doubled by the introduction of every successive factor in the product.

241. Let us now examine some of the consequences of this proposition.

Product of n binomial factors having the same first term.

Let the first terms of the several factors become identical with each other and with x, whilst the second continue different from each other: or in other words, let it be required to find the product of the n factors, $x+a_1$, $x+a_2, \ldots x+a_n$.

In this case, the several combinations of $x_1, x_2, \ldots x_n$, degenerate into powers of x, whose indices are the numbers of letters in each combination.

The first term is, therefore, x^n.

The second term, or the first class of complementary combinations, becomes $A_1 x^{n-1}$ where $A_1 = a_1 + a_2 + \ldots . a_n$, or the sum of the second terms of the binomial factors.

The third term, or the second class of complementary combinations, becomes $A_2 x^{n-2}$ where $A_2 = a_1 a_2 + a_1 a_3 + \&c.$ or the sum of the combinations, two and two, of the second terms of the binomial factors.

The fourth term, or the third class of complementary combinations, becomes $A_3 x^{n-3}$ where $A_3 = a_1 a_2 a_3 + a_1 a_3 a_4 + \&c.$ or the sum of the combinations, three and three, of the second terms of the binomial factors.

The $(1+r)^{\text{th}}$ term, or the r^{th} class of complementary combinations, becomes $A_r x^{n-r}$ where A_r is the sum of the combinations of the second terms of the binomial factors, taken r and r together.

242. This proposition is the fundamental proposition in the Theory of Equations, when their composition is once granted or demonstrated: the following modification of it is the foundation of the *binomial* theorem, which will be more particularly considered in the next chapter.

Binomial theorem.

243. Let us suppose the second terms also of the several binomial factors to become identical with each other and with a. The continued product $(x+a_1)(x+a_2)\ldots(x+a_n)$ becomes, therefore, $(x+a)^n$, where n is the number of factors which are equal to each other and to $x+a$.

The first term is x^n, as before.

The coefficient of the second term, or $A_1 = na$, since all its terms, which are n in number, become equal to each other and to a: the second term is, therefore, nax^{n-1}.

The coefficient of the third term, or $A_2 = \dfrac{n(n-1)}{1.2} a^2$, since all its terms, which are $\dfrac{n(n-1)}{1.2}$ in number, become equal to each other and to a^2: the third term is, therefore, $\frac{n(n-1)}{1.2}a^2x^{n-2}$.

The coefficient of the fourth term, or

$$A_3 = \frac{n(n-1)(n-2)}{1.2.3}a^3,$$

since all its terms, which are $\frac{n(n-1)(n-2)}{1.2\ .\ 3}$ in number, become equal to each other and to a^3: the fourth term is, therefore, $\frac{n(n-1)(n-2)}{1.2\ .\ 3}a^3x^{n-3}$.

The coefficient of the $(1+r)^{\text{th}}$ term, or

$$A_r = \frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ \ldots.\ \ r}a^r,$$

since all its terms, which are $\frac{n(n-1)\ldots(n-r+1)}{1.2\ \ldots\ r}$ in number, are equal to each other and to a^r: the $(1+r)^{\text{th}}$ term is, therefore, $\frac{n(n-1)\ldots(n-r+1)}{1.2\ \ldots\ r}a^rx^{n-r}$.

It follows, therefore, that

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1.2}a^2x^{n-2}$$
$$+ \frac{n(n-1)(n-2)}{1.2\ .\ 3}a^3x^{n-3} + \ldots\ldots \frac{n(n-1)\ldots(n-r+1)}{1.2\ \ldots\ r}$$
$$a^rx^{n-r} + \&c.$$

Complementary combinations of different orders.

244. Before we proceed to the investigation of the composition of the product of any number of polynomial factors, having each of them the same number of terms, it will be proper to complete the theory of *complementary combinations*, when more than two classes of quantities enter into each combination.

In the case which we have already considered, where there are n things in each combination, of which a_1 are of one kind and $n-a_1$ or a_2 of the other, the expression for the number of *complementary* combinations is

Of the second order.

$$\frac{n(n-1)\dots(n-a_1+1)}{1.2 \quad \dots \quad a_1}$$
$$=\frac{n(n-1)\dots\dots(n-a_1+1)(n-a_1)\dots\dots 2.1}{1.2 \quad \dots \quad a_1\times 1.2 \dots\dots (n-a_1)}$$
$$=\frac{n(n-1) \dots\dots\dots 2.1}{1.2\dots a_1\times 1.2\dots a_2}:$$

multiplying the numerator and denominator by the same quantity $(n-a_1)(n-a_1-1)\dots 2.1$ in the first place, and replacing $n-a_1$ by a_2, which is equal to it, in the second.

245. Let us now consider the case where there are three different classes of quantities existing in each combination, and let there be a_1 of one kind, a_2 of a second, and a_3 of a third.

Of the third order.

If we omit the a_1 quantities belonging to the first class, there will remain $n-a_1$ or a_1+a_2 quantities, the number of whose *complementary* combinations is, by the last case,

$$\frac{(n-a_1)(n-a_1-1)\dots\dots 2.1}{1.2 \quad \dots \quad a_2\times 1.2\dots\dots a_3} \qquad (\alpha).$$

The number of combinations of the first class is

$$\frac{n(n-1)\dots(n-a_1+1)}{1.2 \quad \dots\dots \quad a_1} \qquad (\beta):$$

each of which may be combined with all the complementary combinations of the $(n-a_1)$ quantities taken a_2 from one class, and a_3 from the other: the whole number, therefore, of *complementary* combinations of the *third* order, will clearly be found by multiplying the expression (β) into (α), when we get

$$\frac{n(n-1)\dots\dots 2.1}{1.2\dots a_1\times 1.2\dots a_2\times 1.2\dots a_3} \qquad (\gamma).$$

Of the fourth order.

246. Let us now consider the case where there are *four* different classes of quantities in each combination, namely, a_1 of the first kind, a_2 of the second, a_3 of the third, and a_4 of the fourth.

If we omit the a_1 quantities belonging to the first class, there will remain $n-a_1$ or $a_2+a_3+a_4$ quantities of the remaining classes, whose complementary combinations by the last case are

$$\frac{(n-a_1)\,(n-a_1-1)\dots\quad 2.1}{1.2\ .\ a_2\times 1.2\ \dots\ a_3\times 1.2\dots a_4}\qquad (\alpha').$$

If the number (α') be multiplied into the number of combinations of the first class, or

$$\frac{n(n-1)\dots(n-a_1+1)}{1.2\ \dots.\ \ a_1}\qquad (\beta'),$$

we shall get the expression for the number of complementary combinations of the fourth order, which is

$$\frac{n(n-1)\ \dots.\ 2.1}{1.2\ \dots\ a_1\times 1.2\dots a_2\times 1.2\dots a_3\times 1.2\dots a_4}\ (\gamma').$$

Of the n^{th} order.

247. The cases already considered are sufficient to ascertain the law of formation of these expressions, which may be generalized by shewing, that if it be true for the number of the complementary combinations of the $(m-1)^{\text{th}}$ order, it must be true, likewise, for those of the m^{th} order.

For let us suppose the number of these combinations of n things, arranged in $(m-1)$ different classes, the number of quantities in which are severally denoted by a_1, a_2, $\dots a_{m-1}$, to be

$$\frac{n(n-1)\dots 2.1}{1.2\ ..a_1\times 1.2\dots a_2\times\ \dots.\ \times 1.2\dots a_{m-1}}:$$

and let it be required to find the number of complementary combinations of n things, arranged in m classes, the first involving a_1 quantities, the second a_2, and the last a_m: the number of complementary combinations of

$(n-a_1)$ things, arranged in classes of a_2, $a_3 \ldots a_n$ each, is, by the assumed case,

$$\frac{(n-a_1)\,(n-a_1-1)\ldots.2.1}{1.2\ldots a_2 \times 1.2\ldots a_3 \times \ldots\ldots \times 1.2\ldots a_m} \qquad (\alpha''),$$

which may be severally combined with the combinations of the first class, in which a_1 quantities out of n are taken together, the number of which is

$$\frac{n(n-1)\ldots.(n-a_1+1)}{1.2 \qquad\qquad a_1} \qquad (\beta''),$$

and thus form a number of complementary combinations of the n^{th} order, which is expressed by

$$\frac{n(n-1)\ldots. \qquad 2.1}{1.2\ldots a_1 \times 1.2\ldots a_2 \times \ldots. \times 1.2\ldots a_m} \qquad (\gamma'').$$

Polynomial products.

248. The preceding theory of complementary combinations will enable us to investigate, generally, the composition of the products of polynomial factors: we will commence with the product of n factors, $(a_1+b_1+c_1+\ldots l_1)$, $(a_2+b_2+c_2+\ldots.l_2)$, $\ldots.(a_m+b_m+c_m+\ldots.l_m)$, each consisting of m terms, all of which are different from each other.

The product is homogeneous, of n dimensions, and all its terms different from each other, for the same reasons as were given in Art. 239.

There is a term involving a's only (in number n), another involving b's only, and similarly for all the other letters.

The other terms form complementary combinations of all orders not exceeding the m^{th}: and the number of terms corresponding to any assigned order, will be determined by the formula given in the last Article.

Thus, the number of terms in which an a appears α_1 times, a b appears α_2 times, a c appears α_3 times ... and an l appears α_m times, where

$$\alpha_1 + \alpha_2 + \alpha_3 + \dots \alpha_m = n,$$

is expressed by the formula

$$\frac{n(n-1)\dots \qquad 2.1}{1.2\dots\alpha_1 \times 1.2\dots\alpha_2 \times 1.2\dots\alpha_3 \times \dots \times 1.2\dots\alpha_m}.$$

Number of terms $= m^n$.

249. The whole number of terms in the product is m^n: for there are m terms in the first and all other factors, a number of terms equal to m^2 in the product of two factors, a number equal to m^3 in the product of three factors, and so on, until we get to the product of n such factors, in which the number of terms is m^n.

250. If we now suppose all the letters of the same class in the several factors, and therefore the factors themselves to become identical with each other, the product of the n equal factors (omitting the numbers attached to each letter) will become

$$(a + b + c + \dots + l)^n:$$

and the combinations of the letters of the same class, whether of a's or b's or c's or ... l's, will become powers of those letters, whose indices are the numbers of them which appear in each term: all those terms, therefore, will become identical, which involve severally the same number of letters belonging to the same class: if the number, therefore, of one kind (a) be α_1, of the second (b) be α_2, of the third (c) be α_3, and of the m^{th} (l) be α_m, where

$$\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_m = n,$$

then the term which comprehends the whole number of corresponding complementary combinations, which may be called likewise the *general term* of the product, is

$$\frac{n(n-1)\dots \qquad 2.1}{1.2\dots\alpha_1 \times 1.2\dots\alpha_2 \times 1.2\dots\alpha_3 \times \dots \times 1.2\dots\alpha_m} a^{\alpha_1} b^{\alpha_2} c^{\alpha_3} \dots l^{\alpha_m}.$$

Homogeneous products.

251. There are other classes of combinations, the theory of whose formation, with the determination of their number, will lead, like those we have already considered, to very important applications. Of this kind are those combinations, which may be termed *homogeneous products*, which admit of repetitions of the same letter: thus the quantities a, b, c, taken all together, form only *one combination*, properly so called, which is abc: but the same three letters form *ten homogeneous products*, which are the following:

$$a^3+b^3+c^3+a^2b+a^2c+ab^2+ac^2+b^2c+bc^2+abc.$$

Number of homogeneous products

252. It remains to investigate, generally, the formula for the determination of the numbers of such products.

Let the quantities, which, as well as their powers, are to be combined together, be

$$a_1,\ a_2,\ a_3,\ \ldots.\ a_n:$$

of two dimensions,

and let it be required to determine the number of their *homogeneous products*, when taken two and two, three and three, ... and r and r together; or which are of two, three, ... and r dimensions.

The homogeneous products of two dimensions are the squares of all the letters, together with their products or combinations, two and two: and as the number of the first is n, and of the second $\dfrac{n(n-1)}{1.2}$, the whole number required is, therefore,

$$n+\frac{n(n-1)}{1.2}=\frac{n(2+n-1)}{1.2}=\frac{n(n+1)}{1.2}.$$

of three dimensions.

253. In order to determine the number of homogeneous products of three dimensions, let us suppose the several homogeneous products of two dimensions multiplied successively by all the (n) letters, together with the sum of the particular letters which form the factors of each product: the number of these is, in each case, $n+2$, and it may be shewn, that

every homogeneous product of three dimensions will occur three times in the result and no more: for in the product

$$a_1^2(a_1+a_2+\dots+a_n+2a_1)=a_1^2(3a_1+a_2+\dots a_n),$$

the term $3a_1^3$ appears, and in that product only: the term $a_1^2a_2$ occurs once in this product, and twice in the product

$$a_1a_2(a_1+a_2+\dots a_n+a_1+a_2)$$
$$=a_1a_2(2a_1+2a_2+a_3+\dots a_n),$$

and in no other product. Again, the term $a_1a_2a_3$ occurs once in this product, once in

$$a_1a_3(a_1+a_2+a_3+\,.\,.\,a_n+a_1+a_3),$$

and once more in

$$a_2a_3(a_1+a_2+a_3+\,.\,.\,a_n+a_2+a_3),$$

and in no others: and whatever is proved to be true of a_1^3, a_1^2a, and $a_1a_2a_3$, must be equally true of all similar powers or products, inasmuch as the result must be symmetrical with respect to all the letters (Art. 136): it follows, therefore, that the whole number of homogeneous products of three dimensions will be found by multiplying the number of homogeneous products of two dimensions by $n+2$, and dividing the result by 3: and consequently the number required is

$$\frac{n(n+1)(n+2)}{1.2.3}.$$

Law of formation

254. The factors of the numerator of this expression, are the natural numbers *increasing* from n; and those corresponding to them in the denominator are the natural numbers increasing from 1; the number of them being the same in both cases, and equal to the dimension of each product: before we can legitimately generalize the law of the formation of this expression, we must shew, that if it be true for the number of homogeneous products of $(r-1)$ dimensions, it must be true, likewise, for those of the next superior or r^{th} dimension.

255. For this purpose, let us suppose every homogeneous product of $(r-1)$ dimensions successively multiplied by all the (n) letters, together with the sum of the particular letters, whether the same or different, which that product contains: it will be found that every homogeneous product required (of r dimensions), will occur r times in the result, and no more: thus, in the product

For n dimensions.

$$a_1^{r-1}\{a_1+a_2+\ldots a_n+(r-1)a_1\} \text{ or}$$
$$a_1^{r-1}(ra_1+a_2+\ldots a_n) \qquad (a),$$

we find ra_1^r, which occurs in no other: the term $a_1^{r-1}a_2$, which occurs *once* in this product, occurs $(r-1)$ times in

$$a_1^{r-2}a_2\{a_1+a_2+\ldots a_n+(r-2)a_1+a_2\} \text{ or}$$
$$a_1^{r-2}a_2\{(r-1)a_1+2a_2+a_3+\ldots a_n\} \qquad (\beta),$$

and in no others: the term $a_1^{r-2}a_2^2$, which occurs twice in this product, occurs also $(r-2)$ times in

$$a_1^{r-3}a_2^2\{a_1+a_2+\ldots a_n+(r-3)a_1+2a_2\} \qquad (\gamma),$$

and in no others: whilst the term $a_1^{r-2}a_2a_3$, which occurs *once* in (β), *once* in

$$a_1^{r-2}a_3\{a_1+a_2+\ldots a_n+(r-2)a_1+a_3\},$$

will be found $(r-2)$ times in

$$a_1^{r-3}a_2a_3\{a_1+a_2+a_3+\ldots a_n+(r-3)a_1+a_2+a_3\} \text{ or}$$
$$a_1^{r-3}a_2a_3\{(r-2)a_1+2a_2+2a_3+\ldots a_n\} \qquad (\delta),$$

and in no others: it follows, therefore, that a_1^r, $a_1^{r-1}a_2$, $a_1^{r-2}a_2^2$, $a_1^{r-2}a_2a_3$, and consequently all similar products, must occur r times in the result, and no more: and generally, the term

$$a_1^{\alpha_1}a_2^{\alpha_2}\ldots\ldots a_n^{\alpha_n},$$
$$\text{where } \alpha_1+\alpha_2+\ldots\ldots\alpha_n=r,$$

which appears α_1 times in the product

$$a_1^{\alpha_1-1}a_2^{\alpha_2}\ldots\ldots a_n^{\alpha_n}\{a_1+a_2+\ldots a_n+(\alpha_1-1)a_1+\alpha_2a_2+\ldots\alpha_na_n\}$$
$$\text{or } a_1^{\alpha_1-1}a_2^{\alpha_2}\ldots\ldots a_n^{\alpha_n}\{\alpha_1a_1+(\alpha_2+1)a_2+\ldots\ldots(\alpha_n+1)a_n\},$$

will appear α_2 times in the product

$$a_1^{\alpha_1} a_2^{\alpha_2 - 1} \ldots\ldots a_n^{\alpha_n} \{(\alpha_1 + 1) a_1 + \alpha_2 a_2 + \ldots (\alpha_n + 1) a_n\}$$

. . . .

and likewise α_n times in the product

$$a_1^{\alpha_1} a_2^{\alpha_2} \ldots\ldots a_n^{\alpha_n - 1} \{(\alpha_1 + 1) a_1 + (\alpha_2 + 1) a_2 + \ldots \alpha_n a_n\},$$

and in no others: it appears, therefore, $(\alpha_1 + \alpha_2 + \ldots \alpha_n)$ or r times in the result, and no more: the same reasoning which was applied to this *general* term, is equally applicable to all others which can be formed, and which appear, therefore, severally r times in the result, and no more.

It follows, therefore, that the number of homogeneous products of n quantities of r dimensions, will be found by multiplying by $n + r - 1$ and dividing by r, the number of homogeneous products of n quantities of $(r-1)$ dimensions.

If the number of homogeneous products of n letters of $(r-1)$ dimensions be assumed to be

$$\frac{n(n+1) \ldots\ldots (n+r-2)}{1.2 \quad \ldots\ldots \quad (r-1)},$$

the number of homogeneous products of n letters of r dimensions must be

$$\frac{n(n+1) \ldots\ldots (n+r-2)\,(n+r-1)}{1.2 \quad \ldots\ldots \quad (r-1)\,r}:$$

and as this expression has been shewn to be true, when $r=2$ and $r=3$, it must be true, therefore, when $r=4$, 5, &c., as far as any number whatever.* (Art. 226).

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* The general expression for the number of *combinations* of n things taken r and r together, may be found by a method analogous to the preceding, without the previous knowledge of the expressions for the number of *variations* and *permutations*: for if every combination of $(r-1)$ letters, be multiplied by all the other letters which do *not* appear in it, which are $n-r+1$ in number, it may very easily be shewn that every combination of r letters will appear r times in the result: if C_{r-1} be the number of combinations of n things taken $(r-1)$ and $(r-1)$ together, and C_r their number when taken r and r together, it would follow, therefore, that $C_r = C_{r-1} \times \frac{n-r+1}{r}$, from which the complete expression for C_r may be easily obtained.

Series in which homogeneous products of successive dimensions appear.

256. Homogeneous products of one, two, three and all superior dimensions, such as we have just been considering, present themselves in the successive terms of the series which represents the result of the division of 1 by

$$(1-a_1)(1-a_2)(1-a_3)\dots\dots(1-a_n);$$

or, what is the same thing, as the coefficients of the successive powers of x in the series which arises from dividing 1 by $(1-a_1x)(1-a_2x)(1-a_3x)\dots(1-a_nx)$: for (Art. 46. Ex. 22.)

$$\frac{1}{1-a_1x}=1+a_1x+a_2^2x^2+a_1^3x^3+\dots\dots a_1^rx^r+\dots$$

$$\frac{1}{1-a_2x}=1+a_2x+a_2^2x^2+a_2^3x^3+\dots\dots a_2^rx^r+\dots$$

$$\frac{1}{1-a_3x}=1+a_3x+a_3^2x^2+a_3^3x^3+\dots\dots a_3^rx^r+\dots$$

$$\dots\qquad\dots\dots\dots\dots\dots$$

$$\frac{1}{1-a_nx}=1+a_nx+a_n^2x^2+a_n^3x^3+\dots\dots a_n^rx^r+\dots:$$

and if we multiply these several series together, and arrange the results according to powers of x, we shall find

$$\frac{1}{(1-a_1x)(1-a_2x)(1-a_3x)\dots(1-a_nx)}$$

$$=1+H_1x+H_2x^2+H_3x^3+\dots H_rx^r+\dots$$

where H_1 is the sum of the n letters, a_1, a_2, .. a_n: H_2 is the sum of their *homogeneous products* of two dimensions, H_3 the sum of their *homogeneous products* of three dimensions, ... H_r the sum of their *homogeneous products* of r dimensions, and so on for ever.

Series for $\frac{1}{(1-ax)^n}$.

257. If we now suppose all the n letters a_1, a_2, .. a_n to become identical with each other and with a, the fraction

$$\frac{1}{(1-a_1x)(1-a_2x)(1-a_3x)\dots(1-a_nx)}$$

will become

$$\frac{1}{(1-ax)^n},$$

and the successive coefficients H_1, H_2, H_3, .. H_r... will become

$$na,\ \frac{n(n+1)}{1.2}a^2,\ \frac{n(n+1)\,(n+2)}{1.2.3}a^3,\ldots\frac{n(n+1)..(n+r-1)}{1.2\ .\ r}a^r,$$

and consequently

$$\frac{1}{(1-ax)^n} = 1 + nax + \frac{n(n+1)}{1.2}a^2x^2 + \frac{n(n+1)\,(n+2)}{1.2\ .\ 3}a^3x^3$$

$$+\,..\frac{n(n+1)\,..\,(n+r+1)}{1.2\ \ldots\ r}a^rx^r + \text{\&c.}$$

a result which we shall have occasion to consider more particularly in the next chapter.

Further analysis of the formula for the number of homogeneous products.

258. The formula

$$\frac{n(n+1)\ldots(n+r-1)}{1.2\ \ldots.\ r}$$

expresses the different modifications of the product

$$a_1^{\alpha_1}a_2^{\alpha_2}\ \ldots.\ a_n^{\alpha_n},$$

where $\alpha_1 + \alpha_2 + \ldots.\ \alpha_n = r$, and where α_1, α_2, $\alpha_3 \ldots \alpha_n$ admit of all different values from 0 to r: we may in some respects analyse their formation, by separating a_1 from the remaining $(n-1)$ letters, and considering them as arising from the successive multiplication of the homogeneous products of these $(n-1)$ letters, which are of r, $r-1$, $r-2$,...3, 2, 1, and 0 dimensions, into 1, a_1, a^2, ... a_1^{r-2}, a_1^{r-1} and a_1^r; the successive numbers of these terms, determined by putting $n-1$ in the place of n in all cases, and r, $r-1$, $r-2$,...3, 2 and 1 in the place of r, are

$$\frac{(n-1)\ n(n+1)\ldots.(n-r+2)}{1\ .\ 2\ \ldots..\ r}$$

$$\frac{(n-1)\,n(n+1)\ldots(n+r-3)}{1\,.\,2\quad\ldots\ldots\quad(r-1)}$$

$$\frac{(n-1)\,n(n+1)\ldots(n+r-4)}{1\,.\,2\quad\ldots\ldots\quad(r-2)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{(n-1)\,n(n+1)}{1\,.\,2\,.\,3}$$

$$\frac{(n-1)\,n}{1\,.\,2}$$

$$\frac{(n-1)}{1}.$$

It follows, therefore, that

$$\frac{n(n+1)\ldots\ldots(n+r-1)}{1.2\quad\ldots\ldots\quad r}=1+(n-1)+\frac{(n-1)\,n}{1.2}$$
$$+\frac{(n-1)n(n+1)}{1\,.\,2\,.\,3}+\ldots\ldots\frac{(n-1)n(n+1)\ldots(n+r-2)}{1\,.\,2\quad\ldots\ldots\quad r}.$$

In a similar manner we should find, by putting $n-1$ in the place of n, that the $(r+1)^{\text{th}}$ term of this series, or

$$\frac{(n-1)n(n+1)\ldots(n+r-2)}{1\,.\,2\quad\ldots\ldots\quad r}=1+(n-2)+\frac{(n-2)(n-1)}{1\,.\,2}$$
$$+\frac{(n-2)(n-1)n}{1\,.\,2\,.\,3}+\ldots\ldots\frac{(n-2)\,(n-1)\ldots(n+r-3)}{1\,.\,2\quad\ldots\quad r}.$$

By successive substitutions of $n-2$, $n-3$, $n-4$, ...3, 2, in the place of n in the original formula and the terms to which it is equal, we should form successive series of different orders, where the $(r+1)^{\text{th}}$ term of a series of any assigned order would be equal to the sum of $(r+1)$ terms of the series of the order next below it: the last of these series but one, corresponding to $n=3$, (the second when reckoned from the fundamental series), or

$$\frac{3.4\ldots(r+2)}{1.2\ \ldots\ r}=1+2+\frac{2.3}{1.2}+\frac{2.3.4}{1.2.3}+\ldots\ldots\frac{2.3..(r+1)}{1.2\ ..\ r},$$

becomes

$$\frac{(r+1)\,(r+2)}{1\ .\ 2}=1+2+3+4+\ \ldots.\,(r+1),$$

or is the sum of $(r+1)$ terms of the series of natural numbers: whilst the next succeeding or *fundamental* series corresponding to $n=2$, gives us

$$\frac{2.3.4\ldots(r+1)}{1.2\ \ldots.\ r}=1+1+\frac{1.2}{1.2}+\frac{1.2.3}{1.2.3}+\ldots\frac{1.2\ldots r}{1.2\ldots r},$$

or $$(r+1)=1+1+1+1+\ \ldots..\ 1,$$

which is a series of $(r+1)$ units.

Figurate numbers.

259. The numbers which form these successive series are called *figurate* numbers of the 1st, 2d, 3d,....nth order, where the rth term of any one order is equal to the sum of r terms of the order preceding: the following is a table of the eight first terms of the eight first orders.

Fundamental Series and

1st Order.	1,	1,	1,	1,	1,	1,	1,	1.
2d	1,	2,	3,	4,	5,	6,	7,	8.
3d	1,	3,	6,	10,	15,	21,	28,	36.
4th	1,	4,	10,	20,	35,	56,	84,	120.
5th	1,	5,	15,	35,	70,	126,	210,	330.
6th	1,	6,	21,	56,	126,	252,	462,	792.
7th	1,	7,	28,	84,	210,	462,	924,	1716.
8th	1,	8,	36,	120,	330,	792,	1716,	3432.

260. The knowledge of these numbers, or of the law of their formation, will enable us to answer some questions, which we have been previously obliged to pass over without notice: thus, let it be required to determine the number of terms in the product or series corresponding to

Number of terms in the series corresponding to $(a+b+c+..l)^n$.

$$(a+b+c+ \dots l)^n,$$

$$\text{or} \qquad (a_1+a_2+a_3+ \dots a_r)^n,$$

the general term of which we have already determined (Art. 250).

The number of terms depends upon the number of variations of form of the literal product

$$a_1^{\alpha_1} a_2^{\alpha_2} \dots\dots a_r^{\alpha_r},$$

where $\alpha_1 + \alpha_2 + \dots \alpha_r = n$, and where $\alpha_1, \alpha_2, \dots \alpha_n$ admit of all values from 0 to n: it follows, therefore, that the number required is identical with the number of *homogeneous* products of r letters of n dimensions, and is, therefore, found by means of the formula in Art. 255, by merely changing n into r and r into n: in other words, it is equal to

$$\frac{r(r+1) \dots (r+n-1)}{1.2 \quad \dots \quad n} \quad (a).$$

If $r=2$, this expression becomes

For a binomial.

$$\frac{2.3 \dots (n+1)}{1.2 \quad \dots \quad n} = (n+1),$$

which is the number of terms in the series corresponding to

$$(a+b)^n.$$

If $r=3$, the formula becomes

For a trinomial.

$$\frac{3.4 \dots (n+2)}{1.2 \quad \dots \quad n} = \frac{(n+1)\,(n+2)}{1 \quad . \quad 2},$$

which is the number of terms in the series corresponding to

$$(a+b+c)^n.$$

For a tetranomial.

If $r=4$, the formula becomes

$$\frac{4.5 \ldots\ldots (n+3)}{1.2 \ \ldots \ n} = \frac{(n+1)\,(n+2)\,(n+3)}{1\ .\ 2\ .\ 3},$$

which is the number of terms corresponding to the series for

$$(a+b+c+d)^n.$$

Generally.

Generally, if r is any number less than n, the general formula (α) is equivalent to

$$\frac{(n+1)\,(n+2)\,\ldots\,(n+r-1)}{1\ .\ 2\ \ldots\ (r-1)} \quad (\beta),$$

inasmuch as the product $r\,(r+1) \ldots\ldots (n-1)\,n$, is, in this case, common to its numerator and denominator.

Resolution of a number into parts.

261. The formula (β) above given, determines the number of ways in which the number n may be broken into two, three, four or r parts, between 0 and n inclusive, which are either different in themselves or in their arrangement: thus, let it be proposed to find in how many different ways the number 7 may be broken into four parts, zero included, which are either different in the parts themselves or in their arrangement.

The formula gives $\frac{8.9.10}{1.2.3} = 120$, and these arise from the different permutations of

7, 0, 0, 0, which are 4 in number.
4, 1, 1, 1, ... 4 ....
2, 2, 2, 1, ... 4 ....
6, 1, 0, 0, ... 12 ....
5, 2, 0, 0, ... 12 ....
4, 3, 0, 0, ... 12 ....
5, 1, 1, 0, ... 12 ....
3, 3, 1, 0, ... 12 ....
3, 2, 2, 0, ... 12 ....
3, 2, 1, 1, ... 12 ....
4, 2, 1, 0, ... 24 ....

These numbers, including zero, in the order of their successive permutations, express the indices of a, b, c and d, in the several terms of the series or form which is equivalent to $(a+b+c+d)^7$.

Combinatorial analysis.

262. The theorems and problems which we have given in the preceding articles of this chapter, are amongst the most important of those which present themselves in the Theory of Permutations and Combinations, or in the Combinatorial Analysis, as it is sometimes called: this subject, indeed, when considered in its most general form, is much too extensive to be comprehended within the proper limits of a work of this kind; and we have felt ourselves compelled, therefore, to confine ourselves in the selection which we have made, to fundamental propositions, and such others as are necessary in the investigation and establishment of the binomial and multinomial theorems, or useful in facilitating the developements effected by means of them.

The foundation of the theory of chances.

263. There are few subjects which admit of more varied illustrations by means of problems, than the Theory of Combinations; inasmuch as such problems resolve themselves almost immediately into problems for the calculation of chances or probabilities, and consequently spring from them: it will be expedient, therefore, for us to subjoin such an explanation of the mathematical meaning of the terms which are most commonly used in the Doctrine of Chances, as well as of the most simple first principles of the science, as may be requisite to shew its connection with the Theory of Combinations, at least in one of its departments, and may enable the student to apply most of the propositions in this chapter to a very extensive class of questions, which are full of instruction and interest.

Popular meanings of the term chance.

264. The term *chance*, in popular language, has various meanings attached to it, whether primitive or derived, though it is not always very easy, nor in this instance

very important, to distinguish one from the other: it sometimes means an *event*, whose occurrence is uncertain, whether under the influence or not, of determined or determinable laws: on other occasions, it is used to express the *cause* which influences the happening of an event: and sometimes it is used as the expression of our opinion of the intensity of the cause which determines or influences the event, whether it be according to expectation or the contrary.

Its mathematical meaning.

265. It is in this latter sense that it approaches most nearly to its mathematical meaning, where it is used as synonymous with *probability;* and the chance of the happening, or the probability of the happening of an event, or its contrary, is measured, and *therefore defined*, by the ratio which exists between the number of events which *must* happen, or of cases which *must* exist, and the whole number of events which both *must* and *may* happen, or of cases which both *must* and *may* exist, and which are all of them similarly circumstanced.

Mode of expressing it.

266. This ratio may be expressed by means of a fraction, whose numerator is the number of favourable events or cases, and whose denominator is the number of *all* the events or cases, whether favourable or unfavourable: for all ratios, as will be seen in a subsequent chapter, are expressed and measured by means of fractions, whose numerators are the antecedents and whose denominators are the consequents of the ratios.

Thus, if a expresses the number of favourable events or cases, and b the number of those which are unfavourable, the *chance* of the event's happening or of the case existing, is

$$\frac{a}{a+b};$$

whilst the *chance* of its failure or of its not existing, is expressed by

$$\frac{b}{a+b}.$$

267. From such a mode of representation, it will follow that *certainty*, which supposes all the events or cases favourable, in the first case, when $b=0$, or all of them unfavourable, in the second case, when $a=0$, will be expressed by 1: the ratio, therefore, of the *chance* to *certainty*, or of the degree of probability (as it is sometimes expressed) to certainty, will be the ratio of the fraction by which it is denoted to unity, or the ratio of its numerator to its denominator.

Certainty represented by unity.

268. The ratio of the chance of success to that of failure, or the ratio of the *odds for or against*, as expressed in popular language, will be that of a to b, or of b to a, which are the numerators of the fractions by which the respective chances are denoted.

Odds.

269. Chances may be separated into two great classes, as *absolute* and *moral*: the first are those, where the numerator and denominator of the fractions by which they are expressed, admit of absolute determination: the second are those, where the numerator and denominator, one or both of them, admit not of absolute determination, but are inferred from experiment or observation: the *certainties* also, in which these different classes of *chances* may be said to terminate, as the limit of their different values, may be distinguished from each other, in a similar manner, as *absolute* and *moral*.

Absolute and moral chances.

Absolute and moral certainties.

The chances considered in the following problems, belong chiefly to the first of these classes, and their determination requires no principle which is foreign to the theory of combinations: in most cases, however, moral chances are convertible, as far as their estimation is concerned, into absolute chances; and a few examples will be given, in order to shew the nature of the reasoning which is employed for this purpose.

270. To find the chance of throwing an ace with a single die.

Problems.

Simple chances: of throwing an ace with one die,

There is only one face, which *can* be uppermost, though there are six which are equally likely to be so:* the chance, therefore, that this face is the *ace*, is $\frac{1}{6}$.

of not throwing an ace,

(α) The chance that this face is not the *ace*, is $\frac{5}{6}$: for there are five out of six equally possible cases, which are favourable to this hypothesis.

of throwing an ace or deuce,

(β) The chance that the face thrown is either an ace or a deuce, is $\frac{2}{6}$ or $\frac{1}{3}$: for there are here two favourable cases out of six which are equally likely to happen: the chance of failure, or that it is neither an ace nor a deuce, is $\frac{4}{6}$ or $\frac{2}{3}$.

of throwing an ace with a tetrahedial die.

(γ) If the die had been a regular tetrahedron, whose faces were marked with the numbers 1, 2, 3, 4, the chance of its *resting upon* an ace would be $\frac{1}{4}$; the chance of its not doing so, would be $\frac{3}{4}$.

Of drawing an ace from a pack of cards.

(δ) The chance of drawing the ace of spades from a pack of 52 cards, is $\frac{1}{52}$: the chance of drawing any one of the four aces, is $\frac{4}{52}$ or $\frac{1}{13}$: for there are four favourable cases out of fifty-two, which are both favourable and unfavourable, and all of them equally likely to happen.

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* The phrases *equally likely*, *equally possible*, are used in this and on all other occasions, in the sense which is given to them in Art. 126.

That a given day of a given month in a year not given, is a given day of the week.

(ϵ) The chance that the 14th of November of any year not assigned, falls upon a Friday, is $\frac{1}{7}$: for this is one of seven successive days, one of which, and one only, must be a Friday: and it cannot fall upon a determinate day, since neither 365 nor 366 are multiples of 7, and therefore different and successive years begin upon different days of the week.

Of drawing a white or a black ball from an urn containing a given number of both.

(ζ) If 14 white and 6 black balls be thrown into an urn, the chance of drawing a white ball out of it, at one trial, is $\frac{14}{20}$: the chance of failing, or of drawing a black ball, is $\frac{6}{20}$.

Chances of success and failure supplemental to each other.

271. In this and in all other cases, the chances of success and failure of the *same* event are *supplemental* to each other, their sum being equal to 1, which is the measure and representative of certainty: the knowledge of one, therefore, necessarily determines the other.

Compound chances.

272. The following examples of Compound Chances are introductory to their general theory, which will follow.

Chance of throwing an ace twice with one die.

(α) To find the chance of throwing an ace, twice in succession, with a single die.

There are six cases, which are equally likely to occur at the first throw, and the same number at the second: these may be combined or permuted together in 6×6 or 36 different ways, which are equally likely to happen, and only one of them is favourable: the chance is, therefore, $\frac{1}{36}$.

With two dice.

(β) The chance of throwing *two aces* at one *contemporaneous* throw with two dice, is equally $\frac{1}{36}$: for the succession of time makes no difference whatever in the number of favourable and unfavourable permutations.

Of throwing an ace and deuce.

(γ) The chance of throwing an *ace* at the first throw, and a *deuce* at the second, is also $\frac{1}{36}$: for there is only one favourable permutation out of 36.

(δ) The chance of throwing an *ace* at one throw, and a *deuce* at the other, without reference to their order of succession, is $\frac{1}{18}$: for in this case there are two permutations forming one combination (1, 2 and 2, 1), which are favourable to the hypothesis made, and two only, out of the whole 36.

Proposition respecting compound chances.

273. The chance of an event contingent upon other events, is the continued product of the chances of the separate events.

Let the several chances be

$$\frac{a_1}{a_1+b_1}, \frac{a_2}{a_2+b_2}, \frac{a_3}{a_3+b_3}, \dots \frac{a_n}{a_n+b_n},$$

where $a_1, a_2, a_3, \dots a_n$ represent the numbers of cases which are favourable, and $b_1, b_2, b_3, \dots b_n$ the numbers of cases which are unfavourable, to the particular hypothesis made in each separate event, whether of success or failure.

For two chances.

We will consider, in the first instance, the chance which is dependent upon the two separate chances

$$\frac{a_1}{a_1+b_1} \text{ and } \frac{a_2}{a_2+b_2}.$$

Every case in a_1+b_1 may be combined with every case in a_2+b_2, and thus form $(a_1+b_1)(a_2+b_2)$ combinations of cases, which are equally likely to happen.

The favourable cases in the first (a_1) may be combined severally with the favourable cases in the second (a_2), and thus form $a_1 a_2$ combinations of cases favourable to the compound event.

The compound chance is denoted, therefore, by

$$\frac{a_1 a_2}{(a_1 + b_1)\,(a_2 + b_2)},$$

which is the product of the separate chances.

For three chances.

Let us now pass to the consideration of the chance of the event contingent upon three other events, whose respective chances are

$$\frac{a_1}{a_1 + b_1},\ \frac{a_2}{a_2 + b_2},\ \frac{a_3}{a_3 + b_3}.$$

The several combinations of all the cases in the two first chances, which are, by the last case, $(a_1 + b_1)\,(a_2 + b_2)$ in number, may be severally combined with the $a_3 + b_3$ different cases, both favourable and unfavourable, of the third chance, and thus form $(a_1 + b_1)\,(a_2 + b_2)\,(a_3 + b_3)$ combinations which are equally likely to happen.

The favourable cases in the two first chances, which are $a_1 a_2$ in number, may be combined severally with the a_3 favourable cases of the third chance, and thus form $a_1 a_2 a_3$ cases which are favourable to the compound event.

The chance, therefore, of the compound event, is

$$\frac{a_1 a_2 a_3}{(a_1 + b_1)\,(a_2 + b_2)\,(a_3 + b_3)},$$

which is the product of the simple chances.

For any number of them.

If we now consider any number (n) of chances,

$$\frac{a_1}{a_1 + b_1},\ \frac{a_2}{a_2 + b_2},\ \frac{a_3}{a_3 + b_3},\ldots\ \frac{a_n}{a_n + b_n},$$

and assume the law expressed in the enunciation of the proposition as true for $(n-1)$ of them, it may be proved to be true for n of them: for the chance of the event contingent upon the $(n-1)$ first events being

$$\frac{a_1 a_2 \ldots a_{n-1}}{(a_1+b_1)(a_2+b_2)\ldots(a_{n-1}+b_{n-1})},$$

all the combinations of favourable and unfavourable cases in its denominator, may be severally combined with the a_n+b_n favourable and unfavourable cases in the n^{th} chance, and thus produce

$$(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)$$

favourable and unfavourable cases, for the compound event, which are equally likely to happen.

In a similar manner the $a_1 a_2 \ldots a_{n-1}$ favourable cases of the first $(n-1)$ chances, may be severally combined with the a_n favourable cases of the n^{th} chance, and thus produce $a_1 a_2 \ldots a_n$ favourable cases for the compound event, which are equally likely to happen: the compound chance is, therefore, denoted by

$$\frac{a_1 a_2 \ldots a_n}{(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)},$$

which is the product of all the simple chances.

It follows, therefore, that the law which has been proved to be true for 2 and 3 chances, is necessarily true for 4, 5, and so on, as far as any number whatever.

Its great importance.

This is a most important proposition, and makes the calculation of the chance of any compound event dependent upon the separate and simple chances of the several events, in their assigned order, upon which it is dependent.

Before we proceed to the consideration of some other important propositions which are intimately connected with it, we will illustrate its application by a few examples.

274. To find the chance of throwing an ace in the *first only* of two successive throws.

Chance of throwing an ace in the first only of two successive throws.

The first simple chance is $\frac{1}{6}$.

The second simple chance is $\frac{5}{6}$: for an ace must *not* be thrown the second time, and there are five *favourable** cases for its failure.

The compound chance is, therefore, $\frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$.

275. What is the chance of drawing the four aces from a pack of cards, in four successive trials?

Of drawing the four aces successively from a pack of cards.

The first simple chance is $\frac{4}{52}$.

The second simple chance is $\frac{3}{51}$.

For if an ace be drawn the first time, there remain only 3 aces and 51 cards.

The third simple chance is $\frac{2}{50}$.

For if two aces be drawn the two first times, there remain only 2 aces and 50 cards.

The fourth simple chance is $\frac{1}{49}$.

For if three aces be drawn the three first times, there remain only 1 ace and 49 cards.

The compound chance required is

$$\frac{4.3.2.1}{52.51.50.49} = \frac{1}{270725}.$$

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* It must be kept in mind, that the favourable cases are those which satisfy the particular hypothesis made for each event or trial: it is an ace in the first throw, and any other point in the second.

Of winning two games of whist in succession.

276. What is the chance of winning two games at whist (or of any other game where there are only two equal chances for winning or losing) in succession?

The first simple chance is $\frac{1}{2}$.

The second simple chance is $\frac{1}{2}$.

For the event of the first game does not influence the chance of the second.

The compound chance is, therefore,

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

Of not winning.

The chance of not winning two games in succession, (which is supplemental to the other) is, therefore, $\frac{3}{4}$.

The odds.

The odds against winning two games in succession, are 3 to 1,

Of winning three games successively

The chance of winning three games in succession is $\frac{1}{8}$, being the product of the three simple chances $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$.

or two out of three.

The chance of winning the two first games and losing the third, or of winning the first, losing the second and winning the third, or of losing the first and winning the second and third, is in each case $\frac{1}{8}$, inasmuch as the simple chances are $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$, the same as in the first instance.

The chance of winning two games out of three, without reference to their order of succession, is $\frac{3}{8}$, the sum of the chances of each separate event, which equally answer the conditions of the question.

The chance of losing the two first games and winning the third, or of losing the first, winning the second and losing the third, or of winning the first and losing the second and third, is in each case $\frac{1}{8}$, inasmuch as the three simple chances are $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$. Or one out of three.

The chance of losing two games in succession, without reference to their order of succession, is $\frac{3}{8}$, the sum of the chances of the three events, which equally answer the conditions of the question.

The chance of losing three games in succession is $\frac{1}{8}$, the product of the three simple chances $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$. Or losing three in succession.

277. The same conclusions may be very easily deduced from the theory of combinations: in three successive games, with equal chances for winning or losing, there are $2 \times 2 \times 2$ or 8 different combinations of events which must and may happen: there is only one of these combinations by which they may all of them be won: there are three combinations, by which two of them may be won and one lost: three, also, where two of them may be lost and one of them won: and one only by which they may all of them be lost: the whole sum of these is 8, which is the entire number of combinations. The same conclusions differently obtained.

278. In a similar manner, if the favourable and unfavourable cases for n different events, be represented as in Art. 273, by $a_1, a_2, \dots a_n$, and $b_1, b_2, \dots b_n$ respectively, and their sums by $a_1 + b_1, a_2 + b_2, \dots a_n + b_n$, then the different combinations of the different letters in their continued This process of reasoning generalized.

product, represent all the different modifications of the compound event which can possibly present themselves: for in the first place, the product

$$a_1 a_2 \dots a_n$$

will express the nmmber of ways in which all the events may succeed in succession, in conformity with the hypothesis made with respect to each separate event: every complementary combination of the first class (involving $(n-1)$ a's and one b, Art. 239.) will express all the different ways in which the events supposed may succeed $(n-1)$ times and fail once, and in the particular order of the numbers attached to each letter: and the whole of such complementary combinations (which are n in number) will express every possible way in which $(n-1)$ required events may happen and one of them fail, without reference to their order: again, every complementary combination of the second class will express the number of ways in which $(n-2)$ required events may happen and two of them fail, in the order of the numbers attached to each letter, whether a or b: whilst the whole of those complementary combinations (which are $\frac{n(n-1)}{1.2}$ in number) will express every possible way in which $(n-2)$ required events may happen and two of them fail, without any reference to the order of their succession: and *generally*, every complementary combination of the r^{th} class will express the number of different ways in which $(n-r)$ required events may happen and r of them fail, in the particular order of the numbers attached to each letter: whilst the whole of those complementary combinations (which are $\frac{n(n-1)\,..\,(n-r+1)}{1.2\;\;\dots\;\;r}$ in number) will express every possible way in which $(n-r)$ required events may happen and the others fail, without any reference to the order of their succession.

For repeated trials, where the

279. In all cases of successive trials, where the number of favourable and unfavourable cases for each event remains

the same, the continued product in question becomes $(a+b)^n$ (omitting the numbers attached to each letter), and the different classes of complementary combinations become the several terms of the series for $(a+b)^n$, beginning with the second: it will be very easy to give to each of those terms the interpretation due to the theory.

number of favourable and unfavourable cases are the same.

The first term a^n, expresses the number of combinations by which the required event may happen n times in succession: the corresponding chance, therefore, is

$$\frac{a^n}{(a+b)^n}.$$

The literal part of the second term $a^{n-1}b$, expresses the number of ways in which the required event may happen $(n-1)$ times and fail once, in a specified order: the entire term $na^{n-1}b$, expresses all the possible ways in which it may happen $(n-1)$ times and fail once, without reference to their order: the corresponding chances are

$$\frac{a^{n-1}b}{(a+b)^n} \text{ and } \frac{n\,a^{n-1}b}{(a+b)^n}.$$

Generally, the literal part of the $(1+r)^{\text{th}}$ term or $a^{n-r}b^r$, expresses the number of ways in which the required event may happen $(n-r)$ times and fail r times, in a specified order: whilst the entire term or

$$\frac{n(n-1)\ldots(n-r+1)}{1.2\quad\ldots\quad r}a^{n-r}b^r$$

expresses all the possible ways in which it may happen $n-r$ times and fail r times, without reference to their order of succession: the corresponding chances are

$$\frac{a^{n-r}b^r}{(a+b)^n} \text{ and } \frac{n(n-1)\ldots(n-r+1)}{1.2\quad\ldots\quad r}\cdot\frac{a^{n-r}b^r}{(a+b)^n}.$$

If the question proposed had been to find the number of ways in which an event may happen r times *at least* and fail $(n-r)$ times, without reference to their order of succession, it will be expressed by the sum of those terms of the series, in which the index of a is not less than r: the corresponding chance is, therefore,

$$\frac{a^n + na^{n-1}b + \dots \frac{n(n-1)\dots(r-1)}{1.2\ \dots(n-r)}a^r b^{n-r}}{(a+b)^n}.$$

Examples. 280. The following examples will serve to illustrate the application of these formulæ.

(α) In the game of *cross* and *pile* (*heads* and *tails*), what is the chance that *cross* will come up three times exactly in seven trials?

In each trial, either *cross* or *pile* may equally happen.

Consequently $a=1$, $b=1$, $n=7$ and $r=3$: the chance, therefore,

$$= \frac{7.6.5.4}{1.2.3.4} \cdot \frac{a^3 b^4}{(a+b)^7} = \frac{7.6.5.4}{1.2.3.4} \cdot \frac{1}{2^7} = \frac{35}{128}.$$

The chance that *cross* will come up three times at least, is

$$\frac{a^7 + 7a^6 b + \frac{7.6}{1.2} \cdot a^5 b^2 + \frac{7.6.5}{1.2.3} \cdot a^4 b^3 + \frac{7.6.5.4}{1.2.3.4} a^3 b^4}{(a+b)^7}$$

$$= \frac{1+7+21+35+35}{2^7} = \frac{99}{128}.$$

(β) In five throws with a single die, to find the chance of throwing two aces only in the two last throws.

In this case, the order of the successive events is specified: and since $a=1$, $b=5$, $n=5$ and $r=2$, the chance is

$$\frac{a^2 b^3}{(a+b)^5} = \frac{125}{7776}.$$

The chance that an ace is thrown twice only in five trials, is

$$\frac{5.4}{1.2} \cdot \frac{a^2 b^3}{(a+b)^5} = \frac{1250}{7776}.$$

The chance that an ace is thrown at least twice in any five trials, is

$$\frac{a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3}{(a+b)^5} = \frac{1526}{7776}.$$

(γ) Two persons, *A* and *B*, whose skill is in the proportion of 4 to 3, play at bowls together: what is the chance that *A* will at least win 6 games out of 7?

Meaning of the term *skill.*

By *A*'s *skill*, in this and other cases, as far at least as the purposes of calculation are concerned, we mean the ratio of the cases or events which are favourable to *A*, to the whole number of them, whether favourable or unfavourable: and the *relative* skill of *A* to *B* would be defined by the ratio of the numbers of cases or events, which were respectively favourable to *A* and to *B*: if this ratio be constant and invariable, the term *skill* may be used as equivalent to *chance* for all purposes of calculation: if this ratio be not constant, and admits only of determination within small limits, its approximate value must be used as if it were constant, and the *moral* chance considered equivalent to the absolute chance, which this approximate value would express: in other words, skill in its *ultimate* application must be *assumed* to depend upon fixed and invariable causes.*

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* There are not many games of pure skill, and in most cases the influence of skill compared with that of chance, can only be calculated from their combined effects: thus the natural chances of a game should enable *A* to win five games out of six: but the influence of superior skill is found by experience to enable him to win six games out of seven: then the value of *A*'s skill (estimated absolutely) is $\frac{6}{7} - \frac{5}{6} = \frac{36-35}{42} = \frac{1}{42}$, or is such as would enable him to win 1 game out of 42, by its influence alone.

It

These considerations bring the present problem under the general formula given above (Art. 279), where $a=4$, $b=3$, $n=7$, $r=6$: the chance is, therefore,

$$=\frac{a^7+7a^6b}{(a+b)^7}=\frac{102400}{823543}.$$

(δ) There is a lottery, consisting of a great number of tickets, where the prizes are to the blanks in the ratio of 1 to 7: what is the chance of taking three prizes in five trials?

In this case, we may consider the relation of the prizes to the blanks as not sensibly affected by the tickets which are drawn, inasmuch as they bear a very small ratio to the whole number: in other words, we may consider the problem as one of repeated trials, where the simple chances are the same.

It is not often, however, that such a separation of the effects of chance and skill is required, as we commonly observe their combined effect in the results of repeated trials, and form our estimate accordingly: for there exists a *popular assurance*, independent of any refined mathematical reasoning, that the actual results of repeated trials will be proportional to the combined influence of chance and skill, and the *intensity* of this assurance increases rapidly with the increase of the number of trials.

Thus, if A should win from B 40 games out of 70, it would be probable that A's skill was to B's as 4 to 3, or (in case chance was combined with skill) that the cases in favour of A and B respectively were in that proportion: but if A should win 400 games out of 700, this *assurance* could be much increased, and still more so, if A should win 4000 games out of 7000.

It is not our present object to enquire into the mathematical limits of error of such determinations, an investigation of great difficulty, and requiring some of the most refined artifices of analysis: it is sufficient for us to assert, what such investigations would establish, that the ratio of the numbers of events in favour of A or of B, will *approximate* to the ratio of the chances in favour of each.

In most cases, the mathematical assurance will increase much more rapidly than the popular assurance: if an ace is thrown 10 times successively with a single die, it is extremely probable that the die is loaded: but if an ace is thrown 20 times, the mathematical probability of the truth of this hypothesis is increased in the proportion of 6^{10} to 6^{20}, or as 1 to 60466176, a ratio with which the popular assurance can keep no pace.

We have, therefore, $a=1$, $b=7$, $n=5$ and $r=3$, and consequently the chance

$$=\frac{a^5+5a^4b+10a^3b^2}{(a+b)^5}$$

$$=\frac{1+35+490}{8^5}=\frac{526}{32768}$$

$$=\frac{3}{187} \text{ nearly.}$$

281. It frequently happens, that there are more than two ways in which an event may happen: thus, if an urn contain assigned numbers of black, white and red balls, and the chance of drawing a ball of an assigned colour is required: or, if there be more than two players at a game, who have equal or different chances of winning: and similarly for other cases. The theory of *complementary* combinations, which enabled us to represent and estimate all the chances by which assigned events might happen, when there were two possible species of events in each simple chance, will be found equally serviceable when there are a greater number of them.

Multiple chances, where the event of each trial may happen in more than two ways.

282. Thus, if a, b, c denote the cases favourable to three distinct events, and $a+b+c$ the whole number of cases, then the simple chances are severally

Mode of expressing simple multiple chances.

$$\frac{a}{a+b+c},\ \frac{b}{a+b+c},\ \frac{c}{a+b+c}.$$

Thus, if an urn contains 6 white, 8 black and 10 red balls, the simple chance of drawing a white ball is $\frac{6}{6+8+10}=\frac{6}{24}$ or $\frac{1}{4}$: of drawing a black ball is $\frac{8}{24}$ or $\frac{1}{3}$: and of drawing a red ball is $\frac{10}{24}$ or $\frac{5}{12}$. Their sum or $\frac{1}{4}+\frac{1}{3}+\frac{5}{12}=1$, or certainty.

Compound multiple chances: for two simple chances.

283. If there be two urns, containing respectively a and a' white balls, b and b' black balls, and c and c' red balls, the chance of drawing a white ball, first from one urn and then from the other, in successive or cotemporaneous trials, will be denoted by

$$\frac{aa'}{(a+b+c)\,(a'+b'+c')}.$$

For the whole number of ways in which the balls may be *imagined* to be combined together two and two, is $(a+b+c)\,(a'+b'+c')$, any one combination of which is equally likely to be drawn: the number of such combinations of white balls is aa', the product of the numbers of such balls in each urn: the chance required is found, therefore, by dividing aa' by $(a+b+c)\,(a'+b'+c')$.

The terms of this product, $aa'+bb'+cc'+ab'+a'b+ac'+a'c+bc'+b'c$, express all the different species of binary combinations which can possibly be formed: the number of binary combinations of white balls is aa', of blacks bb', of red balls cc', of white and black balls is $ab'+a'b$, of white and red balls is $ac'+a'c$, of black and red balls is $bc'+b'c$: the chance of drawing any such combination being found by dividing the term or terms which express the number of combinations which are favourable to the hypothesis made, by the whole number of combinations, whether favourable or not.

For any number of multiple simple chances.

284. If there were n such urns, the white balls in which were denoted severally by $a_1, a_2, \ldots a_n$, the black balls by $b_1, b_2, \ldots b_n$, the red balls by $c_1, c_2, \ldots c_n$, the terms of the product $(a_1+b_1+c_1)\,(a_2+b_2+c_2)\ldots(a_n+b_n+c_n)$ would represent all the combinations which could be formed by taking a ball from each urn: and the terms in this product, which would represent all the different combinations, containing α white balls, β black balls and γ red balls, where $\alpha+\beta+\gamma=n$, would be the sum of those which involve severally α a's, β b's, γ c's, the number of which is (Art. 245.)

$$\frac{n(n-1)\dots 2.1}{1.2\dots\alpha\times 1.2\dots\beta\times 1.2\dots\gamma}.$$

For repeated trials.

285. If we should suppose that there was one urn only, containing a white balls, b black balls and c red balls, and if a ball was drawn from it n times successively and replaced before the next trial, the combinations which might arise would be precisely the same as if there were n different urns, containing the same number of each kind of balls, with the first: in this case, however, the continued product considered in the last article, would become $(a+b+c)^n$, and the different classes and orders of complementary combinations would degenerate into the terms of its developement: the general term

$$\frac{n(n-1)\dots 2.1}{1.2\,..\,\alpha\times 1.2\,..\,\beta\times 1.2\,..\,\gamma}\,a^\alpha b^\beta c^\gamma$$

would express the whole number of combinations, in which there were α white balls, β black balls and γ red balls, and the chance of drawing such a combination would be

$$\frac{n(n-1)\dots 2.1}{1.2\,..\,\alpha\times 1.2\,..\,\beta\times 1.2\,..\,\gamma}\cdot\frac{a^\alpha b^\beta c^\gamma}{(a+b+c)^n}.$$

It is not necessary to point out the mode of investigating the formulæ corresponding to the different problems considered in Art. 279: nor to shew in what manner corresponding formulæ may be investigated and constructed, when the urn contains any number of different species of balls, or when the heap or collection of things contains any number of classes of things which are of the same kind.

286. The following are examples.

(1) An urn contains 3 white, 4 black and 5 red balls: what is the chance of drawing 1 white, 2 black and 3 red balls, in six successive trials?

In this case, $a=3$, $b=4$, $c=5$, $\alpha=1$, $\beta=2$, $\gamma=3$ and, therefore, $n=6$: the chance, therefore, is,

$$\frac{6.5.4.3.2.1}{1.2\times 1.2.3}\cdot\frac{a\,b^2c^3}{(a+b+c)^6}=\frac{625}{5184}.$$

(2) There is a heap of 20 cards, of which 7 are spades, 8 are clubs, and the rest hearts and diamonds: what is the chance of drawing 2 spades and 2 clubs in four trials, the card drawn being replaced before the next trial?

In this case, $a=7$, $b=8$, $c=5$, $\alpha=2$, $\beta=2$ and $n=4$: the chance, therefore, is,

$$\frac{4.3.2.1}{1.2\times1.2}\cdot\frac{a^2b^2}{(a+b+c)^4}=\frac{147}{1250}$$

$$=\frac{2}{17}\text{ nearly.}$$

(3) There are three gamesters, *A*, *B* and *C*, whose respective skills are as 3, 2 and 1: what is the chance of *A*'s winning *at least* 2 games out of 4?

In this case, $a=3$, $b=2$, $c=1$, $n=4$: the favourable terms in $(a+b+c)^4$ are all those which involve a^2, a^3 or a^4: the chance is, therefore,

$$=\frac{a^4+4a^3b+4a^3c+6a^2b^2+6a^2c^2+12a^2bc}{(a+b+c)^4}$$

$$=\frac{891}{1296}=\frac{9}{13}\text{ nearly.}$$

Miscellaneous problems illustrating the theory of combinations.

287. The following are miscellaneous problems, admitting of solution by the aid of the theory of combinations, and those general principles of chances which we have already had occasion to make use of.

(1) In the French lottery there are 90 numbers, 5 of which are drawn at a time: what is the chance that two and two only of five specified numbers will be drawn?

The whole number of quinary combinations, which are equally likely to be drawn, is

$$\frac{90.89.88.87.86}{1.2.3.4.5} \qquad (\alpha).$$

The number of ternary combinations of 88 numbers (omitting the two), is

$$\frac{88.87.86}{1.2.3} \qquad (\beta).$$

The number of binary combinations of 5 numbers (any of which may come up), is

$$\frac{5.4}{1.2} \qquad (\gamma).$$

The whole number of favourable quinary combinations is

$$\frac{88.87.86}{1.2.3} \times \frac{5.4}{1.2};$$

for any one binary combination in (γ) may be joined with any one ternary combination in (β).

The chance, therefore, is,

$$\frac{88.87.86}{1.2.3} \times \frac{5.4}{1.2} \times \frac{1.2.3.4.5}{90.89.88.87.86} = \frac{1}{40} \text{ nearly.}$$

(a) The chance that two, at *least*, of five numbers, would be drawn, which is the sum of the separate chances that five, four, three and two of them will be severally drawn, is, therefore,

$$\frac{\left\{1 + \frac{86}{1} \times \frac{5.4.3.2}{1.2.3.4} + \frac{87.86}{1.2} \times \frac{5.4.3}{1.2.3} + \frac{88.87.86}{1.2.3} \times \frac{5.4}{1.2}\right\} \times 1.2.3.4.5}{90.89.88.87.86}$$

$$= \frac{1135201}{43949243} = \frac{4}{155} \text{ nearly.}$$

(*β*) The chance that two specified numbers will be drawn (an *amb* or *bine*, in the lottery phrase) is

$$\frac{88.87.86}{1.2.3} \times \frac{1.2.3.4.5}{90.89.88.87.86} = \frac{2}{801}.$$

(*γ*) The three last problems, and all others relating to this species of lottery, will be the same, whether we suppose the five numbers drawn at once or successively, one by one.

For the combinations, upon which the chances depend, will be precisely the same in both cases.

(2) What is the chance of dealing, in the game of whist, an ace to each hand?

The number of different ways in which the cards may be dealt, is

$$\frac{52.51\ldots\ldots40}{1.2\ldots\ldots13} = 635013559600.$$

If the four aces be removed, the different ways in which the remaining 48 may be dealt, is

$$\frac{48.47\ldots\ldots\ldots37}{1.2\ \ldots.\ 12}.$$

Any one of the four aces may be united with these, thus producing

$$4 \times \frac{48.47\ \ldots\ 37}{1.2\ldots\ldots12}$$

hands, in which one ace and only one ace appears.

The chance required, therefore, is

$$= \frac{48.47\ldots.\ 37 \times 13 \times 4}{52.51\ \ldots.\ 40}$$

$$= \frac{39.38.37}{51.50.49} = \frac{36516}{83300} = \frac{32}{73} \text{ nearly.}$$

(3) To find the chance, in the game of whist, of the dealer and his partner having the four honours.

There are two cases to be considered: 1st, when the dealer turns up an honour: 2d, when he does not turn up an honour.

In both cases, the dealer and his partner have 25 cards out of the remaining 51, which may be any one of

$$\frac{51 . 50 \ldots . 27}{1 . 2 \ \ldots . 25} \qquad (\alpha)$$

different combinations.

The number of these combinations, which contain the 3 honours, will be the same as the number of combinations of 48 cards, taken 22 and 22 together, which is

$$\frac{48 . 47 \ldots\ldots 27}{1 . 2 \ \ldots \ 22} \qquad (\beta).$$

If the first supposition be true, the chance of the remaining 3 honours being found with the dealer and his partner, is

$$= \frac{(\alpha)}{(\beta)} = \frac{25 . 24 . 23}{51 . 50 . 49} = \frac{92}{833}.$$

The chance of an honour being turned up by the dealer, is $\frac{4}{13}$.

The compound chance of these two events is, therefore,

$$\frac{92}{833} \times \frac{4}{13} = \gamma.$$

If an honour is not turned up, which is the second supposition, the chance of the four honours being found with the dealer and his partner, is

$$\frac{25 . 24 . 23 . 22}{51 . 50 . 49 . 48} = \frac{253}{4998}.$$

The chance that an honour is not turned up, or that the second supposition is true, is $\frac{9}{13}$.

The compound chance of these two events is

$$\frac{253}{4998} \times \frac{9}{13} \qquad (\delta).$$

The entire chance, that the dealer and his partner have the four honours in one way or the other, is, therefore,

$$= \gamma + \delta = \frac{92}{833} \times \frac{4}{13} + \frac{253}{4998} \times \frac{9}{13}$$

$$= \frac{115}{1666} \text{ or } \frac{2}{29} \text{ nearly.}$$

The chance of the two other partners having the four honours, will be found to be $\frac{69}{1666}$.

(4) If we draw 4 cards out of a whole pack, what is the chance that one of them will be a heart, another a diamond, the third a club, and the fourth a spade?

The whole number of quaternary combinations is

$$\frac{52 . 51 . 50 . 49}{1 . 2 . 3 . 4}.$$

Every one of the 13 hearts may be combined with every one of the 13 diamonds, and these binary combinations with every one of the 13 clubs, and these ternary combinations with every one of the 13 spades, and thus form $13 \times 13 \times 13 \times 13$, or 13^4 quaternary combinations, which are alone favourable to the hypothesis made.

The chance, therefore, is

$$\frac{13^4 \times 1.2.3.4}{52.51.50.49} = \frac{2197}{20825} = \frac{2}{19} \text{ nearly.}$$

(5) If an urn contain 26 balls, of which 5 are white, 6 black, 7 red and 8 blue: what is the chance of drawing, when 10 are drawn at a time, 2 white, 3 black and 4 red balls?

The whole number of denary combinations is

$$\frac{26.25\,.\,.\,.\,.\,.\,17}{1.2\ \ldots\ 10}.$$

The number of binary combinations of 5 white balls

$$=\frac{5.4}{1.2}=10\quad(\lambda_1).$$

The number of ternary combinations of 6 black balls

$$=\frac{6.5.4}{1.2.3}=20\quad(\lambda_2).$$

The number of quaternary combinations of 7 red balls

$$=\frac{7.6.5.4}{1.2.3.4}=35\quad(\lambda_3).$$

The number of blue balls (taken one by one)

$$=8\qquad(\lambda_4).$$

The whole number of denary combinations which answer the conditions of the question $=\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4$: and the chance is, therefore,

$$=\frac{10\times20\times35\times8\times1.2\,.\,.\,.\,.\,10}{26.25\quad.\,.\,.\,.\,.\quad17}$$

$$=\frac{11200}{1062347}=\frac{7}{664}\text{ nearly.}$$

◆

CHAP. X.

ON THE BINOMIAL AND POLYNOMIAL THEOREMS.

What constitutes the binomial theorem.

288. IN Art. 243, we have deduced, as a consequence of the law of formation of binomial products and the theory of combinations, the law of formation likewise of the terms of the series for $(x+a)^n$ or $(a+x)^n$, where n is a whole number: it is the algebraical expression of this law, which will be found to be true, when n expresses any quantity whatsoever, which constitutes the celebrated binomial theorem.

Properties of the series when the index is a whole number.

289. Before we proceed to consider this theorem under this very general form, we will examine the properties of the series and of its terms, when the index is a whole number, and when the series terminates after an assignable number of terms.

The series.

290. We thus find (Art. 243),

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2$$

$$+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \dots\dots + \frac{n(n-1)\dots(n-r+1)}{1.2 \quad \dots. \quad r}a^{n-r}x$$

$$+ \&c.$$

Its law.

Where the powers of a in each term decrease by unity, the powers of x increase by unity, and the several coefficients, beginning with the second term, are the numbers of combinations of n things, taken 1 and 1, 2 and 2, 3 and 3, ... r and r together.

291. It is sometimes convenient, when the explicit expression of each coefficient is not required, to make use of a conventional symbol to represent them successively, with different numbers subjoined to them, to indicate their order of succession in the series: thus C_1, C_2, C_3, ..C_r, &c. might be assumed to represent the successive combinations of n things, which are the coefficients of the series, and, therefore,

Conventional notation to represent the series.

$$(a+x)^n = a^n + C_1 a^{n-1} x + C_2 a^{n-2} x^2 + C_3 a^{n-3} x^3 \\ + \dots . C_r a^{n-r} x^r + \dots .$$

The transition from any one coefficient to its complete algebraical expression is immediate: for, generally,

$$C_r = \frac{n(n-1) \dots (n-r+1)}{1.2 \quad \dots \quad r},$$

which is the coefficient of the $(1+r)^{\text{th}}$ term of the series.

292. Inasmuch as

$$(a+x)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left\{1+\frac{x}{a}\right\}^n,$$

we have, therefore,

$$a^n\left\{1+\frac{x}{a}\right\}^n = a^n + C_1 a^{n-1} x + C_2 a^{n-2} x^2 + \dots . C_r a^{n-}\ a^r + \dots$$

$$= a^n\left\{1 + C_1 . \frac{x}{a} + C_2 . \frac{x^2}{a^2} + \dots . C_r . \frac{x^r}{a^r} + \dots\right\}.$$

If we now divide both sides by a^n, we get

$$\left(1+\frac{x}{a}\right)^n = 1 + C_1 . \frac{x}{a} + C_2 . \frac{x^2}{a^2} + \dots . C_r . \frac{x^r}{a^r} + \dots$$

If we further replace $\frac{x}{a}$ by u, we find

Series for $(1+u)^n$.

$$(1+u)^n = 1 + C_1 u + C_2 u^2 + \dots C_r u^r + \dots$$

a series, in which the ascending and integral powers of u or of the second term of the binomial, enter into the successive terms of the series, beginning with the second: the same is the case with the second term of the binomial, in the original series, where the successive powers of x increasing from 1 appear united with the successive powers of a, descending from n.

Importance of attending to this form of the series.

It is convenient to make use of this form of the series, when we are considering its general properties, on account of its greater simplicity: and the observation of its form will be found hereafter of great importance, when we come to consider the series corresponding to any index whatever.

Examples.

292. We will now give a few examples of the developement of the series, corresponding to particular values of the index.

(1) $(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$.

(2) $(a+x)^6 = a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$.

(3) $(a+x)^7 = a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$.

(4) $(a+x)^8 = a^8 + 8a^7x + 28a^6x^2 + 56a^5x^3 + 70a^4x^4 + 56a^3x^5 + 28a^2x^6 + 8ax^7 + x^8$.

The numbers which form the coefficients of this developement, will be found in the table of *figurate numbers* given in Art. 259, by drawing a line from the 1st term of the 8th order to the 8th term of the 1st order: a similar remark applies to the coefficients in Ex. 1, 2 and 3.

(5) $(5+4x)^4 = 625 + 2000x + 2400x^2 + 1280x^3 + 256x^4$.

(6) $(a-x)^9 = a^9 - 9a^8x + 36a^7x^2 - 84a^6x^3 + 126a^5x^4 - 126a^4x^5 + 84a^3x^6 - 36a^2x^7 + 9ax^8 - x^9$.

In this and all similar developements, the 2d, 4th, 6th and all the other *even* terms will be *negative*, inasmuch as they involve *odd* powers of $-x$.

(7) $(a^2-ax)^{10}=a^{20}\left(1-\frac{x}{a}\right)^{10}$

$$=a^{20}\left\{1-10.\frac{x}{a}+45.\frac{x^2}{a^2}-120.\frac{x^3}{a^3}+210.\frac{x^4}{a^4}\right.$$

$$-252.\frac{x^5}{a^5}+210.\frac{x^6}{a^6}-120.\frac{x^7}{a^7}+45.\frac{x^8}{a^8}-10.\frac{x^9}{a^9}$$

$$\left.+\frac{x^{10}}{a^{10}}\right\}.$$

(8) $\left(\frac{1}{2}x-2y\right)^7=\frac{x^7}{128}\left(1-\frac{4y}{x}\right)^7=\frac{x^7}{128}-\frac{7}{32}.x^6y$

$$+\frac{21}{8}x^5y^2-\frac{35}{2}x^4y^3+70x^3y^4-168x^2y^5+224xy^6-128y^7.$$

In this case, we may either expand $\left(\frac{1}{2}x-2y\right)^7$ at once, or begin by expanding $\left(1-\frac{4y}{x}\right)^7$, and afterwards multiply every term into $\frac{x^7}{128}$.

(9) $(\sqrt{a}+\sqrt{b})^4=a^2+4a^{\frac{3}{2}}b^{\frac{1}{2}}+6ab+4a^{\frac{1}{2}}b^{\frac{3}{2}}+b^2$
$=a^2+6ab+b^2+4(a+b)\sqrt{ab}.$

(10) $(\sqrt[3]{x}+\sqrt[3]{y})^6=x^2\left(1-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)^6$

$$=x^2\left\{1-\frac{6y^{\frac{1}{3}}}{x^{\frac{1}{3}}}+15.\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}-20.\frac{y}{x}+15.\frac{y^{\frac{4}{3}}}{x^{\frac{4}{3}}}-6.\frac{y^{\frac{5}{3}}}{x^{\frac{5}{3}}}+\frac{y^2}{x^2}\right\}$$

$$=x^2-20xy+y^2-6x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{4}{3}}+y^{\frac{4}{3}})+15x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}}).$$

(11) The 5th term of $(a^2-b^2)^{12}=495a^{16}b^8$.

(12) The 6th term of $(a-x)^{30}=-142506a^{25}x^5$.

(13) The 10th term of $(\sqrt{a}-\sqrt{b})^{19}=-92378a^5b^{\frac{9}{2}}$.

(14) The 4th term of $\left\{\frac{2ac}{b^2} + \frac{1}{4}bc^2d\right\}^6 = \frac{5}{2}\cdot\frac{a^3c^9d^3}{b^3}$.

(15) The 7^{th} term of $(a^3+3ab)^9 = 61236a^{15}b^6$.

Determination of an assigned term

293. Any assigned term of the series, such as the $(1+r)^{\text{th}}$, may be determined from the formula

$$\frac{n(n-1)\ \ldots\ (n-r+1)}{1.2\ \ \ldots.\ \ r}A^{n-r}B^r,$$

where n is the index of the binomial, A its first term and B its second: thus, in finding the 7^{th} term of $(a^3+3ab)^9$, we have $1+r=7$, $n=9$, $A=a^3$ and $B=3ab$, whence the assigned term which is required in the last of the examples just given, may be determined: and similarly with respect to the four examples preceding it.

Its sign.

294. If B or the second term of the binomial be negative, any assigned term is negative, if in an even place, and positive, if in an odd place.

Number of terms.

295. The number of terms in the entire series is $(n+1)$, or one greater than the index of the binomial (Art. 260): thus, the number of terms in the series for $(a+x)^3$ is 4, in that for $(1-x)^{13}$ is 14; and the same fact may be observed in all the examples given in Art. 292.

Coefficients the same from the beginning and the end of the series.

296. The coefficients of the series are the same when taken from the beginning and the end: for the coefficients of the terms equidistant from each end, form *supplementary* combinations (Art 237): thus, if the series be written as follows,

$$(1+u)^n = 1 + C_1u + C_2u^2 + \ldots.\ C_ru^r + \ldots.C_{n-r}u^{n-r}$$
$$+ \ldots.\ C_{n-2}u^{n-2} + C_{n-1}u^{n-1} + C_nu^n,$$

we have $C_n = 1$, since C_n is the number of combinations of n things taken all together:

$$C_{n-1} = C_1,\ C_{n-2} = C_2,\ \ldots\ C_{n-r} = C_r,$$

which are severally pairs of supplementary combinations, and also the coefficients of terms which are equidistant from the end and the beginning of the series.

In speaking of this property of the coefficients, we do not take into account the portion of the several coefficients, which in some cases may arise from particular numerical coefficients of A or B in the binomial: instances of this effect may be seen in Ex. 5 and 8, in Art. 292.

Middle term

297. When the index n is even, the number of terms $(n+1)$ is odd, and the middle term is the $\left(\frac{n}{2}+1\right)^{\text{th}}$, whose coefficient is the greatest in the series (Art. 238): the term itself in the series for $(a+x)^n$ is

$$\frac{n(n-1)\ldots\ldots\left(\frac{n}{2}+1\right)}{1.2\quad\ldots\ldots\quad\frac{n}{2}}\,a^{\frac{n}{2}}x^{\frac{n}{2}}.\text{*}$$

or terms.

If the index n of $(a+x)^n$ be odd, the number of terms $(n+1)$ is even; and there are two middle terms, whose coefficients are supplementary combinations, and therefore equal: these are

$$\frac{n(n-1)\ldots\left(\frac{n+3}{2}\right)}{1.2\ \ldots\ \left(\frac{n-1}{2}\right)}.a^{\frac{n+1}{2}}x^{\frac{n-1}{2}}$$

and $$\frac{n(n-1)\ldots\left(\frac{n+3}{2}\right)\left(\frac{n+1}{2}\right)}{1.2\ \ldots\ldots\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)}.a^{\frac{n-1}{2}}x^{\frac{n+1}{2}}\dagger.$$

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[*] The coefficient of this term may be put under a different and equivalent form, which is

$$\frac{1.3.5\ldots\ldots(n-1)}{1.2.3\ \ldots\ldots\ \frac{n}{2}}.2^{\frac{n}{2}}.$$

[†] The coefficient of either of these terms may be put under the different and equivalent form

$$\frac{1.3.5\quad\ldots\ldots\quad n}{1.2\ \ldots\ldots\ldots\ \frac{n-1}{2}}.2^{\frac{n-1}{2}}.$$

Examples.

Thus, the middle term of $(a^2-x^2)^{14}$ is

$$-\frac{14.13 \ldots . 8}{1.2 \ldots . 7}a^{14}x^{14} = -3432a^{14}x^{14}.$$

The two middle terms of $(a^2x+ax^2)^{17}$ are

$$\frac{17.16 \ldots . 10}{1.2 \ldots . 8}a^{26}x^{25} \text{ and } \frac{17.16 \ldots . 10.9}{1.2 \ldots . 8.9}a^{25}x^{26},$$

or $24310\,a^{26}x^{25}$ and $24310\,a^{25}x^{26}$.

Number of terms $=2^n$.

298. We have already seen that the sum of the coefficients of all the terms (which would be the number of all the terms in the product or developement of $(a+x)^n$, if *like* terms were *not* collected into one) is 2^n (Art. 240): the same conclusion may be otherwise obtained as follows: for $(1+u)^n$

$$=1+C_1u+C_2u^2+ \ldots C_ru^r+ \ldots C_nu^n:$$

and if we make $u=1$, we find

$$(1+1)^n=2^n=1+C_1+C_2+ \ldots C_r+ \ldots C_n,$$

which is the sum of all the coefficients of the series.

Sum of coefficients of odd terms equal to the sum of coefficients of even terms.

299. If in the same series for $(1+u)^n$, we make $u=-1$, all the even terms will become negative, and we shall find

$$(1-1)^n=1-C_1+C_2-C_3+C_4-C_5+ \ldots \pm C_r \mp C_n=0:$$

or, in other words, the sum of the coefficients of the odd terms will be equal to the sum of the coefficients of the even terms, and therefore each of them will be equal to 2^{n-1}, which is half the sum of all the terms.

Numbers of odd and even combinations.

300. The coefficients of the *odd* terms, omitting the first, express the sum of all the *even* combinations of n things; whilst the coefficients of the *even* terms express the sum of all the *odd* combinations of n things: it follows, therefore, from the last Article, that the sum of all the

odd combinations of n things exceeds, by *unity*, the sum of all the *even* combinations of n things: the number of the first is 2^{n-1}, and that of the second is $2^{n-1}-1$.*

Modes of denoting an ambiguous sign.

301. The $(1+r)^{\text{th}}$ term of the series for $(1-u)^n$ or $C_r u^r$, is negative when $1+r$ is even, and positive when $1+r$ is odd: in order to express the ambiguity which exists with regard to its sign, when r is not assigned, the double sign is put before it, and the term written thus $\pm C_r u^r$.

The same term may be otherwise represented, as follows;

$$(-1)^r C_r u^r:$$

a form which involves no ambiguity: for $(-1)^2=1$, $(-1)^3=-1$, $(-1)^4=1$, $(-1)^5=-1$, $(-1)^6=1$, and so on, as far as $(-1)^r$, which is -1, when r is odd or $1+r$ even, and $+1$, when r is even or $1+r$ odd.

Extension of conventional notations.

302. The equation

$$(1+u)^{m+n}=(1+u)^m \times (1+u)^n,$$

when viewed in connection with the series or developements corresponding to $(1+u)^{m+n}$, $(1+u)^m$ and $(1+u)^n$ will lead to many important consequences: before, however, we proceed to their more particular consideration, it will be expedient to *enlarge* the conventional notation which we have hitherto employed to represent the series for $(1+u)^m$.

Instead, therefore, of denoting the series for $(1+u)^m$ by

$$1+C_1 u+C_2 u^2+C_3 u^3+\ldots C_r u^r,$$

we will express it by

$$1+C_1(m)\,u+C_2(m)\,u^2+C_3(m)\,u^3+\ldots\, C_r(m)\,u^r+\ldots.$$

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* The chance of taking an *odd* number from a bag containing n things (supposing any combination from 1 to n equally likely to be taken), is $\frac{2^{n-1}}{2^n-1}$: the corresponding chance of taking an *even* number is $\frac{2^{n-1}-1}{2^n-1}$.

writing the index of the binomial after C_1, C_2, C_3, ... C_r, which were before employed, without any such addition, to denote those coefficients.

This notation, though more complicated than the former, will be found convenient when the series corresponding to binomials with different indices, are considered with reference to each other: thus, $C_r(m)$ denotes the coefficient of the $(1+r)^{\text{th}}$ term of the series for $(1+u)^m$, $C_r(n)$ denotes the coefficient of the $(1+r)^{\text{th}}$ term of the series for $(1+u)^n$, $C_r(m+n)$ denotes the coefficient of the $(1+r)^{\text{th}}$ term of the series for $(1+u)^{m+n}$; whilst C_r denotes the coefficient of the $(1+r)^{\text{th}}$ term of the series corresponding to any binomial, whether its index be m, n, $m+n$, or any other quantity whatsoever.

Consequences of the equation $(1+u)^{m+n} = (1+u)^m \times (1+u)^n$.

303. Inasmuch as $(1+u)^m \times (1+u)^n = (1+u)^{m+n}$, the same relation must exist between the series which are severally *equal* as well as equivalent to the binomials $(1+u)^m$, $(1+u)^n$ and $(1+u)^{m+n}$: or, in other words,

$$\begin{aligned}
&\{1 + C_1(m)u + C_2(m)u^2 + \dots C_r(m)u^r + \dots\} \\
&\times \{1 + C_1(n)u + C_2(n)u^2 + \dots C_r(n)u^r + \dots\} \\
&= 1 + C_1(m+n)u + C_2(m+n)u^2 + \dots C_r(m+n)u^r + \dots
\end{aligned}$$

In a similar manner, if we introduce into the product an additional binomial $(1+u)^p$, we shall find

$$\begin{aligned}
&\{1 + C_1(m)u + C_2(m)u^2 + \dots C_r(m)u^r + \dots\} \\
&\times \{1 + C_1(n)u + C_2(n)u^2 + \dots C_r(n)u^r + \dots\} \\
&\times \{1 + C_1(p)u + C_2(p)u^2 + \dots C_r(p)u^r + \dots\} \\
&= 1 + C_1(m+n+p)u + C_2(m+n+p)u^2 \\
&\qquad + \dots C_r(m+n+p)u^r + \dots.
\end{aligned}$$

and the same remark is clearly applicable to the continued product of the series corresponding to any number of such binomials.

Conventional notation: the series for $(1+u)^m$ denoted by $S(m)$.

304. A conventional notation, extremely convenient from its brevity, may be made use of to express the proposition contained in the last Article: if the series corresponding to $(1+u)^m$ (or any binomial whose index is m) be denoted by $S(m)$, the series for $(1+u)^n$ will be denoted by $S(n)$, and that for $(1+u)^{m+n}$ by $S(m+n)$: the conclusions in the last article will thus assume the form

$$S(m) \times S(n) = S(m+n).$$

$$S(m) \times S(n) \times S(p) = S(m+n+p)^*.$$

Corresponding terms in $S(m+n)$ and $S(m) \times S(n)$.

305. In comparing the several terms of the developement of $S(m+n)$ with those corresponding to them in the actual product of $S(m)$ and $S(n)$, we must consider all those terms in this product as one, which involve the same power of u: and such corresponding terms in $S(m+n)$ and $S(m) \times S(n)$ must be equal to each other; otherwise the developement of $(1+u)^{m+n}$ would not be equivalent to (and in this case identical with) the product of the series for $(1+u)^m$ and $(1+u)^n$: as an example, let us compare the coefficients of the terms involving u^r, or of the $(1+r)^{\text{th}}$ term in each.

$$S(m+n) = (1+u)^{m+n} = 1 + C_1(m+n)u + C_2(m+n)u^2 \\ + \ldots C_r(m+n)u^r + \ldots$$

The coefficient of the $(1+r)^{\text{th}}$ term or $C_r(m+n)$

$$= \frac{(m+n)\,(m+n-1)\,\ldots\ldots\,(m+n-r+1)}{1\,.\,2\quad\ldots\ldots\quad r} \text{ (Art. 291).}$$

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* Such conventional forms must be interpreted in accordance with the particular convention made respecting them, and not according to the general principles of the interpretation of algebraical forms: it is on this account that they should be sparingly and cautiously used; for unless their meaning has been previously explained, and their equivalent and proper algebraical form exhibited, so that the transition from one to the other may be readily and immediately effected, they would only serve to encumber instead of abbreviating algebraical language.

Again,

$$S(m) = 1 + C_1(m) \,.\, u + C_2(m) \,.\, u^2 + \ldots\ldots C_r(m) \,.\, u^r + \ldots\ldots$$

$$S(n) = 1 + C_1(n) \,.\, u + C_2(n) \,.\, u^2 + \ldots\ldots C_r(n) \,.\, u^r + \ldots\ldots ;$$

the coefficient of the $(1+r)^{\text{th}}$ term in $S(m) \times S(n)$ or $C_r(m+n)$

$$= 1 \times C_r(n) + C_1(m) \times C_{r-1}(n) + C_2(m) \times C_{r-2}(n) + \ldots\ldots C_{r-1}(m) \times C_1(n) + C_r(m) \times 1$$

$$= 1 \times \frac{n(n-1)\ldots(n-r+1)}{1 \,.\, 2 \;\;\ldots\ldots\;\; r} + \frac{m}{1} \times \frac{n(n-1)\ldots(n-r+2)}{1 \,.\, 2 \;\;\ldots\;\; (r-1)}$$

$$+ \frac{m(m-1)}{1 \,.\, 2} \times \frac{n(n-1)\ldots\ldots(n-r+3)}{1 \,.\, 2 \;\;\ldots\ldots\;\; (r-2)}$$

$$+ \ldots\ldots \frac{m(m-1)\ldots(m-r+2)}{1 \,.\, 2 \;\;\ldots\;\; (r-1)} \times \frac{n}{1}$$

$$+ \frac{m(m-1)\ldots\ldots(m-r+1)}{1 \,.\, 2 \;\;\ldots\ldots\;\; r} \times 1 :$$

replacing the several coefficients of the series for $S(m)$ and $S(n)$ by their *explicit* algebraical values.*

The middle term of $(1+u)^{2n}$.

306. If $m = n = r$, we have $S(m+n) = S(2n) = (1+u)^{2n}$, and the $(1+r)^{\text{th}}$ term becomes the $(1+n)^{\text{th}}$ or middle term of the series: consequently

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* The formulæ contained in this article, furnish the means of solving the following problem, which is similar in its character to some of those solved in Art. 288:

"An urn contains m white and n black balls: what is the chance that in drawing $r\,(=p+q)$ balls from the urn at one trial, there will be p white balls and q black balls?"

The chance required is

$$= \frac{C_p(m) \times C_q(n)}{C_r(m+n)} = \frac{\dfrac{m(m-1)\ldots\ldots(m-p+1)}{1 \,.\, 2 \;\;\ldots\ldots\;\; p} \times \dfrac{n(n-1)\ldots(n-q+1)}{1 \,.\, 2 \;\;\ldots\;\; q}}{\dfrac{(m+n)(m+n-1)\ldots\ldots(m+n-r+1)}{1 \,.\, 2 \;\;\ldots\ldots\;\; r}} .$$

$$C_n(2n) = \frac{2n(2n-1)\,..\,(n+1)}{1\,.\,2\;\;....\;\;n}$$

$$= \frac{2n(2n-1)\,.....\,2\,.\,1}{(1\,.\,2\,...\,n)^2}.$$

Sum of the squares of the coefficients of $(1+x)^n$.

Under the same circumstances, $S(m)$ becomes identical with $S(n)$, and the coefficient of the $(1+n)^{\text{th}}$ term of their product or $C_n(2n) = 1 \times C_n + C_1 \times C_{n-1} + C_2 \times C_{n-2} + \ldots C_{n-1} \times C_1 + C_n \times 1 = 1^2 + C_1^{\,2} + C_2^{\,2} + \ldots. C_{n-1}^{\;2} + C_n^{\,2}$: for $1 = C_n$, $C_1 = C_{n-1}$, $C_2 = C_{n-2}$, $C_r = C_{n-r}$, &c., inasmuch as they severally form *supplementary* combinations, or are the coefficients of terms equidistant from the beginning and end of the series (Art. 296).

It follows, therefore, that

$$\frac{2n(2n-1)\,.....\,2\,.\,1}{(1\,.\,2\,...\,n)^2}$$

$$=1^2+\left(\frac{n}{1}\right)^2+\left\{\frac{n(n-1)}{1\,.\,2}\right\}^2+....\left\{\frac{n(n-1)...(n-r+1)}{1\,.\,2\;\;...\;\;r}\right\}^2+...*$$

307. Before we proceed to the consideration of the series which is equivalent to $(a+x)^n$, when n is any quantity whatsoever, it may be proper to observe, that we may

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* The formulæ in this article, furnish the means of solving the following problem:

"An urn contains the same number (n) of black and white balls: what is the chance of drawing the same number of black and white balls from the urn, every *even* combination of the whole number being equally likely to be drawn?"

The number of combinations which answer the conditions of the question

$$=\left(\frac{n}{1}\right)^2+\left\{\frac{n(n-1)}{1\,.\,2}\right\}^2+\left\{\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\right\}^2$$

$$+...\left\{\frac{n(n-1)...(n-r+1)}{1\,.\,2\;\;...\;\;r}\right\}^2+\text{\&c.}=\frac{2n(2n-1)...2\,.\,1}{(1\,.\,2\,...\,n)^2}-1=(\alpha):$$

and the whole number of *even* combinations of any kind which can be formed $=2^{2n-1}-1.=(\beta)$.

The chance is, therefore, $=\frac{(\alpha)}{(\beta)}$.

Reduction of binomials.

confine our attention altogether to the more simple form $(1+u)^n$: for if we make $u=\frac{x}{a}$, and therefore $a+x=a+au=a(1+u)$, we shall have $(a+x)^n=a^n(1+u)^n$: the following examples are in addition to those which are given in Art. 181.

$$(1) \quad \sqrt{(ax-x^2)}=(ax-x^2)^{\frac{1}{2}}=a^{\frac{1}{2}}x^{\frac{1}{2}}\left(1-\frac{x}{a}\right)^{\frac{1}{2}}$$

$$=a^{\frac{1}{2}}x^{\frac{1}{2}}(1+u)^{\frac{1}{2}}, \text{ if } u=-\frac{x}{a}.$$

$$(2) \quad (a^2x^3+a^3x^2)^{-\frac{1}{3}}=a^{-\frac{2}{3}}x^{-1}\left(1+\frac{a}{x}\right)^{-\frac{1}{3}}$$

$$=\frac{1}{a^{\frac{2}{3}}x}(1+u)^{-\frac{1}{3}}, \text{ if } u=\frac{a}{x}.$$

$$(3) \quad (x^{\frac{1}{3}}-a^{\frac{1}{3}})^{-3}=\frac{1}{x}\left(1-\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)^{-3}$$

$$=\frac{1}{x}(1+u)^{-3}, \text{ if } u=-\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

$$(4) \quad \left(\frac{a}{x}+\frac{x}{a}\right)^{\frac{1}{\sqrt{2}}}=\frac{a^{\frac{1}{\sqrt{2}}}}{x^{\frac{1}{\sqrt{2}}}}\left(1+\frac{x^2}{a^2}\right)^{\frac{1}{\sqrt{2}}}$$

$$=\frac{a^{\frac{1}{\sqrt{2}}}}{x^{\frac{1}{\sqrt{2}}}}(1+u)^{\frac{1}{\sqrt{2}}}, \text{ if } u=\frac{x^2}{a^2}.$$

Generalization of the form of the series for $(1+u)^n$.

308. The series for $(1+u)^n$, when n is a whole number, expressed by a general symbol, is

$$1+\frac{n}{1}u+\frac{n(n-1)}{1.2}u^2+\frac{n(n-1)(n-2)}{1.2.3}u^3+\ldots.$$

$$\frac{n(n-1)\ldots(n-r+1)}{1.2\ \ldots\ r}u^r+\ldots.$$

a form which may be continued indefinitely, according to the same law of formation, without reference to the specific value of n: such an indefinite form of the series would be true for integral and positive values of n, inasmuch as all the terms after the $(n+1)^{\text{th}}$ would involve a factor in the numerator of their coefficients, which is equal to zero: such a series, therefore, is general in form, though the quantity n is specific in value[*].

This series true for all values of n.

309. The law of the *permanence of equivalent forms*, (Art. 132) would enable us to conclude that the series which was equivalent to $(1+u)^n$, when the index was *general* in its form, though *specific* in its value, must be equivalent to it likewise, when the index is *general* both in form and value[†].

The great importance, however, of the proposition which is thus established, will justify us in detailing the reasoning as applicable to this case, upon which the law is founded.

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[*] In this series, the symbol r, though general in form, is essentially specific in value: for it determines the position of any assigned term, and must therefore be an integral and positive number: in speaking, therefore, of the symbols of this formula becoming general in signification as well as in form, our observations apply to the index of the binomial, the only quantity which makes the series for one binomial different from that for another.

[†] In speaking of the application of the law of the permanence of equivalent forms, to the generalization of forms, corresponding to a symbol which is general in form though specific in value, we must take care that they be not exhibited under a form which is essentially connected with its specific value: thus, if n be a whole number,

$$\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots ab^{n-2}+b^{n-1},$$

an equivalent form, which is general in one sense only, inasmuch as the last terms are essentially dependent upon the integral values of n: the equivalent form

$$\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b^2+a^{n-3}b^3+\dots\dots a^{n-r}b^r+\dots$$

is equally significant with the former, and is not connected with the specific value of n: it is, therefore, under a form adapted to the application of the law.

There can be only one general form of the series for $(1+u)^n$.

310. In the first place, if there exists a *general* equivalent form or series for $(1+u)^n$, it must be the form or series in question: for if that form be general, it must comprehend equally all values of n, whether whole numbers or any other quantities whatsoever, and whether they be expressed by a general symbol or are specifically assigned: it must necessarily coincide, therefore, with the general form which has been proved to exist, when n was any whole number.

This general form consistent with the interpretation given to particular values of the index.

311. In the second place, the *assumption* of the existence of such a general equivalent form or series, will lead to no consequences which are inconsistent with the interpretation already given to particular values of the index of $(1+u)^n$, whether it be a positive fraction, or a negative whole number or a negative fraction.

For the equation

$$S(m+m') = S(m) \times S(m'),$$

which is true when m and m' are whole numbers, and when the series corresponding to them are expressed without reference to their specific values, must be true, also, when m and m' are any quantities whatsoever: for the *symbolical* result of their multiplication is obviously independent of the specific values of the general symbols.

In other words, if

$$S(m) = 1 + \frac{m}{1} . u + \frac{m(m-1)}{1.2} u^2 + \dots .$$
$$\frac{m(m-1) \dots . (m-r+1)}{1.2 \quad \dots .. \quad r} u^r + \dots ..$$

$$S(m') = 1 + \frac{m'}{1} . u + \frac{m'(m'-1)}{1.2} u^2 + \dots .$$
$$\frac{m'(m'-1) \dots (m'-r+1)}{1.2 \quad \dots \quad r} u^r \dots ..$$

then, also,

$$S(m) \times S(m') = S(m+m') = 1 + \frac{(m+m')}{1} u +$$

$$+ \frac{(m+m')\,(m+m'-1)}{1\,.\,2} u^2 + \ldots.$$

$$\frac{(m+m')\,(m+m'-1)\ldots.(m+m'-r+1)}{1\,.\,2 \quad \ldots.. \quad r} u^r + \ldots$$

whatever the values of m and m' may be: for the laws for the incorporation of those symbols and the resulting forms, are perfectly independent of the specific values of the symbols themselves.

When the index is a negative whole number,

It follows from this proposition, that $(1+u)^{-m} = S(-m)$: for if $m' = -m$, we have $S(m+m') = S(m-m) = S(o) = (1+u)^o = 1$: and the $S(m) \times S(m')$ can only become $= 1$, by putting $-m$ in the place of m', in the series for $S(m')$ and $S(m+m')$: we thus get

$$(1+u)^{-m} = S(-m) = 1 - mu + \frac{m(m+1)}{1\,.\,2} u^2$$

$$- \frac{m(m+1)(m+2)}{1\,.\,2\,.\,3} u^3 + \ldots.(-1)^r \frac{m(m+1)\ldots(m+r-1)}{1\,.\,2 \ \ldots \ r} u^r + \ldots{}^{*}$$

or a positive numerical fraction,

Again,

$$S(m+m) = S(2m) = (1+u)^{2m} = \{S(m)\}^2$$

$$S(2m+m) = S(3m) = (1+u)^{3m} = \{S(m)\}^3$$

$$\ldots\ldots\ldots\ldots$$

$$S\{(q-1)m+m\} = S(qm) = (1+u)^{qm} = \{S(m)\}^q.$$

Consequently, if $qm = p$, and therefore $m = \frac{p}{q}$, we find

$$(1+u)^p = \{S(m)\}^q = \left\{S\left(\frac{p}{q}\right)\right\}^q;$$

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[*] The same form of the series for $(1+u)^{-m}$ is otherwise obtained in Art. 257.

and therefore, extracting the q^{th} root on both sides,

$$(1+u)^{\frac{p}{q}} = S\left(\frac{p}{q}\right) = 1 + \frac{p}{q}u + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1\,.\,2}u^2$$

$$+ \ldots\ldots\ldots \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\ldots\ldots\left(\frac{p}{q}-r+1\right)}{1\,.\,2 \quad \ldots \quad r}u^r + \ldots\ldots$$

Or under another and equivalent form, which is very convenient when real numerical expansions are required to be made under their reduced and most simple form,

$$(1+u)^{\frac{p}{q}} = 1 + \frac{p}{q}.u + \frac{p(p-q)}{q^2}.\frac{u^2}{1.2} + \frac{p(p-q)(p-2q)}{q^3}.\frac{u^3}{1.2.3}$$

$$+ \ldots\ldots\ldots \frac{p(p-q)\ldots\ldots\{p-(r-1)q\}}{q^r}.\frac{u^r}{1\,.\,2\ldots\ldots r} + \ldots$$

where the fractional factors are reduced to others equivalent to them, having the common denominator q; and the successive products of the natural numbers, $1\,.\,2$, $1\,.\,2\,.\,3, \ldots\, 1\,.\,2\ldots r$ are written underneath the successive powers of u.

or a negative numerical fraction.

In a similar manner, we shonld find

$$S(-m-m) = S(-2m) = (1+u)^{-2m} = \{S(-m)\}^2$$

$$S(-2m-m) = S(-3m) = (1+u)^{-3m} = \{S(-m)\}^3$$

. . . .

$$S\{-(q-1)m-m\} = S(-qm) = (1+u)^{-qm} = \{S(-m)\}^q.$$

Consequently, if $-qm = -p$, and therefore $-m = \frac{p}{q}$, we find

$$(1+u)^{-p} = \{S(-m)\}^q = \left\{S\left(-\frac{p}{q}\right)\right\}^q$$

and, therefore, extracting the q^{th} root on both sides,

$$(1+u)^{-\frac{p}{q}} = S\left(-\frac{p}{q}\right) = 1 - \frac{p}{q}u + \frac{\frac{p}{q}\left(\frac{p}{q}+1\right)}{1.2}u^2$$

$$-\frac{\frac{p}{q}\left(\frac{p}{q}+1\right)\left(\frac{p}{q}+2\right)}{1.2.3}u^3 + \dots (-1)^r \frac{\frac{p}{q}\left(\frac{p}{q}+1\right)\dots\left(\frac{p}{q}+r-1\right)}{1.2\;.\;.\;.\;r}u^r + ..$$

or under another and equivalent form,

$$(1+u)^{-\frac{p}{q}} = 1 - \frac{p}{q}u + \frac{p(p+q)}{q^2}\cdot\frac{u^2}{1.2} - \frac{p(p+q)\,(p+2q)}{q^3}\cdot\frac{u^3}{1.2.3}$$

$$+\;.\;.\;.\;.\;.\;(-)^r\cdot\frac{p(p+q)\;.\;.\;.\;\{p+(r-1)q\}}{q^r}\cdot\frac{u^r}{1.2\dots r} + \dots$$

We have thus shewn the perfect agreement of the general series for $(1+u)^n$, with the meaning which has been given to it in those cases, where the index admitted of interpretation, and where the operation which it denoted was capable of being defined and determined.

Examples. 312. Before we proceed to any further observations on the connection between $(1+u)^n$ and its developement, we shall give a few examples of the actual developement of the series corresponding to particular values of the index.

(1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots.$

This is the indefinite quotient of the division of 1 by $1+x$.

(2) $$(1+x)^{-2} = 1 - 2x + \frac{2.3}{1.2}\cdot x^2 - \frac{2.3.4}{1.2.3}x^3$$

$$+\frac{2.3.4.5}{1.2.3.4}x^4 - \&c.$$

$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \&c.$$

The coefficients are the second order of figurate numbers (Art. 259).

$$(3)\quad (1-x)^{-3}=1+3x+\frac{3.4}{1.2}x^2+\frac{3.4.5}{1.2.3}x^3$$
$$+\frac{3.4.5.6}{1.2.3.4}x^4+\&c.$$
$$=1+3x+6x^2+10x^3+15x^4+\&c.$$

The coefficients are the third order of figurate numbers.

The terms, in this case, are all positive, inasmuch as the even terms, which are *naturally* negative, involve odd powers of $-x$.

$$(4)\quad (a+x)^{-4}=a^{-4}\left\{1-4.\frac{x}{a}+\frac{4.5}{1.2}\cdot\frac{x^2}{a^2}\right.$$
$$\left.-\frac{4.5.6}{1.2.3}\cdot\frac{x^3}{a^3}+\&c.\right\}$$
$$=\frac{1}{a^4}\left\{1-4.\frac{x}{a}+10.\frac{x^2}{a^2}-20.\frac{x^3}{a^3}+35.\frac{x^4}{a^4}-\&c.\right\}$$
$$=\frac{1}{a^4}-4.\frac{x}{a^5}+10.\frac{x^2}{a^6}-20.\frac{x^3}{a^7}+35.\frac{x^4}{a^8}-\&c.$$

The coefficients are the figurate numbers of the fourth order.

$$(5)\quad \left(\frac{1}{x}-\frac{1}{a}\right)^{-5}=x^5\left(1-\frac{x}{a}\right)^{-5}$$
$$=x^5\left\{1+5.\frac{x}{a}+15.\frac{x^2}{a^2}+35.\frac{x^3}{a^3}+70.\frac{x^4}{a^4}+\&c.\right\}$$
$$=x^5+5.\frac{x^6}{a}+15.\frac{x^7}{a^2}+35.\frac{x^8}{a^3}+70.\frac{x^9}{a^4}+\&c.$$

The coefficients are the figurate numbers of the fifth order.

$$(6)\quad (1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1.1}{2^2}\cdot\frac{x^2}{1.2}+\frac{1.1.3}{2^3}\cdot\frac{x^3}{1.2.3}$$
$$-\frac{1.1.3.5}{2^4}\cdot\frac{x^4}{1.2.3.4}+\&c.$$
$$=1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}+\frac{5x^4}{128}-\frac{7x^5}{256}+\&c.$$

The terms of the arithmetical series

$$1,\ -1,\ -3,\ -5,\ -7,\ -\&c.\ldots$$

are the factors of the numerators of the coefficients.

$$(7)\quad \frac{1}{\sqrt{(1+x)}}=(1+x)^{-\frac{1}{2}}=1-\frac{1}{2}x+\frac{1.3}{2^2}\cdot\frac{x^2}{1.2}$$

$$-\frac{1.3.5}{2^3}\cdot\frac{x^3}{1.2.3}+\frac{1.3.5.7}{2^4}\cdot\frac{x^4}{1.2.3.4}-\&c.\ldots$$

$$=1-\frac{1}{2}x+\frac{3x^2}{8}-\frac{5x^3}{16}+\frac{35x^4}{128}-\frac{63x^5}{256}+\&c.$$

The terms of the arithmetical series

$$-1,\ -3,\ -5,\ -7,\ -\&c.\ldots$$

are the factors of the numerators of the coefficients.

$$(8)\quad (a+x)^{\frac{1}{3}}=a^{\frac{1}{3}}\left\{1+\frac{1}{3}\cdot\frac{x}{a}-\frac{1.2}{3^2}\cdot\frac{x^2}{1.2.a^2}\right.$$

$$\left.+\frac{1.2.5}{3^3}\cdot\frac{x^3}{1.2.3.a^3}-\frac{1.2.5.8}{3^4}\cdot\frac{x^4}{1.2.3.4.a^4}+\&c.\right\}$$

$$=a^{\frac{1}{3}}+\frac{x}{3a^{\frac{2}{3}}}-\frac{x^2}{9a^{\frac{5}{3}}}+\frac{5x^3}{81a^{\frac{8}{3}}}-\frac{10x^4}{243a^{\frac{11}{3}}}+\&c.$$

The terms of the arithmetical series

$$1,\ -2,\ -5,\ -8,\ -\&c.\ldots$$

are the factors of the numerators of the coefficients.

$$(9)\quad (a+x)^{-\frac{1}{3}}=a^{-\frac{1}{3}}\left\{1-\frac{1}{3}\cdot\frac{x}{a}+\frac{1.4}{3^2}\cdot\frac{x^2}{1.2.a^2}\right.$$

$$\left.-\frac{1.4.7}{3^3}\cdot\frac{x^3}{1.2.3.a^3}+\frac{1.4.7.10}{3^4}\cdot\frac{x^4}{1.2.3.4.a^4}-\&c.\right\}$$

$$=\frac{1}{a^{\frac{1}{3}}}-\frac{1}{3}\cdot\frac{x}{a^{\frac{4}{3}}}+\frac{2}{9}\cdot\frac{x^2}{a^{\frac{7}{3}}}-\frac{14}{81}\cdot\frac{x^3}{a^{\frac{10}{3}}}+\frac{35}{243}\cdot\frac{x^4}{a^{\frac{13}{4}}}-\&c.$$

The terms of the arithmetical series

$$-1,\ -4,\ -7,\ -10,\ \&c.\ldots$$

are the factors of the numerators of the coefficients.

$$(10)\quad (a^2-ax)^{\frac{3}{10}}=a^{\frac{3}{5}}\left\{1-\frac{3}{10}\cdot\frac{x}{a}+\frac{3.7}{10^2}\cdot\frac{x^2}{1.2.a^2}\right.$$

$$\left.-\frac{3.7.17}{10^3}\cdot\frac{x^3}{1.2.3.a^3}+\frac{3.7.17.27}{10^4}\cdot\frac{x^4}{1.2.3.4.a^4}-\&c.\right\}$$

The terms of the arithmetical series

$$3,\ -7,\ -17,\ -27,\ -37,\ -\&c.\ldots$$

are the factors of the numerators of the coefficients.

$$(11)\quad (a^2-ax)^{-\frac{3}{10}}=\frac{1}{a^{\frac{8}{5}}}\left\{1+\frac{3}{10}\cdot\frac{x}{a}+\frac{3.13}{10^2}\ \frac{x^2}{1.2.a^2}\right.$$

$$\left.+\frac{3.13.23}{10^3}\cdot\frac{x^3}{1.2.3.a^3}+\frac{3.13.23.33}{10^4}\cdot\frac{x^4}{1.2.3.4.a^4}+\&c.\right\}$$

$$(12)\quad (\sqrt[4]{a}+\sqrt[4]{x})^{\frac{4}{5}}=a^{\frac{1}{5}}\left\{1+\frac{4}{5}\cdot\frac{x^{\frac{1}{4}}}{a^{\frac{1}{4}}}-\frac{4.1}{5^2}\cdot\frac{x^{\frac{1}{2}}}{1.2.a^{\frac{1}{2}}}\right.$$

$$\left.+\frac{4.1.6}{5^3}\cdot\frac{x^{\frac{3}{4}}}{1.2.3.a^{\frac{3}{4}}}-\frac{4.1.6.11}{5^4}\cdot\frac{x}{1.2.3.4.a}+\&c.\right\}$$

$$=a^{\frac{1}{5}}+\frac{4}{5}\cdot\frac{x^{\frac{1}{4}}}{a^{\frac{1}{20}}}-\frac{2}{25}\cdot\frac{x^{\frac{1}{2}}}{a^{\frac{3}{10}}}+\frac{4}{125}\cdot\frac{x^{\frac{3}{4}}}{a^{\frac{11}{20}}}-\frac{11}{625}\cdot\frac{x}{a^{\frac{4}{5}}}+\&c.$$

$$(13)\quad (1+x)^{\frac{1}{\sqrt{2}}}=1+\frac{1}{\sqrt{2}}\cdot x+\frac{1(1-\sqrt{2})}{2}\cdot\frac{x^2}{1.2}$$

$$+\frac{1(1-\sqrt{2})(1-2\sqrt{2})}{2\sqrt{2}}\cdot\frac{x^3}{1.2.3}$$

$$+\frac{1(1-\sqrt{2})(1-2\sqrt{2})(1-3\sqrt{2})}{2^2}\cdot\frac{x^4}{1.2.3.4}-\&c.$$

(14) The 11th term of the series for $(a^3-x^3)^{\frac{7}{3}}$

$$=-\frac{2618}{4782969}\cdot\frac{x^{30}}{a^{23}}.$$

(15) The 11th term of the series for $(a^3-x^3)^{-\frac{7}{3}}$

$$=\frac{100230130}{4782969}\cdot\frac{x^{30}}{a^{37}}.$$

(16) The 17th term of the series for $(a+x)^{\frac{1}{2}}$

$$=-\frac{9694845}{1891371008}\cdot\frac{x^{16}}{a^{\frac{31}{2}}}.$$

(17) The 13th term of the series for $(a-x)^{\frac{1}{3}}$

$$=-\frac{1179256}{129140163}\cdot\frac{x^{12}}{a^{\frac{35}{3}}}.$$

(18) The 10th term of the series for $(\sqrt{3}-\sqrt{2})^{\frac{1}{5}}$

$$=-\frac{121771}{9765625}\cdot\frac{2^{\frac{9}{2}}}{3^{\frac{22}{5}}}=-\frac{1948336\times\sqrt{2}}{791015625\times 3^{\frac{2}{5}}}.$$

313. We shall now proceed to make some observations upon the general algebraical equivalence of $(1+u)^n$ and the series corresponding to it, and to point out the circumstances under which they may be considered as *arithmetically* as well as algebraically equivalent to each other.

Observations on the meaning of $(1+u)^n$ and its equivalent series.

Our interpretation of the meaning of $(1+u)^n$, as we have already seen, was founded upon the general principle of indices, which is symbolized by the equation

$$(1+u)^m\times(1+u)^n=(1+u)^{m+n},$$

whatever m and n may be: we were thence enabled to give a determinate and consistent interpretation of $(1+u)^n$, whenever n was a positive or negative integer

It is not necessary to assign its meaning for all values of n.

or a positive or negative numerical fraction: for other values of n (and n is a symbol which is equally general with any other symbol employed in Algebra) we have given no interpretation, and it does not necessarily follow that any discoverable interpretation exists for such values: the *algebraical form* however exists, independently of any interpretation of its meaning and the determination of the algebraical equivalence of any other form or series, must depend upon such *equivalent* form or series satisfying the requisite algebraical conditions.

Condition to be satisfied by the series for $(1+u)^m$.

314. Thus, if we take $S(m)$ as the representative of the series corresponding to $(1+u)^m$ (whether equivalent or not), we have found under all circumstances

$$S(m) \times S(n) = S(m+n),$$

an equation which strictly corresponds to

$$(1+u)^m \times (1+u)^n = (1+u)^{m+n}:$$

if, therefore, $S(m)$ be equivalent to $(1+u)^m$, under any discoverable circumstances where m is general in form, even though it may be specific in value, it must likewise, in virtue of this equation and the law of the permanence of equivalent forms, be equivalent also when m is general both in form and value.

Test of equivalence.

This general proposition therefore, furnishes the proper algebraical test of the equivalence of $(1+u)^m$ and $S(m)$, inasmuch as that series, and no other, satisfies the algebraical conditions upon which that equivalence depends.

Series given by the binomial theorem, identical with those derived by other operations denoted by the index.

315. The series given by this theorem are identical with those derived by other operations, for assigned and particular values of n.

Thus, if n be a positive whole number, the series for $(1+u)^n$ is equal as well as equivalent, to $(1+u)^n$, being derived from the repeated multiplication of $1+u$ into itself, and being made the foundation of the theorem.

If n be a negative whole number, the series for $(1+u)^{-n}$ or $\frac{1}{(1+u)^n}$, is the result of the division of 1 by the series for $(1+u)^n$.

If n be a positive fractional number, such as $\frac{p}{q}$, then $(1+u)^{\frac{p}{q}}$ denotes the q^{th} root of $(1+u)^p$: and if n be a negative fractional number, such as $-\frac{p}{q}$, then $(1+u)^{\frac{-p}{q}}$ denotes the q^{th} root of $(1+u)^{-p}$ or of $\frac{1}{(1+u)^p}$.

Rules for extracting roots derived from this theorem.

In neither of these cases can any general rule be formed for the performance of the operations denoted by the indices $\frac{p}{q}$ or $-\frac{p}{q}$, which is not derived from the theorem itself: and the results of such operations, derived from this theorem, must be consistent with the meaning attached to them, inasmuch as

$$\left\{S\left(\frac{p}{q}\right)\right\}^q = S(p) \text{ and } \left\{S\left(\frac{-p}{q}\right)\right\}^q = S(-p):$$

or in other words, the q^{th} power of the series corresponding to the q^{th} root of $S(p)$ in one case and of $S(-p)$ in the other, will reproduce those quantities which are respectively equivalent to $(1+u)^p$ and $(1+u)^{-p}$.

Square and cube roots.

316. The particular rules, considered in Chap. VII. for the extraction of the square and cube roots, are founded upon the same principle; and the series which they respectively give, must therefore coincide with those given by the theorem: for the square of the series in one case and its cube in the other, must reproduce the original quantity or expression, of which it was the square or cube root.

Cases in which $\overline{(1+u)}|^n$ is arithmetically equivalent to the series.

317. It thus appears that, the binomial $(1+u)^n$ and its corresponding series as given by the binomial theorem, are in all cases *equivalent* to each other, in the algebraical sense which is attached to that term. It remains to consider under what circumstances they are arithmetically equivalent or rather equal to each other: or when the arithmetical value of the result of the operation which is indicated by the index of arithmetical binomials, is equal to or may be made to differ from, the sum or aggregate of a determinable number of the terms of the corresponding series, by a quantity less than any which can be assigned.*

When the series is divergent.

318. In the first place, if the series is *divergent* for any *assignable* number of its terms, the sign = does not indicate the arithmetical equality of the quantities between which it is placed, inasmuch as the aggregation of any number of its terms, however great, will never approximate to a fixed and determinate value.

When the series is convergent, or becomes so after a certain number of terms.

319. We must confine our attention, therefore, to those series which are *convergent*, either from the beginning or after an assignable number of terms: in the first case, we approximate more nearly to the arithmetical value of the series, the greater the number of terms which we collect into one: in the second case, we commence with the aggregation of all the *divergent* terms, to which we add the approximate arithmetical value of the convergent part of the series, determined as in the first case.

Compared with geometric series.

320. It is by means of a comparison of the terms of these series with those of a class of series which are called *geometric*, that we are enabled to judge of the accuracy of such approximations, and to determine the limits of error to which they are subject: for this purpose it will

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* Quantities are arithmetically equal and equivalent, which are identical with each other, or which differ from each other by a quantity less than any which is arithmetically assignable.

be necessary to premise the theory of such series, both as regards their generation and the determination of their values.

Geometric series: their generation.

321. *Geometric series* are such as arise from the division of a by $1-x$, where a and x are any quantities whatsoever, whether simple or compound: the quotient

$$a+ax+ax^2+ax^3+\dots ax^r+\dots$$

may be taken as the general representative of such series: and under all circumstances, the fraction $\frac{a}{1-x}$ is algebraically equivalent to the series

$$a+ax+ax^2+ax^3+\dots\dots$$

indefinitely continued, without any reference to the specific values of the symbols a and x.

Sum of n terms,

322. If, however, we confine our attention to a determinate number (n) of terms of this series, we shall find

$$\frac{a-ax^n}{1-x}=a+ax+ax^2+\dots\dots ax^{n-1}:$$

the two members of the equation being equal as well as equivalent, inasmuch as the second member is the *actual quotient* of the division of $a-ax^n$ by $1-x$.

If we make $\frac{a}{1-x}=s$, then $\frac{a-ax^n}{1-x}=s-sx^n$, and the equation assumes the form

$$s-sx^n=a+ax+ax^2+\dots\dots ax^{n-1}.$$

and to infinity.

If the series be supposed to be indefinitely continued, n is infinite, and the term sx^n is no longer capable of algebraical expression: it remains to consider its arithmetical value under the same circumstances, for particular values of x.

Arithmetical sum of the infinite series when divergent,

323. In the first place, if x be greater than 1, x^n increases in value with the increase of n, and becomes infinite when n is so: the omission of the term sx^n will not therefore produce an arithmetical equality (according to the definition) between s and the series

$$a + ax + ax^2 + ax^3 + \ldots.$$

indefinitely continued.

when convergent.

In the second place, if x be less than 1, then sx^n becomes less and less the greater n is, and may thus be made less than any quantity which is assigned: under such circumstances, $s - sx^n$, when n is infinite, is arithmetically equal to s (according to the definition), and there exists, therefore, an arithmetical equality between s or $\frac{a}{1-x}$ and the series

$$a + ax + ax^2 + \ldots.$$

indefinitely continued.*

Meaning of the terms infinite and indefinite.

* The terms *infinite* and *indefinite* are frequently used indiscriminately by mathematical writers, though, if due regard be paid to propriety of language, they should be distinguished from each other: they are *negative* terms, whose meaning must be defined and determined by that of the terms *finite* and *definite*, which are respectively opposed to them.

A *finite* number, a *finite* line, a *finite* space, a *finite* time, would denote any number, line, space or time, which is either assigned or assignable: whilst the term *definite* could properly be applied to such of those quantities only as were assigned and determined: in other words, the term *finite* is more comprehensive than *definite*, being limited only by the power possessed by the mind of conceiving the relations which the magnitudes to which it is applied bear to other magnitudes of the same kind.

An *infinite* number, an *infinite* line, an *infinite* space, an *infinite* time, could bear no conceivable or expressible relation to a *finite* number, a *finite* line, a *finite* space, or a *finite* time: the term *indefinite*, properly speaking, when applied to these quantities, should imply nothing more than that they were not determined or not assignable.

Infinity and zero.

Magnitudes may be *infinitely* great as well as *infinitely* small, and the abstract term *infinity* should be, properly speaking, equally applicable to both, though confined, by the usage of language, exclusively to the former; whilst the term *zero* is exclusively applied to the latter: the general term *infinity* is superseded by the specific terms *immensity* and *eternity* in the case of space and time. The

Series in which the common ratio is $\frac{1}{2}$.

324. If we examine the particular case in which $x=\frac{1}{2}$, and where each succeeding term of the series is one-half of that which precedes it, we shall find

$$\frac{a}{1-x}=2a=a+\frac{a}{2}+\frac{a}{4}+\frac{a}{8}+\frac{a}{16}+\text{\&c.}$$

Symbol of infinity.

The symbol ∞ is used to denote magnitudes which are *infinitely* great, whilst the symbol 0 denotes those which are *infinitely* small: they are connected by the equations $\frac{a}{0}=\infty$ and $\frac{a}{\infty}=0$, where a is a *finite* magnitude: in the first case, we consider ∞ as the *quotient* of the division of a by 0; in the second, we consider 0 as the quotient of the division of a by ∞: and such results may be interpreted by considering the *dividend* as the *product* of the *divisor* and *quotient:* thus there is no *finite* number which, when multiplied into *zero* or an infinitely small number (fractional) will produce a finite product: there is no *finite* line, which multiplied into zero or an infinitely small line, will produce a *finite* area, and similarly in all other cases.

The product of *infinity* into *zero* or of ∞ into 0, in the sense which we have attached to those terms and symbols, *may* produce a finite quantity, but it does not follow that it must do so: the equation $\frac{a}{0}=\infty$, is universally true, when a is *finite:* but the same equation is equally true when a is *infinite:* in other words, $\frac{\infty}{0}=\infty$: in such a case, the *infinity* denoted by the symbol ∞ on one side of the equation is said to be *infinitely* greater than the *infinity* denoted by the same symbol ∞ on the other: such a result is due to the use of symbolical language only, inasmuch as the mind is incapable of conceiving the distinction between different *orders* of infinite magnitudes: we only mean to say, that if the symbol ∞ was employed to denote infinity in the same manner as an absolute magnitude, then the relation between $\infty\times\infty$ and ∞ would be the same as that between ∞ and 1, though we should denote $\infty\times\infty$ and ∞, when interpreted by the same term *infinity*, by the single symbol ∞.

Different orders of infinities, denoted by the same symbol.

Different orders of zeros.

In the same manner, $\infty\times 0$ may be equal to 0 as well as to a finite or infinite quantity: for, if we use the symbol 0 absolutely, we still denote $a\times 0$, $a\times 0\times 0$, &c. equally by the single symbol 0: for $\frac{a}{0}=\infty$, and therefore (symbolically) $\frac{a\times 0}{0\times 0}=\infty$, which, if considered without reference to this particular mode of derivation, would be denoted by $\frac{0}{0}=\infty$: in other words, different orders of *zeros* may appear as a result of symbolical language, as well as different orders of *infinities*, though in both cases it is equally impossible to conceive or express them when they present themselves independently of their symbolical connection.

Values of the expressions $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

The preceding considerations would lead us to the interpretation of the meaning of such expressions as $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which may equally denote a finite magnitude, or the sum of the series is twice the first term: it follows, therefore, that the first term of the series is equal to the

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magnitude, infinity or zero: for such expressions considered absolutely, would denote the quotient of *zero* divided by *zero*, or of *infinity* by *infinity*: and in the first case $\frac{0}{0}$ may be equal to 0, or a or ∞, inasmuch as we have already seen that $0 \times 0 = 0$, $0 \times a = 0$, and also that $0 \times \infty$ may be equal to 0: in the second case $\frac{\infty}{\infty}$ may be equal to 0, or a or ∞, inasmuch as we have already seen that $\infty \times 0$ may be equal to ∞, and also that $\infty \times a = \infty$ and $\infty \times \infty = \infty$: in both cases, therefore, there are three classes of magnitudes which equally answer the required conditions: when however such expressions as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ present themselves as particular states of an algebraical expression, which for other and general values of the symbols, admitted of an *equivalent* form, which under the same circumstances does not become $\frac{0}{0}$ or $\frac{\infty}{\infty}$, there no longer exists any indetermination respecting its value: for the form which is equivalent for all values of the symbols, must be true likewise for those which make the primitive expression $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

The following are examples:

$$(\alpha) \quad \frac{a^3 - x^3}{a^2 - x^2} = \frac{a^2 + ax + x^2}{a + x},$$

for all values of a and x: let $a = x$, then we find

$$\frac{0}{0} = \frac{3a^2}{2a} = \frac{3a}{2}.$$

$$(\beta) \quad \frac{a - ax^n}{1 - x} = a + ax + ax^2 + \dots\dots ax^{n-1},$$

for all values of a and x: let $x = 1$, then

$$\frac{0}{0} = a + a \times 1 + a \times 1^2 + \dots a \times 1^{n-1} = na.$$

$$(\gamma) \quad \frac{3a^2 - 2ax - 21x^2}{9a^3 - 24a^2x - 3ax^2 - 18x^3} = \frac{3a + 7x}{9a^2 + 3ax + 6x^2}.$$

If $x = \frac{a}{3}$, we have

$$\frac{0}{0} = \frac{1}{2a}.$$

$$(\delta) \quad \frac{\sqrt{(a^2 - x^2)} + \sqrt{(ax - x^2)}}{\sqrt{(a^3 - x^3)}} = \frac{\sqrt{(a + x)} + \sqrt{x}}{\sqrt{(a^2 + ax + x^2)}}.$$

If $x = a$, we have

$$\frac{0}{0} = \frac{\sqrt{2} + 1}{\sqrt{(3a)}}.$$

(ϵ)

sum of all those which follow it; and the same observation applies to any other term of the series.

325. The relation which exists between any one term and the sum of all those which follow it, in this particular series, where the *inverse* ratio of two consecutive coefficients is $\frac{1}{2}$, affords a very convenient measure of convergency for other series, where that ratio is greater or less than $\frac{1}{2}$; if it be greater than $\frac{1}{2}$, any given term is less than the sum of all those which follow it: but if it be less than $\frac{1}{2}$, then any given term is greater than the sum of all those that follow it.

Series in which the common ratio is less than $\frac{1}{2}$.

326. In many cases, where the common ratio or multiplier x, is variable and assignable at pleasure, a value of it may be found, which will make the sum of the series differ from its first term by a quantity less than any which may be assigned: thus, if a value of x be required, which makes the sum of all the terms of the series, after the first, equal to δ, then we shall have

Series in which the common ratio is assignable at pleasure.

$$\frac{a}{1-x}=a+ax+ax^2+ax^3+\&c.$$
$$=a+\delta.$$

Therefore $a=a+\delta-(a+\delta)x$, and consequently

$$\delta=(a+\delta)x, \text{ and also}$$

$$x=\frac{\delta}{a+\delta}.$$

$$(\epsilon)\quad \frac{a\sqrt{(ax)}-x^2}{a-\sqrt{(ax)}}=\frac{ax^{\frac{1}{2}}+a^{\frac{1}{2}}x+x^{\frac{3}{2}}}{\sqrt{a}}.$$

If $x=a$, we have

$$\frac{0}{0}=3a.$$

$$(\eta)\quad \frac{\frac{1}{1-x}}{1+\frac{1}{1-x}}=\frac{1}{2-x}.$$

If $x=1$, we have

$$\frac{\infty}{\infty}=1.$$

It is obvious, that any value of x which is less than $\frac{\delta}{a+\delta}$ $\left(\text{for example } \frac{\delta}{2(a+\delta)}\right)$, will make the sum of the series differ from its first term by a quantity less than δ: in other words, the sum of the series may be made to differ from its first term, by a quantity less than any which can be assigned.

The same conclusion would follow, if the series proceeded according to powers of rx, where r is a quantity either given or determinate, and x is assignable at pleasure: for in this case, we should find $rx = \frac{\delta}{a+\delta}$, and therefore $x = \frac{\delta}{r(a+\delta)}$; and consequently in this case also, as well as in the former, the sum of the series may be made to differ from its first term, by a quantity less than any which can be assigned.

Formation of limiting series.

327. The following proposition, combined with the conclusions given in the preceding articles, will enable us to ascertain, generally, under what circumstances we may pass from the *algebraical* to the *arithmetical* sum of a series, whose terms follow any assignable law.

Proposition.

"The terms of the geometric series

$$a + arx + ar^2x^2 + ar^3x^3 + \&c. \qquad (\alpha)$$

are severally either equal to or greater than, the corresponding terms of the series

$$a + a_1x + a_2x^2 + a_3x^3 + \&c. \qquad (\beta)$$

if r be the greatest inverse ratio of any two of its consecutive coefficients."

Proof. First case.

In the first place, let $\frac{a_1}{a}$ be greater than $\frac{a_2}{a_1}$, $\frac{a_3}{a_2}$, and all subsequent ratios of a similar kind: then we have

$\frac{a_1}{a} = r$, and therefore $a_1 = a\,r$: $\frac{a_2}{a_1}$ is less than r, and therefore $a_2 \angle a_1 r$ (Art. 23), and therefore also $\angle a\,r^2$, since $a_1 = a\,r$: $\frac{a_3}{a_2} \angle r$, and therefore $a_3 \angle a_2 r \angle a\,r^3$, since $a_2 \angle a\,r^2$; and similarly for all subsequent coefficients of the series (β): it follows, therefore, that the first and second terms of the geometric series (α), are equal to the first and second terms of the series (β), but that all the subsequent terms of the first series are severally greater than those corresponding to them in the second.

Second case.

In the second place, let any other inverse ratio, such as $\frac{a_n}{a_{n-1}}$, and not the first $\frac{a_1}{a}$, be the greatest, and therefore equal to r: we have, consequently, $\frac{a_1}{a} \angle r$, and therefore $a_1 \angle a\,r$: $\frac{a_2}{a_1} \angle r$, and therefore $a_2 \angle a_1 r \angle a\,r^2$, since $a_1 \angle a\,r$: and so on, until we come to a_{n-1}, which is less than $a\,r^{n-1}$: the next ratio, $\frac{a_n}{a_{n-1}} = r$, and therefore $a_n = a_{n-1} r \angle a\,r^n$, since $a_{n-1} \angle a\,r^{n-1}$; and similarly for all subsequent coefficients: it follows, therefore, that the terms of the series (α) after the first, are severally greater than those corresponding to them in the series (β).

Third case.

If the successive ratios $\frac{a_1}{a}$, $\frac{a_2}{a_1}$, $\frac{a_3}{a_2}$, as far as the n^{th} ratio $\frac{a_n}{a_{n-1}}$, be equal to each other, but greater than all those which follow them, then the n first terms of the series (α) and (β) are equal to each other; but all the subsequent terms of (α) are severally greater than those corresponding to them in the series (β).

328. The following proposition, though not dependent upon the one just given, may be demonstrated in a similar manner: we shall confine ourselves, therefore, to its simple enunciation.

Second proposition.

"The terms of the geometric series

$$a + a\rho x + a\rho^2 x^2 + a\rho^3 x^3 + \&c. \qquad (\alpha')$$

are severally equal to or less than the corresponding terms of the series

$$a + a_1 x + a_2 x^2 + a_3 x^3 + \&c. \qquad (\beta)$$

if ρ be the least inverse ratio of any two of its consecutive coefficients."

Superior and inferior limits.

329. The sums of the series (α) and (α') may be considered as *superior* and *inferior limits* to the sum of the series, (β): in other words, if S be the sum of the first series, σ of the second, and s of the third, then S is greater and σ less than s, and consequently s differs from either S or σ by a quantity less than $S - \sigma$.

Limits of a series converging from its first term.

330. It remains to examine the other consequences of this relation of the series to each other.

If rx and ρx be severally less than 1, the geometric series (α) and (α') are convergent, and their respective *arithmetical* sums are $\dfrac{a}{1 - rx}$ and $\dfrac{a}{1 - \rho x}$: it follows, therefore, that the sum of the series (β) differs from $\dfrac{a}{1 - rx}$ or $\dfrac{a}{1 - \rho x}$ by a quantity less than $\dfrac{a}{1 - rx} - \dfrac{a}{1 - \rho x}$ or $\dfrac{a(r - \rho)x}{(1 - rx)(1 - \rho x)}$.

Limits of a series converging after any number of terms.

331. Again, let us suppose the sum of any assigned number of the terms of the series (β) to be determined by actual aggregation, or by any other means: let T be the first of the following terms, and r and ρ the

greatest and least inverse ratios of their coefficients: it would follow, as in the first case, that the series

$$T + Trx + Tr^2x^2 + Tr^3x^3 + \&c.$$
$$T + T_1x + T_2x^2 + T_3x^3 + \&c.$$
$$T + T\rho x + T\rho^2x^2 + T\rho^3x^3 + \&c.$$

are arranged in the order of their magnitudes; and that the sum of the second is, therefore, intermediate in value to $\frac{T}{1-rx}$ and $\frac{T}{1-\rho x}$, which are the sums of the first and third series, if rx and ρx be severally less than 1.

If u, therefore, be the sum of all the terms of the original series which precede T, and s the sum of the whole series indefinitely continued, then $u + \frac{T}{1-rx}$ or $u + \frac{T}{1-\rho x}$ differs from s by a quantity less than

$$\frac{T}{1-rx} - \frac{T}{1-\rho x} \text{ or } \frac{T.(r-\rho)x}{(1-rx)(1-\rho x)}.$$

Conclusions from the preceding theory.

332. In converging series, or in those which become so, T grows less and less, the further it is removed from the beginning of the series, or from the term from which the convergency commences: and it is always possible to find a value of T, and then consequently of $\frac{T}{1-rx}$, $\frac{T}{1-\rho x}$, or of their difference $\frac{T(r-\rho)x}{(1-rx)(1-\rho x)}$, all of which bear a determined relation to T, which is less than any which may be assigned.

It follows, therefore, that we *approximate* more nearly to the sum of an indefinite series of convergent terms, the greater the number of them which we aggregate into one: and also, that such approximations may be carried to any required degree of accuracy, by the aggregation of a finite and determinable number of terms.

Their application to the series for $(1+x)^n$.

333. Let us consider the application of the conclusions just deduced, to different cases of the series for $(1+x)^n$.

The inverse ratio of the coefficients of any two consecutive terms, such as C_{t-1} and C_t, is

$$\frac{C_t}{C_{t-1}} = \frac{n-t+1}{t} = \frac{n+1}{t} - 1:$$

its greatest value, therefore, is n: its least value, to which it constantly approximates, as t increases, is -1*: it follows, therefore, that the three series,

$$1 + nx + n^2x^2 + n^3x^3 + \&c.$$

$$1 + nx + \frac{n(n-1)}{1.2}.x^2 + \frac{n(n-1)(n-2)}{1.2.3}.x^3 + \&c.$$

$$1 - x + x^2 - x^3 + x^4 - \&c.$$

are arranged in the order of their several magnitudes.

It is only, however, when the first and third series are convergent, or when nx and x are both less than 1, that their respective arithmetical sums are $= \frac{1}{1-nx}$ and $\frac{1}{1+x}$: and under such circumstances, the value of the series for $(1+x)^n$ differs from $\frac{1}{1-nx}$ by a quantity less than $\frac{1}{1-nx} - \frac{1}{1+x}$ or $\frac{(n+1)x}{(1-nx)(1+x)}$.

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* Positive quantities are considered as greater than negative quantities, and magnitudes generally are considered as following the order of the series

3, 2, 1, 0, —1, —2, —3, &c.

Point of convergency ascertained in series which are not convergent, from their first term.

334. If x be less than 1, the series for $(1+x)^n$ will become convergent, either from the beginning or after a determinable number of terms: in order to ascertain, in such cases, the *point* of convergency, we must determine when the inverse ratio of two consecutive terms is 1 or less than 1: for this purpose, we must make the general value of that ratio or $\left(\frac{n+1}{t}-1\right)x=1$, and therefore $t=\frac{n+1}{1+\frac{1}{x}}$; and the whole number which is either equal to t or next greater than t (when the value of t is fractional) will determine the point of convergency required.

Thus, if $n=\frac{3}{2}$ and $x=\frac{9}{10}$, we shall find $t=\frac{\frac{3}{2}+1}{1+\frac{10}{9}}$ $=\frac{45}{38}$: the whole number next greater than this value of t is 2, and the convergency commences, therefore, from the second term.

The preceding formula applies for all positive values of n, and for such negative values of n as are less than -1: but if the negative value of n be greater than -1, then the general inverse ratio of any two consecutive terms is necessarily negative if x be positive, and we must make, therefore, $\left(\frac{n+1}{t}-1\right)x=-1$: and consequently, $t=\frac{-n-1}{\frac{1}{x}-1}$. Thus, if $n=-3$, and $x=\frac{11}{12}$, then $t=\frac{3-1}{\frac{12}{11}-1}=22$; or the convergency commences after the 22d term.

Limits of the series for $(1+x)^n$.

335. If we find the sum of the first t terms of the series for $(1+x)^n$, by aggregation, and if we represent the $(1+t)^{\text{th}}$ term of the series by T; then the sum of the remaining terms of the series is included between the sums of the series

$$T - T\,.\left(1-\frac{n+1}{t}\right)x + T\,.\left(1-\frac{n+1}{t}\right)^2 x^2 - \&c.$$

$$\text{and } T - Tx + Tx^2 - \&c.$$

or between the limits $\dfrac{T}{1+\left(1-\dfrac{n+1}{t}\right)x}$ and $\dfrac{T}{1+x}$, when $\left(1-\dfrac{n+1}{t}\right)x$ and x are both of them less than 1: under these circumstances, the sum of the whole series for $(1+x)^n$ differs from the sum of its t first terms, by a quantity less than

$$\frac{\dfrac{n+1}{t}\,.\,Tx}{\left\{1+\left(1-\dfrac{n+1}{t}\right)x\right\}\{1+x\}}.$$

Thus, let it be required to determine within what limits of error, the square root of 5 may be determined from the aggregation of the first five terms of the series for $2\sqrt{(1+\frac{1}{4})}$.

The sixth term of the series or $T = .0000534$ nearly, and the sum or aggregate of the first five terms or 2.236026 exceeds the $\sqrt{5}$ by a quantity less than $\dfrac{12T}{235}$ or $.0000027225$.

Algebraical and Arithmetical summation of series: in what respects different.

336. It is of some importance to point out, in connection with the preceding theory, the essential difference which there is in the object proposed to be effected by the *algebraical and arithmetical summation* of series.

In the first case, we seek to determine the algebraical expression, by the *developement*[*] of which the series is generated.

In the second case, we seek to determine the accurate or approximate arithmetical value of the algebraical sum of the series, under a form which is independent of the particular operation which leads to the developement of the series.

Thus the algebraical sum of the series

$$1+\frac{1}{2}\cdot\frac{1}{4}-\frac{1}{2^2.1.2}\cdot\frac{1}{4^2}+\frac{1.3}{2^3.1.2.3}\cdot\frac{1}{4^3}-\frac{1.3.5}{2^4.1.2.3.4}\cdot\frac{1}{4^4}+\text{\&c.}$$

$$\text{is}\quad \sqrt{\left\{1+\frac{1}{4}\right\}}\quad\text{or}\quad\frac{1}{2}\sqrt{5},$$

the quantity from which it is generated by means of the binomial theorem; whilst the arithmetical sum can be determined *approximately* only, by the *aggregation* of its successive terms, the accuracy of the approximation increasing with the number of convergent terms which are thus collected into one.

Cases in which one sum may be derived directly from the other.

337. If the algebraical sums present themselves under rational forms, such as

$$\frac{a}{1-x},\quad (1+x)^{10},\quad \frac{1}{(a-x)^4},\quad \frac{1+x}{1-x},\quad \text{\&c.}$$

their arithmetical values, when specific rational values are assigned to the symbols, may be determined without reference to the series produced by their developement: under such circumstances, the arithmetical sums of the resulting series would be thus completely determined, in case such series were convergent.[†]

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[*] The term *developement*, in its most general sense, means the performance of an operation or of operations, which in the finite expression or algebraical sum are indicated only and not performed. Developement.

[†] When we speak of convergent series in this place, we mean not only those series where each succeeding term is less than the one before it, but also such as proceed according to the powers of some letter (with a specific value) or Convergent series.

number

Cases in which one sum cannot be derived directly from the other.

338. If, however, the algebraical sums do not present themselves under a rational form, or are not explicitly assigned nor assignable, their arithmetical values or sums can only be approximately determined by reference to the convergent series which result from their developement, or which can by any other means be shewn to correspond to them;* that is, their values or sums do not admit of determination, either accurately or approximately, independently of an operation or operations, either indicated or understood, which must lead to an indefinite series: thus, $\sqrt{3}$ is arithmetically as well as algebraically equivalent to

$$2 . \left\{1 - \frac{1}{2} \cdot \frac{1}{4} - \frac{1.1}{2^2} \cdot \frac{1}{1.2.4^2} - \frac{1.1.3}{2^3} \cdot \frac{1}{1.2.3.4^3} - \&c. \right\};$$

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number, or combination of such letters or numbers; for convergent and indefinite series which want this essential condition, *may*, in some cases, have no finite sums to which we may approximate by the aggregation of their terms; of this kind is the series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$$

of the reciprocals of the natural numbers, the sum of which is infinite, as may be readily shewn as follows: form the terms in succession into periods of

$$2, \ 2, \ 2^2, \ 2^3, \ 2^4, \ \&c.$$

the sum of the first period is greater than twice the second term, or greater than 1 or $\frac{1}{2}$; the sum of the second period is greater than twice its last term, which is $\frac{1}{2}$; the sum of the third period is greater than its last term multiplied into 2^2, which is $\frac{1}{2}$; the sum of the fourth period is greater than its last term multiplied into 2^3, which is $\frac{1}{2}$; and we shall thus find, that the last term in each period multiplied into the number of terms in that period, will always be $\frac{1}{2}$, which is less than the sum of all the terms in that period: and as the number of such periods is infinite, the sum of the series must exceed $\frac{1}{2} \times \infty$, and must therefore be infinite.

* There are innumerable cases where algebraical series may be obtained for quantities, which are not expressible by any finite form, in which the symbols are connected by ordinary algebraical signs: such quantities, whether symbolically exhibited or not, are called *transcendental*, and their properties or values are considered through the medium of the series which can be shewn to be equivalent to them.

but it is only by the aggregation of the terms of the series, that we can obtain an approximate value for $\sqrt{3}$, under a finite and rational form; that is, under a form which is independent of the operation which the sign $\surd$ indicates.

Different series correspond to different equivalent forms.

339. The same algebraical expression will generally admit of different equivalent forms, the developement of which will lead to very different series, which will possess very different degrees of convergency or divergency, when specific values are given to the symbols: the algebraical sums of such series will in all cases be the algebraical expressions from which they are severally generated, and will therefore be reducible to identity with each other: thus the series

$$(1) \quad 1 + n \cdot \frac{x}{1+x} + \frac{n(n+1)}{1.2} \cdot \frac{x^2}{(1+x)^2} + \frac{n(n+1)(n+2)}{1.2.3} \cdot \frac{x^3}{(1+x)^3} + \&c.$$

$$(2) \quad x^n \left\{ 1 + n \cdot \frac{1}{1+x} + \frac{n(n+1)}{1.2} \cdot \frac{x}{(1+x)^2} + \frac{n(n+1)(n+2)}{1.2.3} \cdot \frac{1}{(1+x)^3} + \&c. \right\}$$

$$(3) \quad 2^n \left\{ 1 + n \cdot \left(\frac{x-1}{x+1}\right) + \frac{n(n+1)}{1.2} \cdot \left(\frac{x-1}{x+1}\right)^2 + \frac{n(n+1)(n+2)}{1.2.3} \cdot \left(\frac{x-1}{x+1}\right)^3 + \&c. \right\}$$

$$(4) \quad 2^n x^n \left\{ 1 + n \cdot \left(\frac{1-x}{1+x}\right) + \frac{n(n+1)}{1.2} \cdot \left(\frac{1-x}{1+x}\right)^2 + \frac{n(n+1)(n+2)}{1.2.3} \cdot \left(\frac{1-x}{1+x}\right)^3 + \&c. \right\}$$

result severally from

$$\left(1-\frac{x}{1+x}\right)^{-n}, \quad x^n\left(1-\frac{1}{1+x}\right)^{-n}, \quad 2^n\left(1-\frac{x-1}{x+1}\right)^{-n}$$

$$\text{and} \quad 2^n x^n\left(1-\frac{1-x}{1+x}\right)^{-n},$$

which are equivalent forms of $(1+x)^n$; by such transformations we are sometimes enabled to convert diverging series into others which are convergent, or convergent series into others which are more so.

The arithmetical summation of series, as we have already stated, regards generally the result of an operation which is most conveniently and expeditiously performed by means of it, or the nature of which is known only, as far as it admits of being ascertained, from the series itself, and by means of which alone the operation in question can be performed; in all cases, we should naturally select such forms of the series, when more than one presented itself, as converged most rapidly, and were at the same time independent of the particular operation which was required to be performed; though it may sometimes happen that a less convergent series would be preferred, in consequence of its terms admitting of more rapid arithmetical calculation. It will be found in the practical applications of analysis, that there are few subjects of greater importance than the invention of such series, and the selection of such forms as shorten as much as possible operations of very frequent occurrence; we are not yet prepared, however, for the further prosecution of such enquiries, and we shall conclude therefore the subject which has already detained us so long, by a few observations upon diverging series.

Divergent series: their Algebraical sums.

340. The algebraical sums of series, which become divergent when particular values are given to the symbols, are the algebraical expressions by which the series are generated, and are therefore perfectly independent of their divergency

or convergency; but it is in the latter case only, when the series admits of arithmetical summation, that we are at liberty to neglect the peculiar form of the expression, with which the series is essentially connected; thus, $\frac{1}{1-x}$ is the algebraical and also the arithmetical sum of the series

$$1 + x + x^2 + x^3 + x^4 + \&c.$$

when x is less than 1; but if x be greater than 1, then $\frac{1}{1-x}$ is the algebraical sum of the series

$$1 + x + x^2 + x^3 + \&c.$$

only so long as it preserves its primitive form; thus, $\frac{1}{1-2}$ is algebraically equivalent to

$$1 + 2 + 2^2 + 2^3 + 2^4 + \&c.;$$

but it is no longer so, if $\frac{1}{1-2}$ be reduced to the equivalent algebraical quantity $\frac{1}{-1}$ or -1: for in this latter case there is no peculiar and essential connection between the algebraical sum and the series generated.

In a similar manner, $\frac{1}{1-1}$ and $\frac{1}{0}$ are algebraically equivalent to each other and to ∞; but it is the first of these expressions only, which is algebraically equivalent to the series

$$1 + 1 + 1 + 1 + \&c.$$

inasmuch as it is the algebraical form by which the series is generated.

No Arithmetical sums of divergent series.

341. In the cases which we have just been considering, the algebraical sums do not give arithmetical values, upon the substitution of arithmetical values of the symbols: but there are diverging series, whose terms are alternately

positive and negative, whose algebraical sums give real arithmetical values, upon such substitution of arithmetical values of the symbols: of this kind are the series corresponding to $\frac{1}{1+x}$, for all values of x which are greater than 1: thus,

$$\frac{1}{1+2} = 1 - 2 + 4 - 8 + 16 - 32 + \&c.;$$

in this case, however, it is $\frac{1}{1+2}$, and not $\frac{1}{3}$ which is arithmetically equal to it, which is the algebraical sum of the series, inasmuch as the series is essentially connected with this peculiar form of the expression: neither is $\frac{1}{3}$ the arithmetical sum of this series, at least if proper regard be had to the meaning which we have hitherto attached to the term: for in the first place, in a system of arithmetical algebra, the division of 1 by 3, or by $1+2$, would be impossible, and no such series could be generated: and in the second place, if we allowed the existence of the series, we should not approximate to $\frac{1}{1+2}$ or $\frac{1}{3}$, or any other quantity, by the aggregation of any number of its terms.

In speaking, therefore, of the sums of such series, it must be kept in mind, that we mean algebraical and not arithmetical sums: and also, that we are not at liberty to change the forms of the expressions by which such series are generated.*

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* The series

$$1 - 1 + 1 - 1 + 1 - \&c. \ldots\ldots$$

which is generated by $\frac{1}{1+1}$, has been considered by Leibnitz as equal to $\frac{1}{2}$, for the following reasons. If the number of terms of the series be even, its

Developement of a polynomial.

342. We shall now proceed to consider the developement of a polynomial, and to ascertain the law of its general term, for all values of the index.

Let

$$(a + aa_1 + aa_2 + \dots aa_m)^n$$

be the polynomial whose index n is a general symbol; make

$$u = a_1 + a_2 + \dots . a_m,$$

when the polynomial will assume the form

$$(a + au)^n = a^n (1 + u)^n;$$

the developement of which, by the binomial theorem, gives us

$$a^n(1 + u)^n = a^n \left\{ 1 + nu + \frac{n(n-1)}{1 . 2} u^2 \right.$$

$$\left. + \dots \frac{n(n-1) \dots (n-r+1)}{1 . 2 \quad \dots . \quad r} u^r + \&c. \right\}.$$

The $(1 + r)^{\text{th}}$ term of this series is

$$\frac{n(n-1) \dots (n-r+1)}{1 . 2 \quad \dots . \quad r} . a^n u^r$$

$$= \frac{n(n-1) \dots . (n-r+1)}{1 . 2 \quad \dots \quad r} a^n (a_1 + a_2 + \dots a_m)^r.$$

Inasmuch as r is necessarily a whole number, the general term of

$$(a_1 + a_2 + \dots a_m)^r$$

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its sum is 0; if their number be odd, its sum is 1: but an infinite number is neither odd nor even, but partakes equally of the nature of both: the sum of the series, therefore, must be equally related to its sums upon both hypotheses, and must therefore be $\frac{1}{2}$, which is the arithmetical mean between them. Such reasonings as these would be admissible, if Algebra was a science composed of a series of insulated results, each of which was established upon independent principles, and not a complete system of necessary conclusions from the same common assumed principles, connected by demonstrative evidence.

may be derived immediately from the theorem given in Art. 250; and if

$$\alpha_1 + \alpha_2 + \dots \alpha_m = r$$

be the sum of the indices of a_1, a_2, .. a_m respectively in any particular term, the term itself will be

$$\frac{r(r-1)\dots 2.1}{1.2..\alpha_1 \times 1.2..\alpha_2 \times \dots\dots \times 1.2..\alpha_m} . a_1^{\alpha_1} a_2^{\alpha_2} \dots\dots a_m^{\alpha_m};$$

consequently, the general term of the polynomial will be

Its general term

$$\frac{n(n-1)\dots\dots(n-r+1)}{1.2 \quad \dots\dots \quad r}$$

$$\times \frac{r(r-1)\dots\dots 2.1}{1.2\dots\alpha_1 \times 1.2\dots\alpha_2 \times \dots\dots \times 1.2\dots\alpha_m} a^n a_1^{\alpha_1} a_2^{\alpha_2} \dots a_m^{\alpha_m}$$

$$= \frac{n(n-1)\dots(n-r+1)}{1.2..\alpha_1 \times 1.2..\alpha_2 \times \dots \times 1.2..\alpha_m} a^n a_1^{\alpha_1} a_2^{\alpha_2} \dots\dots a_m^{\alpha_m}.$$

If the original polynomial had presented itself under the form

$$(a + a_1 + a_2 + \dots a_m)^n,$$

under different circumstances.

its general term would have been

$$\frac{n(n-1)\dots\dots(n-r+1)}{1.2\dots\alpha_1 \times 1.2\dots\alpha_2 \times \dots\dots \times 1.2\dots\alpha_m} a^n . \left(\frac{a_1}{a}\right)^{\alpha_1} \left(\frac{a_2}{a}\right)^{\alpha_2} \dots \left(\frac{a_m}{a}\right)^{\alpha_m}$$

$$= \frac{n(n-1)\dots\dots\dots(n-r+1)}{1.2\dots\alpha_1 \times 1.2\dots\alpha_2 \times \dots\dots \times 1.2..\alpha_m} . a^{n-r} a_1^{\alpha_1} a_2^{\alpha_2} \dots a_m^{\alpha_m}.$$

If we had supposed, as is most commonly the case, that the terms of the polynomial were arranged according to the powers of some one letter, such as x, as appears in the following form,

$$(a + a a_1 x + a a_2 x^2 + \dots\dots a a_m x^m)^n,$$

(where the index of x is equal to the distinctive number which is attached to each letter), then the general term which we are now considering would become

$$\frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots a_1 \times 1.2\ldots a_2 \times \ldots\ldots \times 1.2\ldots a_m} . a^n a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_m^{\alpha_m} . x^{\alpha_1 + 2\alpha_2 + \ldots m\alpha_m}.$$

Meaning of general term when the series is arranged according to the powers of one symbol.

343. In this last case, however, the general term of the series will generally have an extended signification, as comprehending all those terms in the developement, which involve the same assigned power of x; for it is usual and in most cases necessary, to arrange the terms of the series resulting from such developements, according to the powers of some one letter: such a term, which is necessarily compound, may be formed from the general term already determined, in the following manner.

Rule for its formation.

"Let the index of x in the required term be t: break the number t into parts, whether equal or different, between 1 and m inclusive, and not exceeding n in number, as often as possible: the numbers in these several *fractures* will correspond to the several particular terms which form the compound term required."

For the several numbers in these fractures, whether repetitions or not, will correspond to the indices of x, in the terms of the polynomial, which enter into each *particular* term, and their sum will form the index of x, which characterizes the compound term: the coefficients of these several terms will be determined from the law which has been assigned, when the terms or their powers, of the polynomial are given, which enter in each combination.

When the index n *is* a whole number, the number of parts in each *fracture* must not exceed n, otherwise $a_1 + a_2 + \ldots a_m$ would exceed n, and the term corresponding to them therefore could not exist in the developement.

344. The law of formation of these and all other series is included in that of their general terms; and it is for

Law of this developement true for all values of the index.

this reason that we have confined our attention to the general term of the developement of polynomials in the preceding investigations: the process which we have followed in deriving it, shews that it is equally true for all values of the index n, the developement being necessarily co-extensive with that of the binomial $(1+u)^n$.

It likewise appears, that integral powers only of all terms of the polynomial after the first, present themselves in the resulting series, the first term a being the only one which is affected with the general index n: this may be always separated from the rest of the series, being treated as a common factor of all its terms, precisely in the same manner as has been so frequently exemplified in the case of binomials (Art. 307.): thus,

(1) $(x - x^2 + x^3)^n = x^n (1 - x + x^2)^n$.

(2) $(a - 3bx + 5cx^2 - 7dx^3)^{\frac{2}{3}}$

$$= a^{\frac{2}{3}}\left(1 - \frac{3bx}{a} + \frac{5cx^2}{a} - \frac{7dx^3}{a}\right)^{\frac{2}{3}}.$$

(3) $(4 - 12x + 20x^2 - 32x^3 + 48x^4)^{\frac{3}{2}}$

$= 8(1 - 3x + 5x^2 - 8x^3 + 12x^4)^{\frac{3}{2}}$.

(4) $\left(\frac{2ab^m}{c} - \frac{3c^2}{4a^2b^{2m}} - \frac{2ab^2}{3c} - \frac{5b}{3a}\right)^3$

$$= \frac{8a^3b^{3m}}{c^3}\left(1 - \frac{3c^3}{8a^3b^{3m}} - \frac{c}{3b^{m-2}} - \frac{5c}{6a^2b^{m-1}}\right)^3.$$

Examples.

345. The following are examples, both of the deduction of the series corresponding to polynomials, and of the determination of assigned terms of them, whether simple or compound.

(1) $(a+b+c)^5 = a^5 + 5a^4b + 5a^4c + 10a^3b^2 + 20a^3bc$ $+ 10a^3c + 10a^2b^3 + 30a^2b^2c + 30a^2bc^2 + 10a^2c^3 + 5ab^4$ $+ 20ab^3c + 30ab^2c^2 + 20abc^3 + 5ac^4 + b^5 + 5b^4c + 10b^3c^2$ $+ 10b^2c^3 + 5bc^4 + c^5$.

The number of terms in this developement $=\frac{6\times 7}{1\times 2}=21$. (Art. 260.)

(2) $$\left(2a-\frac{3b}{2}+\frac{2c}{3b}\right)^4=16a^4-48a^3b+54a^2b^2$$

$$-27ab^3+\frac{81b^4}{16}+\frac{64a^3c}{36}-48a^2c+36abc-9b^2c$$

$$+\frac{32a^2c^2}{3b^2}-\frac{16ac^2}{b}+6c^2-\frac{64ac^3}{27b^3}+\frac{16c^3}{9b^2}+\frac{16c^4}{81b^4}.$$

The number of terms $=\frac{5.6}{1.2}=15$.

(3) To find the term in the developement of

$$(a-b-c)^7,$$

which involves $a^2b^3c^2$.

The term required

$$=\frac{7.6.5.4.3}{1.2.3.1.2}a^2(-b)^3(-c)^2=-21\,a^2b^3c^2.$$

(4) To find the term in the developement of

$$(a-b+c-d+e)^{15},$$

which involves $ab^2c^3d^4e^5$.

The term required

$$=\frac{15.14\,.....\,3.2}{1.2\,.\,1.2.3\,.\,1.2.3.4\,.\,1.2.3.4.5}a(-b)^2c^3(-d)^4e^5$$

$$=5405400ab^2c^3d^4e^5.$$

(5) To find the term in the developement of

$$(1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5)^{10},$$

which involves x^7.

The number 7 may be broken into parts between 1 and 5 in the following different ways,

(1) 5 . 2.
(2) 5 . 1 . 1.
(3) 4 . 3.
(4) 4 . 2 . 1.
(5) 4 . 1 . 1 . 1.
(6) 3 . 3 . 1.
(7) 3 . 2 . 2.
(8) 3 . 2 . 1 . 1.
(9) 3 . 1 . 1 . 1 . 1.
(10) 2 . 2 . 2 . 1.
(11) 2 . 2 . 1 . 1 . 1.
(12) 2 . 1 . 1 . 1 . 1 . 1.
(13) 1 . 1 . 1 . 1 . 1 . 1 . 1.

The term required will therefore be

$$\begin{aligned}
&\Big\{\frac{10.9}{1.1}\cdot a_2a_5+\frac{10.9.8}{1.2.1}a_1^2a_5+\frac{10.9}{1.1}\cdot a_3a_4\\
&+\frac{10.9.8}{1.1.1}\cdot a_1a_2a_4+\frac{10.9.8.7}{1.2.3.1}\cdot a_1^3a_4+\frac{10.9.8}{1.1.2}\cdot a_1a_3^2\\
&+\frac{10.9.8}{1.2.1}\cdot a_2^2a_3+\frac{10.9.8.7}{1.2.1.1}a_1^2a_2a_3+\frac{10.9.8.7.6}{1.2.3.4.1}\,a_1^4a_3\\
&+\frac{10.9.8.7}{1.1.2.3}a_1a_2^3+\frac{10.9.8.7.6}{1.2.3.1.2}a_1^3a_2^2\\
&+\frac{10.9.8.7.6.5}{1.1.2.3.4.5}\cdot a_1^5a_2+\frac{10.9.8.7.6.5.4}{1.2.3.4.5.6.7}\cdot a_1^7\Big\}x^7.
\end{aligned}$$

(6.) $(1+x+x^2)^6=1+6x+21x^2+50x^3+90x^4+126x^5+141x^6+126x^7+90x^8+50x^9+21x^{10}+6x^{11}+x^{12}$.

This developement is effected by breaking the numbers 1, 2, 3, 4, 5, 6, (for the coefficients afterwards recur in an inverse order) into parts between 1 and 2 inclusive, and collecting the corresponding terms severally into one: thus, the middle term involving x^6 is formed of the terms (after the first) corresponding to

2 . 2 . 2.
2 . 2 . 1 . 1.
2 . 1 . 1 . 1 . 1.
1 . 1 . 1 . 1 . 1 . 1,

which are as follows:

$$\frac{6.5.4}{1.2.3}(x^2)^3 + \frac{6.5.4.3}{1.2.1.2}(x)^2(x^2)^2 + \frac{6.5.4.3.2}{1.1.2.3.4}(x)^4 x^2$$

$$+\frac{6.5.4.3.2.1}{1.2.3.4.5.6}(x)^6 = 20x^6 + 90x^6 + 30x^6 + x^6 = 141x^6.$$

(7) To find the middle term of $(1+x+x^2)^{12}$. This term will involve x^{12}, and will be found to be

$$= 73789x^{12}.$$

More generally, let it be required to find the middle term of $(1+x+x^2)^n$. The middle term involves x^n and the number n, if broken into parts between 1 and 2 inclusive, will present the following series of *fractures*.

(1) n units: n parts.
(2) $(n-2)$ units: one 2 $(n-1)$ parts.
(3) $(n-4)$ units: two 2^s $(n-2)$ parts.
(4) $(n-6)$ units: three 2^s $(n-3)$ parts,
$\{(r+1)\}$ $(n-2r)$ units: $r..2^s$ $(n-r)$ parts.

The successive numerators of the coefficients are

$$n(n-1)\ldots2.1,\ n(n-1)\ldots2,\ n.(n-1)\ldots3$$
$$n(n-1)\ldots4,\ldots\ldots\ n(n-1)\ldots(r+1)\ldots\&c.$$

the successive denominators are

$$1.2\ldots n,\ 1.2\ldots(n-2)\times1.,\ 1.2\ldots(n-4)\times1.2,$$
$$1.2\ldots(n-6)\times.1.2.3.,\ldots1.2\ldots(n-2r)\times1.2\ldots r,\ldots$$

It follows therefore that the coefficient of the middle term* required is

$$1 + \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)(n-3)}{1.2.\quad 1.\quad 2}$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.\quad 3.\quad 1.\quad 2.\quad 3} +$$

$$+ \frac{n(n-1)\ldots\ldots\{n-(2r+1)\}}{1.2\ldots.r \times 1.2\ldots.r} + \&c.$$

(8) $1 + x + x^2 + x^3)^6 = 1 + 6x + 21x^2 + 56x^3 + 120x^4 + 216x^5 + 336x^6 + 456x^7 + 546x^8 + 586x^9 + 546x^{10} + 456x^{11} + 336x^{12} + 216x^{13} + 120x^{14} + 56x^{15} + 21x^{16} + 6x^{17} + x^{18}$.

(9) To find the term of the developement of

$$(1 + x + x^2 + x^3 + x^4)^{10}$$

which involves x^{12}.

The number 12 may be broken into parts, between 1 and 4 inclusive, not exceeding 10 in number, in 31 different ways, the first being 4, 4, 4, and the last

2, 2, 1, 1, 1, 1, 1, 1, 1, 1.

The sum of all the terms corresponding to these *fractures* is $182005x^{12}$†.

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* The square of the coefficient of the middle term of $(1+x+x^2)^{2n}$ is equal to the sum of the squares of the coefficients of the series for $(1+x+x^2)^n$; and the same relation exists between the middle term of $(1+x+x^2+\ldots x^m)^{2n}$ and the sum of squares of the coefficients of the series for $(1+x+x^2+\ldots x^m)^n$: for if we make $(1+x+x^2+\ldots x^m)^n = 1 + a_1 x + a_2 x^2 + \ldots a_{nm} x^{nm}$

$$(1+x+x^2+x^m)^n = 1 + a_1 x + a_2 x^2 + a_{nm} x^{nm},$$

it will readily appear that the term in their product which involves x^{nm}, which is the middle term of $(1+x+x^2+\ldots x^m)^n$, will be

$$= (1 \times a_{nm} + a_1 \times a_{nm-1} + a_2 \times a_{nm-2} + \ldots + a_{nm-1} \times a_1 + a_{nm} \times 1)x^{nm}$$

$$= (1^2 + a^{12} + a_2^2 + \ldots\ldots a_{nm-1}{}^2 + a_{nm}{}^2)x^{nm},$$

since $1 = a_{nm}$, $a_1 = a_{nm-1}$, $a_2 = a_{nm-2}$, and so on, for the coefficients of all the terms which are equidistant from the beginning and the end.

† In this and similar cases, the term in question may be determined, as well as the developement of the series very easily effected, by considering $(1+x+x+x^3+x^4)^{10} = \left(\frac{1-x^5}{1-x}\right)^{10}$ and by multiplying together the series for $(1-x^5)^{10}$ and $(1-x)^{-10}$.

$$(10.)\quad (2 - 5x - 7x^2 + x^3 + 3x^4)^5 = 32 - 400x$$
$$+ 1440x^2 + 680x^3 - 11390x^4 + 1955x^5 + 47025x^6$$
$$+ 5435x^7 - 111845x^8 - 71145x^9 + 108073x^{10}$$
$$+ 119495x^{11} - 36185x^{12} - 86055x^{13} - 8165x^{14}$$
$$+ 31441x^{15} + 9465x^{16} - 5715x^{17} - 2565x^{18} + 405x^{19}$$
$$+ 243x^{20}.$$

Problem in chances.

346. The solution of the following problem is connected with the formulæ and methods we have just been considering:

"What is the *chance* of throwing any assigned number $m + n$ in n throws with one die, or in one throw with n dice?"

The faces of the die being marked with the numbers 1, 2, 3, 4, 5, 6, the *minimum* throw is clearly n, when n aces are thrown.

It would be the same problem therefore if the faces of the die were marked with 0, 1, 2, 3, 4, 5, and the number required to be thrown was not $m + n$ but m.

The whole number of combinations both favourable and unfavourable is 6^n (Art. 245.); but those combinations are alone favourable, the sum of the numbers in which is equal to m: if we represented therefore the several faces of the die by x^0, x^1, x^2, x^3, x^4, x^5, the number of those favourable cases would be that coefficient of the term in the developement of $(1 + x + x^2 + x^3 + x^4 + x^5)^n$, which involved x^m: for that coefficient would denote the number of all those combinations of the faces or their representatives which made up the required sum.

The chance required will be found by dividing this coefficient by 6^n. (Art. 266.)

It is obvious that the same problem would be solved by making

$$(1 + x + x^2 + x^3 + x^4 + x^5)^n = \left(\frac{1 - x^6}{1 - x}\right)^n,$$

and collecting those terms in the product of the series for

$$(1 - x^6)^n \text{ and } (1 - x)^{-n}$$

which involved x^m: the coefficient of this term is as follows;

$$\frac{n(n+1)\ldots(n+m-1)}{1.2\ldots\ldots m} - \frac{n.(n+1)\ldots(n+m-7)}{1.2\ldots(m-6)}.n$$

$$+ \frac{n(n+1)\ldots(n+m-13)}{1.2\ldots(m-12)}.\frac{n(n-1)}{1.2.} - \&c.$$

$$\ldots+(-1)^r.\frac{n(n+1)\ldots(n+m-6r-1)}{1.2\ldots(m-6r)}.\frac{n.(n-1)..(n-r+1)}{1.2\ldots r}.+\&c.$$

the series terminating, when $6r + 1$ is equal to, or greater than, m.

Examples. 347. To find the chance of throwing 17 in three throws with a single die.

Since $17 - 3 = 14$, we must break the number 14 into parts between 1 and 5 inclusive, not exceeding 3 in number: there is only one such fraction, which is

5, 5, 4,

and the coefficient corresponding is

$$\frac{3.2.1}{1.2.1} = 3.$$

The chance therefore $= \dfrac{3}{6^3} = \dfrac{1}{72}$.

348. To find the chance of throwing 18 in one throw with 5 dice.

Since $18 - 5 = 13$, the number 13 must be broken into parts, between 1 and 5 inclusive, not exceeding 5 in number: the number of such *fractures* is 20, beginning with 5, 5, 3, and ending with 3, 3, 3, 2, 2: the sum of the coefficients of the corresponding terms in the developement of $(1 + x + x^2 + x^3 + x^4 + x^5)^5$ is 780, which denotes the number of cases favourable to the hypothesis made: the chance therefore required

$$= \frac{780}{6^5} = \frac{780}{7776}.$$

The same conclusion may be much more readily deduced from the second formula, which gives us for the favourable cases

$$= \frac{5.6\ldots.17}{1.2\ldots\ 13} - \frac{5.6\ldots11}{1.2..7}.5$$

$$+ 5.\frac{5.4}{1.2} = 780.$$

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CHAP. XI.

ON RATIOS AND PROPORTIONS.

The term Ratio.

349. THE term *Ratio*, in ordinary language, is used to express the relation which exists between two quantities of the same kind with respect to magnitude: thus we speak of the ratio of two numbers, of two lines, of two areas, of two forces, of two periods of time and of any other concrete quantities, the relation of whose magnitudes to each other admits of being estimated.

Importance of ascertaining its ordinary meaning.

350. The definition of ratio in Algebra, like all other definitions in that science, must be an *assumption* adapted as much as possible to its popular usage, in Arithmetic or Geometry or in both, in order that the conclusions which are founded upon it or represented by means of it, may be transferred to those subordinate sciences: in order properly to effect this object, it will be necessary to commence by endeavouring to ascertain what the meaning and usage of this term in those sciences really is.

In what manner represented.

351. A *ratio*, (the word is here used absolutely) consists of two terms or members, which are denominated the *antecedent* and the *consequent:* it is denoted in arithmetic as well as in geometry, by writing the antecedent before the consequent, with two dots, one above the other, between them: thus the ratio of 3 to 5 is written as follows; $3 : 5$. In a similar manner, if a and b denoted any two other numbers, lines or other magnitudes of the same kind, their ratio, whatever it may mean, would be denoted by $a : b$.

Geometrical representation of ratio.

352. Such a mode of representing a ratio, merely exhibits its terms to the eye as objects of comparison, and consequently conveys to the mind no idea of *absolute* magnitude: it may be called the *geometrical* representation of ratio, being the only one which is used in that science: for

No geometrical definition of ratio.

though in geometry we may agree upon various modes of representing ratios, yet they must all of them be equally arbitrary or independent of each other: for there is no geometrical definition of ratio, by which the equivalence of different modes of representation may be ascertained as necessary consequences of it: for ratio is said to be the relation of quantities, of the same kind with respect to magnitude, a description of its meaning much too vague to be considered as a *definition*, and therefore not capable of being made the foundation of other propositions respecting it: it is for this reason that ratios in geometry are only considered in connection with each other, as constituting or not constituting a *proportion*.

Popular meaning of the term ratio.

353. A little examination however of some of the conditions which ratios, taken according to the popular usage of the term, must satisfy, will lead to an arithmetical mode of representing them, by which their absolute magnitude may be ascertained, and which will thus conduct us to an arithmetical and also to an algebraical definition of ratio, which will be independent of the connection of ratios with each other: for it is perfectly conformable to our common idea of ratios, to consider them in the first place, as necessarily the *same* for the *same* magnitudes, in whatever manner they may be represented; and in the second place, as independent of the specific affections or properties (of the same kind) of the magnitudes themselves.

Changes which the terms of the same ratio may undergo.

354. Thus, if two lines admitted of resolution into 3 and 5 parts respectively, which were equal to each other, the lines themselves might be correctly represented by the numbers 3 and 5, and their ratio by $3 : 5$. But the common unit of their primary division is itself divisible into 2, 3 or m equal parts, and the numbers of these successive parts which the original lines would contain, would be severally 6 and 10, 9 and 15, $3m$ and $5m$, which might denote them equally with the original numbers 3 and 5; their ratio therefore, which remains the same, in conformity with the principle referred to, would be equally represented by $6 : 10$, $9 : 15$, $3m : 5m$.

Again, this mode of representing lines and their ratio, which possess this particular relation to each other, is equally applicable to any other magnitudes of the same kind which possess the same relation to each other: thus two areas, two solids, two forces, two periods of time, may be so related to each other, as to admit of resolution into 3 and 5 parts or units respectively, which are equal to each other: under such circumstances they must admit likewise of resolution into numbers of parts or units equal to each other, which are any equimultiples of 3 and 5: such pairs of numbers therefore, will equally represent those magnitudes, and will likewise equally form the terms of the ratio which expresses their relation to each other.

Conclusions thence deduced.

355. The preceding observations will conduct us naturally to the following conclusions:

(1) Magnitudes of the same kind, which admit of resolution into any numbers of parts or units, which are equal to each other, may be properly represented by such numbers, or by any equimultiples* of them.

(2) The numbers which represent two magnitudes of the same kind will form the terms of the ratio, which expresses their relation to each other: and this ratio remains unaltered, when its terms are replaced by any equimultiples of them.

(3) Such ratios are dependent upon the numbers which form their terms only, and are the same, whatever be the nature and magnitude of the concrete unit of which those numbers may be respectively composed.

A ratio may be represented by means of a fraction.

356. All these conditions will be fully satisfied, if we *agree* to denote a ratio by means of a fraction, of which the antecedent is the numerator, and the consequent the denominator: for the value of this fraction is deter-

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* We shall generally give an enlarged signification to the term *multiple*, as denoting the result of multiplication by fractions as well as by whole numbers.

mined solely by the numbers which form its numerator and denominator, and is entirely independent of the specific value or nature of the units of the same kind, of which they are respectively composed: and it remains unaltered, when its numerator and denominator are multiplied or divided by the same number, that is, when the terms of the ratio corresponding, are replaced by any equimultiples of them.

Arithmetical definition of ratio.

In arithmetic, therefore, a ratio may be *defined*, as the fraction whose numerator is the antecedent, and denominator is the consequent of the ratio.

Algebraical definition of ratio.

357. In algebra likewise we *adopt* the same definition, considering the ratio of two quantities expressed by symbols a and b, as a phrase synonymous with the fraction $\frac{a}{b}$: in this science, therefore, there is a meaning to be attached to the ratio of a to b, whenever there is a meaning to be attached to the fraction $\frac{a}{b}$, whether the quantities which they denote are of the same or a different kind, or possess the same or different algebraical signs: in other words, there is no limit to the interpretation of the term ratio, when thus applied, which is different from that which belongs to the interpretation of the equivalent fraction.

It appears, therefore, that both in Arithmetic and Algebra, the theory of ratios becomes identified with the theory of fractions.

Includes geometrical as well as other magnitudes.

358. The symbols of algebra represent geometrical as well as other quantities, and the lines, areas and solids of geometry, are thus brought within the range of this definition: it must be kept in mind however, that it is only by considering geometry as a science subordinate to algebra, that such quantities admit of the mode of representation which that definition renders necessary: for there is no geometrical mode of representing the division of one line by

Reason why there is no definition of ratio in geometry.

another, or the result of such a division: for this result can bear no analogy to the quantities which produce it, being essentially numerical and consequently not capable of being represented by a line, unless in a symbolical sense, which under all circumstances must be different from that in which the other lines are used. It is of great importance to attend to this distinction, as it serves not only to explain the reason why there is no independent definition of ratio in geometry, but also why in comparing different ratios of geometrical lines or areas with each other, with reference to their identity or diversity, we are not at liberty to avail ourselves of the algebraical definition of ratio, unless we first change the mode of representing the quantities which are the objects of the investigation, and resort to the use of symbolical language.

It is indifferent whether the same ratio presents itself under an arithmetical or a symbolical form.

359. The identity of the definitions of ratio in arithmetic and algebra, makes it indifferent whether the quantities which compose the terms of a ratio, present themselves under an arithmetical or symbolical form: for the transition from one to the other is immediate, so long as it can be effected by an operation, which in one is indicated and not performed: thus the ratio of $\sqrt{4}$ to the $\sqrt{9}$ or $\frac{\sqrt{4}}{\sqrt{9}}$ is identical with $\frac{2}{3}$, though one ratio is symbolical or algebraical, and the other arithmetical: the same transition may be likewise made, whenever there is a symbolical common measure of the terms of a ratio, which leaves, when removed, an arithmetical result: thus

$$\frac{\sqrt{12}}{\sqrt{147}} = \frac{\sqrt{3}\times\sqrt{4}}{\sqrt{3}\times\sqrt{49}} = \frac{\sqrt{4}}{\sqrt{49}} = \frac{2}{7} : \frac{\sqrt{18}}{\sqrt{50}}$$

$$= \frac{\sqrt{2}\times\sqrt{9}}{\sqrt{2}\times\sqrt{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5} : \frac{\sqrt[3]{81}}{\sqrt[3]{375}} = \frac{\sqrt[3]{3}\times\sqrt[3]{27}}{\sqrt[3]{3}\times\sqrt[3]{125}}$$

$$= \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5}:$$

and similarly in other cases.

360. We shall now proceed to the statement of some of the more common propositions concerning ratios, which, though merely properties of arithmetical and other fractions, admit of many important applications, and have given rise to some most important theories.

Ratios: how compared.

"Ratios are compared with each other, by comparing the fractions by which they are denoted."

Thus, the ratios of 3 to 5 and of 5 to 8, are denoted by the fractions $\frac{3}{5}$ and $\frac{5}{8}$: these are identical with the fractions $\frac{24}{40}$ and $\frac{25}{40}$: it is the second of these ratios, therefore, which is the greater of the two.

Ratios of greater or less inequality.

361. A ratio of *greater inequality* is one, whose antecedent is greater than its consequent: a *ratio of lses inequality* is one, whose antecedent is less than its consequent: a *ratio of equality* is one, whose antecedent is equal to its consequent: the first corresponds to an improper fraction, the second to a proper fraction, and the third to unity. The following proposition, connected with ratios which are thus denominated, is frequently used.

Proposition respecting them.

362. "A ratio of greater inequality is diminished and one of less inequality increased, by adding the same quantity to both its terms."

Let $\frac{a}{b}$ be the primitive and $\frac{a+x}{b+x}$ the secondary ratio, formed by adding the same quantity x to both its terms: these fractions reduced to a common denominator, become

$$\frac{ab+ax}{b(b+x)}, \text{ and } \frac{ab+bx}{b(b+x)},$$

respectively: if a be greater than b, or if the primitive ratio be one of less inequality, the first fraction is the *greater* of the two, which is diminished, therefore, by the addition of the same quantity to both its terms: but if a be less than b, or if the primitive ratio be one of less inequality, the first fraction is the less of the two, which is increased therefore by the addition of the same quantity to both its terms.

Meaning of the sum of two or more ratios.

363. If there be several ratios, whose antecedents are multiplied together for a new antecedent, and their consequents together for a new consequent, the resulting ratio is called the *sum* of the component ratios: in other words, the sum of two or more ratios is the product of the fractions which denote them.

Examples.

Thus, the sum of the ratios of $a : b$ and $c : d$ is $ac : bd$.

When the consequent of one ratio is the antecedent of the next.

If the consequent of one ratio become the antecedent of the next, the sum of any number of such ratios is the ratio of the first antecedent to the last consequent.

If the ratios be $a : b$, $b : c$, $c : d$, and $d : e$, their sum

$$= \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{e} = \frac{abcd}{bcde} = \frac{a}{e}.$$

A duplicate ratio.

The sum of two equal ratios, or the *double* of any ratio, is the ratio of the square of the antecedent to the square of the consequent.

For the sum of the ratios $a : b$ and $a : b$ is

$$\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}.$$

A triplicate ratio.

In a similar manner, the *triple* of any ratio is the ratio of the cube of its antecedent to the cube of its consequent.

For the sum of the ratio of $a : b$, added thrice together, is $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$.

More generally, the sum of n times any ratio, such as $a : b$, is the ratio of $a^n : b^n$.

For the product of $\frac{a}{b}$ into itself, repeated as a factor n times, is $\frac{a^n}{b^n}$.

In the same sense, $a^n : 1$ or $\frac{a^n}{1}$ or a^n, is n times the ratio of $a : 1$.

Logarithm: its primitive meaning.

364. It is in this sense, that the powers of numbers may be considered as *multiple* ratios of the simple numbers to 1; thus, $10^2 : 1$ or 10^2, may be termed the *double* of the ratio of $10 : 1$; $10^5 : 1$ or 10^5, the *quintuple* of that ratio, and similarly for other cases, the index of the power being the *measure* of the number of simple ratios whose *addition* forms the compound one: the term *logarithm* means *the number of ratios*, and is therefore equivalent in signification to the index of the number, which forms the *fundamental* ratio.

Thus, if the fundamental ratio be $a : 1$ or a, then 2 is the logarithm of $a^2 : 1$ or of a^2; 3 is the logarithm of $a^3 : 1$ or of a^3; and n is the logarithm of $a^n : 1$ or of a^n.

A subduplicate ratio.

365. Again, $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ or $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$, is the *half* or *subduplicate* of the ratio of $a : b$; for the double of the ratio of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is

$$\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} \times \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \frac{a}{b}.$$

A subtriplicate ratio.

The ratio of $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ or $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$, is the *subtriplicate* of the ratio of $a : b$; for the triple of that ratio

$$= \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} \times \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} \times \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} = \frac{a}{b}.$$

A sesquiplicate ratio.

The ratio of $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is the *sesquiplicate** ratio of $a : b$; for the sum of the simple and subduplicate ratio of $a : b$

$$= \frac{a}{b} \times \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}.$$

The ratio of $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ is $\frac{1}{n}$th part of the ratio of $a : b$; for n times the ratio of $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ is the ratio of $a : b$.

Extended use of the term logarithm.

366. In a similar manner, $a^{\frac{1}{n}} : 1$ or $a^{\frac{1}{n}}$ is the n^{th} part of the ratio of $a : 1$; and it is an easy and natural extension of the use of the term *logarithm*, to call $\frac{1}{n}$ the logarithm of the ratio of $a^{\frac{1}{n}} : 1$ or of $a^{\frac{1}{n}}$, the fundamental ratio being $a : 1$.

Proportion.

367. The theory of the ratios of incommensurable quantities and the modes of approximating to their values, involve necessarily the comparison of ratios with each other, with respect to their identity or diversity; and it is for this reason that the consideration of them would most naturally follow the definition of proportion and its dependent propositions: we shall proceed therefore at once to this second subject, which is of such immense importance in Geometry, and which presents itself in so many forms in all the practical applications of mathematics.

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* Before the introduction of fractional and general indices into Algebra, when Geometry was used as a symbolical science to represent the relations of every species of quantity, those modifications of numbers which indices express so briefly and so uniformly, were considered as the results of ratios, and were designated by an extremely complicated nomenclature: of this kind were the terms *sesquiplicate* and others, which have been explained in the text: for the ratios corresponding to other indices, such as $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, and so on, extremely embarrassing combinations of numerical terms were required: it is easy to conceive how much the difficulty even of the most simple investigations must have been increased, when encumbered by such a nomenclature and its corresponding interpretation.

Its definition.

"Proportion consists in the equality of two ratios."

A *proportion* (the term is here used absolutely) is composed of four terms, the first and third of which are the antecedents, and the second and fourth the consequents, of the ratios which are required to be equal to each other.

Mode of representing a proportion in Geometry.

368. A proportion in Geometry is expressed by connecting the two ratios, written geometrically, with the sign ::, which is used to denote their identity with each other; thus, if $a : b$ and $c : d$ be the ratios which constitute a proportion, they are written thus,

$$a : b :: c : d.$$

Used also in Arithmetic and Algebra.

The same method of exhibiting a proportion is very commonly used, both in Arithmetic and Algebra, whatever be the nature of the quantities which compose it.

Algebraical test of proportionality.

369. The definition of a ratio which is given in Arithmetic and Algebra, will necessarily lead to a mode of exhibiting a proportion which is appropriate to those sciences: for if $\frac{a}{b}$ and $\frac{c}{d}$ denote the ratios respectively which constitute a proportion, the proportion itself must be expressed by the equation

$$\frac{a}{b} = \frac{c}{d}.$$

It is the equality of these fractions which is the test of proportionality, or of the *equality* of the ratios, which is essential to their forming a proportion: it may be ascertained, therefore, whenever the values of these fractions can be calculated or determined.

Product of the extremes equal to product of the means.

370. There is another form of this equation, which is deducible from it, and which expresses therefore the same relation of the quantities involved in it, and which is, generally speaking, more convenient for the purposes of calculation:

if the fractions $\frac{a}{b}$ and $\frac{c}{d}$, be reduced to a common denominator, the equation

$$\frac{a}{b}=\frac{c}{d} \text{ becomes } \frac{ad}{bd}=\frac{bc}{bd};$$

or, omitting the common denominator,

$$ad = bc.$$

It appears, therefore, that if four quantities constitute a proportion, the product of the extremes is equal to the product of the means.

Its converse.

371. Conversely, if the product of any two quantities be equal to the product of two others, they will be convertible into a proportion, by making the terms of one product the extremes, and the terms of the other product the means.

For if $ad = bc$, we obtain, by dividing these equal quantities by bd,

$$\frac{a}{b}=\frac{c}{d}.$$

which is the algebraical mode of expressing the proportion

$$a : b :: c : d,$$

or $$c : d :: a : b;$$

in the first of which, a and d are the extremes, and b and c the means; and in the second, c and b are the extremes, and d and a the means.

Continued proportion.

372. If the two mean terms of a proportion are equal, or if the consequent of one ratio be the antecedent of the next, the product of the extremes is equal to the square of the mean: for in this case,

$$\frac{a}{b}=\frac{b}{c}, \text{ or } \frac{ac}{bc}=\frac{b^2}{bc},$$

and therefore

$$ac = b^2.$$

In this case, the terms a, b, c are said to be in *continued* proportion.

373. Before we proceed further with the examination of the consequences of our definition of proportion, it may be proper to consider how far this definition, and therefore the test of proportionality which it leads to, is consistent with the notions which we attach to the term *proportion* in ordinary language.

The terms ratio and proportion confounded with each other.

In the first place, we frequently confound in ordinary language, the terms *ratio* and *proportion* with each other: thus it is equally common to say, that two lines or other quantities are in the *ratio* of 3 to 5, or that they are in the *proportion* of 3 to 5: both these forms of expression are elliptical, though in different degrees: the first is equivalent to saying, that the ratio of the two lines or quantities is equal to the ratio of 3 to 5, and leaving it to be inferred from thence, that they constitute a proportion: the second may be considered as an abbreviated mode of asserting, that the two lines and the two numbers, form the terms of a proportion. In both cases there is the same tacit reference made to the conditions requisite to establish a proportion.

Usage of the term proportional.

374. The adjective *proportional* is still more variously used, though in all cases it will be found to be equivalent to the assertion of a proportion between certain quantities, either expressed or understood: thus, when we say that the *interest* of money is proportional to the *principal* we merely express in a very elliptical form, the following proposition: "The rate of interest and the time for which it is due being the same, any two principals and their corresponding amounts of interest will constitute the terms of a proportion." Thus, if £100. in one year will produce £5., £700. in the same time and at the same rate will

produce £35.; or in other words, 100 and 700, 5 and 35, will form the terms of the proportion

100 : 700 :: 5 : 35.

Usage of the phrase "in the same proportion."

375. Another mode of expressing the same proposition will serve to illustrate the efforts made in ordinary language to avoid the tediousness which statements, strictly conformable to the arithmetical definition, would render more or less necessary. When speaking of the dependence of the interest upon the principal, when the rate and time is given, we should say, that "In whatever *proportion* the principal was increased, the *interest* would be increased in the same proportion;" a proposition which might be applied to the preceding example in the following manner: "If there are two principal sums of money, £100. and £700., which form a proportion with the numbers 1 and 7: then the amounts of interest, 5 and 35, of those two principals, will likewise form a proportion with the same numbers 1 and 7: and also the two principals and the two amounts of interest will form a proportion with each other."

The effect proportional to its cause.

376. There is no proposition which is more frequently referred to, or which is pregnant with more important consequences than the following: "The effect is always *proportional* to its cause." As this proposition is intimately connected with the subject of our present discussion, we shall endeavour to explain and illustrate its meaning and applications.

When the dependence is physical.

377. When the connection between cause and effect is physical, and therefore not necessary, the proposition expresses the great law of the permanence and uniformity of the operations of nature, our conviction of the truth of which is the foundation of all our reasonings concerning them: thus, if a given force support a given weight, then the double of that force will support the double of the weight, and any multiple of the force will support the same multiple of the weight: the ratio between any two

forces, therefore, will be the same as the ratio between the corresponding weights supported; and such forces and weights will form, therefore, the terms of a proportion. It is in this sense that the force would be considered as proportional to the weight, the force corresponding to the cause, and the weight to the effect in the general proposition we are illustrating.

Cause and effect convertible terms.

378. The ratios, however, which form the proportion, are convertible, and the terms which correspond to them in the proposition must be convertible likewise. In the particular case, therefore, which we are considering, if the force be considered as proportional to the weight, the weight must likewise be considered as proportional to the force: in whatever sense, therefore, the proportion may require to be interpreted, in order to answer to this change in its form, a corresponding change must take place likewise in the connection between the terms of the proposition.

When the dependence is mathematical.

379. When the connection between the terms of the proposition is mathematical, and therefore determined by definition, the proposition may be considered as merely an abbreviated form of expressing the corresponding proportion: thus, when we say that the area of a triangle whose altitude is given, is proportional to its base, we merely assert that the ratio of the area of one triangle to the area of any other triangle possessing the same altitude, is the same as the ratio of the base of the first triangle to the base of the second.

When the dependence is hypothetical.

380. When, however, the connection of the terms of the proposition is neither physically nor mathematically necessary, we must invest them with a necessary connection by means of hypotheses, either expressed or understood, which will be equivalent to definitions: thus, when we say that the "Work done by labourers is proportional to their number," we take it for granted that each individual labours with the same effect: it is in virtue

of such hypotheses, that we conclude with mathematical certainty, that twice or any multiple of the work will be done by twice or the same multiple of the number of men, and consequently, that the quantities of work done in any two cases and the corresponding numbers of men, will constitute the terms of a proportion.

The effect varies as the cause.

381. The relation of cause and effect, expressed by saying that one is proportional to the other, is more frequently designated by an equivalent phrase, which is, that the *effect varies as the cause* or conversely: in both cases, an equally enlarged signification is given to the terms cause and effect, and the connection between them is rendered necessary, either by definition or hypothesis or by inferences drawn from observation of the general laws of the physical world.

Or as the causes or agents conjointly.

382. When the causes or agents which produce an effect are more than one, the effect is said to *vary* conjointly with the causes which produce it. It remains to consider the law of the dependence of the effect and agents in such cases, particularly with reference to the proportion which they form.

Example of joint variation.

383. Thus we should say, that the work done would *vary* conjointly as the number of men employed and the number of days they worked: for if the number of days during which the men worked remained the same, m times the same work would be done by m times the same number of men: and if the number of men remained the same, n times the same work would be done in n times the same number of days: it would follow, therefore, that mn times the original work would be done by m times the original number of men, working for n times the original number of days: and in a similar manner, $m'n'$ times the original work would be done by m' times the original number of men working for n' times the original number of days: the ratio, therefore, of the quantities of work done under these different circumstances, would

be that of mn to $m'n'$, which is the ratio of the product of the number of men into the number of days in the first case, to the product of the number of men into the number of days in the second; and these four quantities, therefore, would form the terms of a proportion.

Independent of the nature and magnitude of the primitive effect and agents.

It is obvious, likewise, that the existence and value of the ratios which form this proportion, are perfectly independent of the particular value or peculiar nature of the primitive effect and agents: for if we denote the original work done by w, the original number of men by M, and the original number of days by N, the terms of the first ratio would be mnw and $m'n'w$, and those of the second $mnMN$ and $m'n'MN$, which are clearly identical with each other, inasmuch as w is a measure of the terms of the first ratio, and MN a measure of the terms of the second.

Where the agents are more than two.

384. Other conditions or other agents might be introduced into the question we have just been considering, contributing to the effect produced: we might consider, for instance, variations, not merely in the number of men and number of days, but likewise in the number of hours during which they should each of them work, and also in the intensity of their labour: the effect produced or work done would vary as all these causes or agents conjointly: for if we double or otherwise increase the number of hours of labour during each day, we should double or increase at the same rate the work done: and if we should double or otherwise increase the effective industry of the workmen, we should double or increase at the same rate the work done under the same circumstances. If, therefore, $m : m'$ expresses the ratio of the number of men in any two cases, $n : n'$ the ratio of the number of days, $p : p'$ the ratio of the number of working hours each day, and $q : q'$ the ratio of the effective industry of the men under such circumstances, then the ratio of the work done in the two cases would be that of $mnpq$ to $m'n'p'q'$.

Law of dependence of one agent upon the other agents and effect.

385. We have hitherto considered the law of dependence of the effect upon its several agents or causes; and it very rarely happens that they are not sufficiently distinguished from each other by the nature of the question proposed: it is frequently, however, a matter of equal importance, to ascertain the law of dependence of one of the agents upon the other agent or agents, when the effect is the same, or upon the other agents and effect, when the effects in different cases are different from each other.

Example:

Thus, the same work is done by different numbers of men in different numbers of days: what is the nature of the relation between the number of men and the number of days?

when the effect is the same.

Let the numbers of men in any two cases be m and m', or let their ratio be that of m to m': let the corresponding numbers of days be n and n'. Then, since the effect or work done varies conjointly as the number of men and the number of days; and since the ratio of the work done under these circumstances, is that of 1 to 1, it follows that

$$1 : 1 :: mn : m'n',$$

and therefore

$$1 = \frac{mn}{m'n'}, \text{ and } \frac{m}{m'} = \frac{n'}{n};$$

and consequently,

$$m : m' :: n' : n,$$

Inverse variation.

or the ratio of the numbers of men in the two cases is the inverse of the ratio of the numbers of days: in other words, the number of men is said to *vary inversely* as the number of days.

When the effect is different.

If, however, we should suppose the work done in the two cases different, and in the ratio of w to w', the first proportion would become

$$w : w' :: mn : m'n';$$

and therefore

$$\frac{w}{w'} = \frac{mn}{m'n'}, \text{ and } \frac{m}{m'} = \frac{w}{w'} \times \frac{n'}{n} = \frac{\left(\frac{w}{n}\right)}{\left(\frac{w'}{n'}\right)};$$

and consequently

$$m : m' :: \frac{w}{n} : \frac{w'}{n'}.$$

Under such circumstances, the number of men in each case is said *to vary* as the work to be done *directly*, and as the number of days *inversely*.

Principle of solution of questions in the Rule of Three, &c.

386. In the innumerable questions which present themselves as examples of the Rule of Three, direct and inverse, simple and compound, and in most of the applications of Arithmetic, it is required to determine the value of one unknown quantity from its connection with three or more known quantities, when there exists amongst all of them a relation of *cause* or *agent* or *agents* and *effect*, such as we have been considering: the common principle of solution in all such questions is, to reduce all the quantities to the form of a proportion, of which the unknown quantity shall form one term: and if three terms of a proportion be given, the fourth may always be determined: thus, in the proportion

$$a : b :: c : d,$$

we have

$$a = \frac{bc}{d}, \quad d = \frac{bc}{a}, \quad b = \frac{ad}{c}, \quad \text{and } c = \frac{ad}{b};$$

and therefore either of the extreme terms of the proportion may be found by dividing the product of the means by the other extreme; and either of the means may be found by dividing the product of the extremes by the other mean.

Arrangement of the terms of the proportion.

387. In order to arrange the quantities in the question, in the proportion which is necessary for its solution, we

must consider, in the first place, whether the unknown quantity can be considered as an effect or an agent; and also, whether it expresses the entire effect or a part of it only: if it be an effect, it must vary as the agent, or jointly as the agents: if it be an agent, it must vary directly as the effect: but if it be one amongst several agents, it must vary as the effect directly, and inversely as the other agents.

Primary agents.

The effect may, in some cases, have a separate dependence upon its own agents, as connected with the estimation of its magnitude: thus, the effect may be an area, dependent upon its length and breadth; or a solid, dependent upon its length, breadth and height: and the object of the problem may sometimes require the determination of one of these subordinate agents of the entire effect produced by the primary agents: it will vary directly as the primary agents, and inversely as the other subordinate agents.

Subordinate agents.

Examples.

388. The following are examples:

(1) If 3 guns in a battle kill 21 men, how many men would 11 guns kill in the same time?

The *effect* is the number of men killed: the *agent* is the number of *equally* efficient guns.

The effect varies as the agent, and therefore

$$\begin{array}{ccccccc} \text{Men.} & & \text{Men.} & & \text{Guns.} & & \text{Guns.} \\ 21 & : & x & :: & 3 & : & 11; \end{array}$$

$$\text{and} \quad x = \frac{11 \times 21}{3} = 77.$$

(2) An elephant consumes 400lbs. of rice in 3 days: how many lbs. of rice would an elephant, which is $\frac{1}{4}$th greater in every dimension, consume in 5 days?

It is assumed as a mathematical hypothesis, that the consumption of rice by the elephant will vary with his bulk and the number of days, and also that his bulk will vary with his length, breadth and height.

The effect is the consumption, and the unknown quantity is one of the effects.

The agents are the length, breadth, and height of the elephant, and the number of days.

$$\begin{array}{ccccccccccc} \text{lbs.} & & \text{lbs.} & & \text{l.} \quad \text{b.} \quad \text{h.} \quad \text{days.} & & \text{l.} \quad \text{b.} \quad \text{h.} \quad \text{days.} \\ 400 & : & x & :: & 1 \times 1 \times 1 \times 3 & : & \frac{5}{4} \times \frac{5}{4} \times \frac{5}{4} \times 5, \end{array}$$

and therefore

$$x = \frac{625 \times 400}{192} = 1302\frac{1}{12} \text{ lbs.}$$

(3) How much in length which is 4 inches in breadth, will make a yard square?

The effect is the area formed, which is the same in both cases.

The agents are the length and breadth, and one therefore varies inversely as the other:

therefore,

$$\begin{array}{ccccccc} \text{1st length.} & & \text{2d length.} & & \text{2d breadth.} & & \text{1st breadth.} \\ x & : & 36 & :: & 36 & : & 4; \end{array}$$

$$\therefore x = \frac{36 \times 36}{4} = 324 \text{ inches.}$$

(4) If 132 men can dig a trench which is 100 yards long, 3 deep and 2 wide in 7 days, working during 10 hours each day, how many men will it require to dig a trench 320 yards long, 4 deep and 3 wide in 11 days, working during 12 hours each day?

The effect is the trench, whose subordinate agents are its length, depth and breadth.

The primary agents are the number of men, the number of days and the number of hours in each day.

It is the number of men in the second case which is the agent to be determined.

The number of men varies as the subordinate agents of the effect directly, and as the other primary agents inversely.

Therefore,

$$132 : x :: \frac{100 \times 3 \times 2}{7 \times 10} : \frac{320 \times 4 \times 3}{11 \times 12}$$

$$:: 33 : 112;$$

$$\therefore x = \frac{112 \times 132}{33} = 448.$$

(5) If 10 men can reap a field in 3 days, whose length is 1200 feet and breadth 800 feet, what is the breadth of a field whose length is 1000 feet, which 12 men can reap in 4 days?

The effect is the area of the field, the subordinate agents of which are its length and breadth: it is the breadth in the second case which is the unknown term of the proportion.

The breadth varies as the number of men and the number of days directly and inversely as the length:

$$800 : x :: \frac{10 \times 3}{1200} : \frac{12 \times 4}{1000}$$

$$:: 25 : 48:$$

$$\therefore x = \frac{800 \times 48}{25} = 1536.$$

389. The subject of proportion presents itself so continually not only in the business but likewise in the language of common life, that we have felt it important not merely

to interpret correctly the very various phrases by which it is expressed, but likewise to shew the general coincidence of such interpretations with the general definition of proportion which we have adopted: it only remains to consider, in conclusion, its coincidence likewise with the peculiar form of the definition of proportion which is used in geometry.

The algebraical definition of proportion not transferable to Geometry.

The same reason which prevented the transfer of the algebraical definition of ratio to Geometry, will likewise prevent the similar transfer of the definition of proportion: for it is impossible to ascertain the identity of the fractions which constitute a proportion, if we have no geometrical means of ascertaining the values of the fractions themselves: it is no answer to this observation to say that the identity of these fractions exists, when their numerators and denominators form the terms of a proportion, in the case of geometrical as well as other quantities: for if geometry be considered as a practical, as well as a theoretical, science, we can avail ourselves of no properties of the quantities which are the objects of its investigation, which are only discoverable by the aid of another science, and which cannot be verified by any geometrical test.

The geometrical definition of proportion must be a consequence of the algebraical definition.

390. But the algebraical definition of proportion comprehends geometrical as well as other quantities, and whatever may be therefore the geometrical definition of proportion which we may adopt, it must coincide with the algebraical definition in its consequences, at least, if not in its form: for though such a definition, like all others, is perfectly arbitrary, when considered merely as the foundation of a system of dependent propositions, it would be worse than useless, if it was so constructed as to lead to conclusions inconsistent with those which result from the definitions of another and more general science by which it is or ought to be comprehended: it is for this reason that it will be expedient to examine the consequences of the algebraical definition of proportion, in order, if possible, to find some one, which admits of

the application of a geometrical test, which is equally applicable to every species of geometrical quantity.

Proposed definition, the rectangle formed by the means equal to the rectangle formed by the extremes.

391. One of the most immediate and remarkable of the consequences of the algebraical definition of ratios, is the equality of the products of the extreme and mean terms of the proportion: if the quantities which were the objects of comparison were geometrical straight lines, the products in question would correspond to rectangles of which these lines were the adjacent sides respectively: and such lines would properly form the terms of a proportion, if those rectangles were equal to each other: and inasmuch as the equality or diversity of such rectangles could be ascertained by practical geometrical means (by their conversion into equal squares), we should thus be put into possession of a geometrical test of proportion, which would necessarily coincide with the test supplied by the algebraical definition.

Objections to it.

392. Without noticing however other objections to the use of such a geometrical definition of proportion, there is one which is fatal to it: it is not applicable to every species of geometrical quantity: for if the terms of the proportion, two or all of them, were areas or solids, the test in question, would no longer admit of application: it does not present itself, in short, under a form which is independent of the specific nature of the quantities which constitute each ratio of the proportion; and it is obvious that the same objection would more or less apply to the adoption of any definition, which was founded upon any consequences of the algebraical definition of proportion, which are independent of or not immediately connected with, the fractional form of the ratios which compose it.

393. It is for this reason that we shall again recur to the original forms of the fractions which constituted the definition of proportion, namely,

$$\frac{a}{b} = \frac{c}{d}:$$

and examine whether there is any modification of their terms, which geometrical quantities could admit of in a system of geometry, which is essentially connected with their equality or identity: among those might be reckoned any multiples whatsoever of the terms of the ratios, whether lines or areas: thus if we take any equimultiples whatsoever (m) of the first and third terms of the proportion, and any equimultiples whatsoever (n) of the second and fourth, we should get

A second consequence of the algebraical definition of proportion examined.

$$\frac{ma}{nb} = \frac{mc}{nd}:$$

and it is obvious that if ma be greater than nb, mc is also greater than nd, and if ma be equal to nb, then mc is also equal to nd, and if ma be less than nb, then mc is also less than nd: for if not, let us suppose mc greater than nb, at the same time that mc is not greater than nd, and it will follow therefore that a ratio of greater inequality may be equal to a ratio of equality or to a ratio of less inequality; or in other words, that an improper fraction may be equal to unity or to a proper fraction: and it may be readily shewn that any other conclusion respecting the relation of the terms ma, nb, mc, nd different from those we have mentioned, will lead to similar contradictions.

394. It will follow therefore if a and b, c and d are pairs of geometrical magnitudes of the same kind respectively, whether lines, or areas, or solids, which form the terms of a proportion, and if any equimultiples whatsoever be taken of the first and third, and any equimultiples whatsoever be taken of the second and fourth, then if the multiple of the first be greater than the multiple of the second, the multiple of third will be greater than the multiple of the fourth: if the multiple of the first be equal to that of the second, the multiple of the third will be equal to that of the fourth: and if the multiple of the first be less than that of the second, the multiple of the third will be less than that

Geometrical definition of proportion.

of the fourth. It is this proposition which is deduced as a necessary consequence of the definition of proportion in algebra, including geometrical as well as other magnitudes, which may be legitimately converted into the definition of proportion in geometry, and which is given for that purpose in the fifth definition of the fifth book of Euclid.

Geometrical definition of quantities which are not proportionals.

395. But this definition is incomplete, unless it can be shewn that it enables us to ascertain not only when four geometrical magnitudes form a proportion, but also when they do not: it follows however as a parallel conclusion, that such magnitudes do not form a proportion, which do not, under all circumstances, fulfil the conditions which the definition prescribes: or, in other words, four magnitudes do not form a proportion, if when any equimultiples *whatsoever* are taken of the first and third, and any equimultiples *whatsoever* of the second and fourth, it can be shewn that the multiple of the first is greater than that of the second, but the multiple of the third *not* greater than that of the fourth; or that the multiple of the first is equal to that of the second, but the multiple of the third *not* equal to that of the fourth: or that the multiple of the first is less than that of the second, but the multiple of the third *not* less than that of the fourth.

A consequence also of the algebraical definition.

396. In order to shew that this definition of quantities which are not proportionals, is likewise a consequence of the algebraical definition of proportion, we will suppose the four quantities a, b, c, d, so related to each other, that $\frac{a}{b}$ is either greater or less than $\frac{c}{d}$: in the first case, if we make

$$\frac{a-x}{b} = \frac{c}{d}, \text{ where } x = \frac{ad-bc}{d},$$

we shall also have

$$\frac{m(a-x)}{nb} = \frac{mc}{nd};$$

if we now suppose $mc = nd$, we shall also have $ma - mx = nb$, and therefore ma is greater than nb, at the same time that mc is equal to nc: or, in other words, equimultiples of the first and third of four quantities are taken and other equimultiples of the second and fourth, and the multiple of the first is greater than that of the second, but the multiple of the third is *not* greater than that of the fourth. If again, we suppose $\frac{a}{b}$ less than $\frac{c}{d}$, then we may make

$$\frac{a+x}{b} = \frac{c}{d}, \text{ where } x = \frac{bc-ad}{b},$$

and we shall also have

$$\frac{m(a+x)}{nb} = \frac{mc}{nd}:$$

if we now suppose $mc = nd$, we shall also have

$$ma + mx = nb,$$

and, therefore, ma is less than nb, at the same time that mc is not less than nd: or in other words, equimultiples of the first and third of four quantities are taken, and equimultiples likewise of the second and fourth, and the multiple of the first is less than that of the second, but the multiple of the third not less than that of the fourth: it thus appears that the geometrical definitions of quantities which are proportionals, as well as of those which are not proportionals, are equally consequences of the algebraical definition of proportion.

The geometrical definition adapted to the forms of geometrical reasonings.

397. The geometrical definition of proportion is adapted not merely to the peculiar nature of the quantities which are considered in geometry, but likewise to the form under which the reasonings concerning their relations present themselves: for there are no processes in Geometry beyond the formation of multiples and submultiples which correspond *generally* to algebraical mul-

tiplication, and none whatever to algebraical division: the operations of Geometry, as far as they can be called so, and the reasonings connected with them, or which are founded upon the definitions, are confined to the comparison of quantities with each other, whether as regards their equality or inequality, or as forming or as not forming the terms of a proportion: the peculiar definition, therefore, of proportion in Geometry has been adopted as much from necessity as from choice, as the only one which was adapted to the form and nature of the processes and reasonings of that science: for we have shewn that there is no other modification of the algebraical definition of proportion, which possessed the necessary conditions.

General propositions concerning ratios and proportions equally consequences of the two definitions.

398. All general propositions concerning ratios and proportions may be considered as equally necessary consequences, both of the algebraical and geometrical definitions; for they are both of them equally applicable to every species of quantity, and one of them may be considered as a proposition deducible from the other: if the only object proposed, therefore, in mathematical investigations, was the most speedy establishment or discovery of truths, without any reference to the form of the demonstrations, we should naturally make use of that definition, which conducted us most rapidly and most easily to the conclusions sought for: but such investigations commonly present themselves as parts of some general system, the completeness of which would require consistency in the form of the demonstrations, and consequently a uniform reference to the same system of definitions as the basis of them: it is for this reason, that we should adhere uniformly to the algebraical definition of proportion in a system of Algebra, and to the geometrical definition in a system of Geometry.

399. The propositions which follow are in substance the same with those which are demonstrated by means of the geometrical definition of proportion in the fifth book of Euclid: the contrast which they present, in respect to

the brevity and facility of the demonstrations in the two cases, is so remarkable, as to have given rise to a very general impression that the difference was less owing to the more tangible nature of the test of proportion in the definition in Algebra, than to the superior strictness and generality of the corresponding definition in Geometry: much, however, of this prejudice ought to be attributed to the very common custom of considering the algebraical as merely *co-extensive* with the arithmetical definition of proportion, and, therefore, comprehending only commensurable magnitudes, whose ratios were expressible by finite numbers: thus, four magnitudes were defined to be proportionals, when the first was the same multiple part or parts of the second, that the third was of the fourth: it was the substitution of a particular consequence of the algebraical definition of proportion, for the definition itself, which really made the propositions founded upon it less comprehensive than those deduced by means of the geometrical definition, in consequence of the non-existence of any algebraical or rather arithmetical test of proportionality, under such circumstances, when the terms of the ratios were incommensurable with each other.

Reason why the geometrical definition has sometimes been considered more general than the algebraical definition.

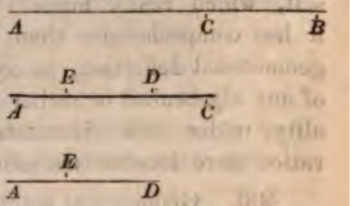
400. Geometrical magnitudes, being subject to the law of continuity, are capable of representing *accurately* the ratios of all other magnitudes of the same kind, whether commensurable or not: when lines, however, are used symbolically, we have no means of ascertaining whether the actual lines assumed for this purpose and exhibited to the eye, possess the same relation to each other with the quantities which they represent, and under such circumstances, therefore, they serve no farther practical use than the general symbols of Algebra employed for the same purpose: it is only, therefore, when they are not used symbolically, but are themselves the magnitudes which they represent, that they become the proper subjects of enquiry as to their commensurability with each other.

Geometrical magnitudes may represent both commensurable and incommensurable ratios.

401. The process for finding the greatest common measure of two lines is identical with that for finding the greatest

Process of ascertaining the commensurability of two lines.

common measure of two numbers: for it merely requires us to cut off the less of two lines (the divisor) from the greater (dividend) as many times as is possible, and then to repeat the same process with the remainder and the last divisor: if the process terminates, the lines are commensurable, and the last divisor is the common measure required: but if the division does not terminate, however long it may be continued, the lines are incommensurable with each other. It may be objected, that this process would not enable us to pronounce two lines to be incommensurable with each other, without an infinite number of operations: but when such incommensurable lines are connected with each other, by some assignable geometrical properties, it is generally possible to shew that the same relations will *recur* between the remainder and the divisor, and consequently that the process can never terminate: thus, if AB and AC represent respectively the diagonal and side of a square: if we cut off AC from AB, the remainder is BC: if again we cut off twice BC from AC, the remainder is AE: it is not very difficult to prove *geometrically* that the relation between AD and DE is the same as that between AB and AC: or in other words, that AD is the diagonal of a square, of which DE or BC is the side: it follows, therefore, that this process can never terminate, in consequence of this continued reproduction of the same relations, and that consequently the diagonal and side of a square are incommensurable with each other.*



402. If we call a the side of a square, the square of the diagonal will be equal to the sum of the squares described upon the two sides, which is consequently $2a^2$: the diagonal,

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[*] The parts of a line divided in extreme and mean ratio may be shewn to be incommensurable with each other in a similar manner.

The ratio of the diagonal and side of a square represented symbolically.

therefore, itself will be denoted by $\sqrt{2} \, . \, a$, and its ratio to its side will be that of $\sqrt{2} : 1$: the representation of this ratio under this form is symbolical only, and is incapable of being replaced by any finite numbers whatsoever: it is the distinction between discrete and continuous quantity, which gives to Geometry a more enlarged power of representation than to Arithmetic, when either science is used symbolically.

Incommensurable ratios lead to continued fractions.

403. The ordinary process for finding the greatest common measure, in all cases of incommensurable ratios, will lead to an indefinite series of quotients, whether recurring by discoverable periods or not, from which a series of converging fractions may be formed, approximating to the values of the ratios themselves to any required degree of accuracy: the theory, however, of the formation of continued fractions from the same series of quotients and of the deduction of the converging fractions from them, will be considered in a subsequent Chapter, to which we shall reserve the further consideration of this subject.*

Approximate ratios of the powers of numbers or quantities which differ very little from each other.

404. In the case, however, of the roots and powers of very large numbers or quantities which differ little from each other when compared with the numbers or quantities themselves, a very ready and useful approximation to the value of their ratios, whether commensurable or incommensurable, may be effected by the following method: if the numbers or quantities themselves be $a + x$ and a, where x is very small compared with a, then

$$\frac{(a+x)^n}{a^n} = \frac{a^n + n a^{n-1} x + \dfrac{n\,(n-1)}{1\,.\,2}\, a^{n-2} x^2 + \&\text{c.}}{a^n}$$

$$= \frac{a + nx + \dfrac{n\,(n-1)}{1\,.\,2} \,.\, \dfrac{x^2}{a} + \&\text{c.}}{a}$$

$$= \frac{a + nx}{a} \text{ nearly,}$$

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* The ordinary process for finding the greatest common measure is not

if we neglect the terms of the numerator after the second, involving $\frac{x^2}{a}$, $\frac{x^3}{a^2}$, $\frac{x^4}{a^3}$, &c. which are very small fractions: thus the ratio of $(101)^2$ to $(100)^2$ is very nearly that of 102 to 100: the ratio of $(137)^3$ to $(135)^3$ is very nearly that of $135 + 3 \times 2$ or 141 to 135: the ratio of $\sqrt{1001}$ to $\sqrt{1000}$ is very nearly that of $1000\frac{1}{2}$ to 1000 or of 2001 to 2000: the ratio of $\sqrt[3]{729}$ to $\sqrt[3]{728}$ is very nearly that of $728\frac{1}{3}$ to 728 or of 2185 to 2184.

General propositions on proportions.

405. We shall now subjoin some of the most important propositions on the subject of proportions, which have not been included in the preceding investigations.

"Magnitudes which have the same ratio to the same magnitudes have the same ratio to one another."

Let $a : b :: e : f$ and also $c : d :: e : f$, then

$$a : b :: c : d.$$

For if $\frac{a}{b} = \frac{e}{f}$ and $\frac{c}{d} = \frac{e}{f}$, then also $\frac{a}{b} = \frac{c}{d}$, and, therefore, $a : b :: c : d$.

Invertendo.

406. "If four magnitudes be proportionals, they shall be proportionals also *invertendo*[*], that is, when the terms of the respective ratios are taken in an inverse order."

Let $a : b :: c : d$; then also

$$b : a :: d : c.$$

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applicable when the terms of the ratio are expressed symbolically, unless, when the operation is performed, we are enabled to pass from symbolical to real quantities or their representatives: of this kind is the ratio of

$$\sqrt{2} \text{ to } 1, \text{ of } \sqrt{3} \text{ to } \sqrt{2}:$$

under such circumstances we must resort to other methods for the formation of the quotients, from which the continued fractions are deduced.

[*] It is usual to quote *technically* this and several other proportions on the subject of proportions by the words or phrase which was used in the Latin translations of Euclid to express the particular modification of the terms of a proportion which it was the object of the proposition to establish.

For if $\frac{a}{b}=\frac{c}{d}$, and if we divide 1 by each of these equal quantities, we shall get

$$\frac{1}{\left(\frac{a}{b}\right)}=\frac{1}{\left(\frac{c}{d}\right)}, \text{ or } \frac{b}{a}=\frac{d}{c},$$

and, therefore,

$$b : a :: d : c.$$

407. "If four magnitudes be proportionals, they shall be proportionals also *alternando*, or when taken alternately." Alternando.

Let $a : b :: c : d$, then also

$a : c :: b : d$.

For if $\frac{a}{b}=\frac{c}{d}$, then also $ad=bc$, and dividing each of these equal quantities by dc, we get

$$\frac{ad}{dc}=\frac{bc}{dc}, \quad \text{or} \quad \frac{a}{c}=\frac{b}{d},$$

and, therefore,

$$a : c :: b : d.$$

408. "If four magnitudes be proportionals, they shall be proportionals also *componendo*, that is, the sum of the first and second shall be to the second as the sum of the third and fourth is to the fourth. Componendo.

Let $a : b :: c : d$, then also

$a+b : b :: c+d : d$.

For if $\frac{a}{b}=\frac{c}{d}$, then also $\frac{a}{b}+1=\frac{c}{d}+1$, and, therefore, $\frac{a+b}{b}=\frac{c+d}{d}$, or $a+b : b :: c+d : d$.

Dividendo. **409.** "If four magnitudes be proportionals, they shall be proportionals also *dividendo*, that is, the difference of the first and second shall be to the second as the difference of the third and fourth is to the fourth."

Let $a : b :: c : d$, then also $a-b : b :: c-d : d$.

$$\text{For if } \frac{a}{b}=\frac{c}{d}, \text{ then also } \frac{a}{b}-1=\frac{c}{d}-1;$$

$$\text{and therefore } \frac{a-b}{b}=\frac{c-d}{d},$$

$$\text{or } a-b : b :: c-d : d.$$

Convertendo. **410.** "If four magnitudes be proportionals, they shall be proportionals also *convertendo*, or the first shall be to the difference of the first and second as the third is to the difference of the third and fourth."

Let $a : b :: c : d$, then also $a : a-b :: c : c-d$.

$$\text{For if } \frac{a}{b}=\frac{c}{d},$$

$$\text{then also } \frac{a-b}{b}=\frac{c-d}{d} \text{ and } \frac{b}{a}=\frac{d}{c};$$

$$\text{therefore } \frac{a-b}{b}\times\frac{b}{a}=\frac{c-d}{d}\times\frac{d}{c},$$

$$\text{or } \frac{a-b}{a}=\frac{c-d}{c}, \text{ and } \textit{invertendo},$$

$$\frac{a}{a-b}=\frac{c}{c-d},$$

$$\text{or } a : a-b :: c : c-d.$$

411. "If four magnitudes be proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference."

$$\text{Let } a : b :: c : d,$$

$$\text{then also } a+b : a-b :: c+d : c-d.$$

$$\text{For if } \frac{a}{b} = \frac{c}{d},$$

$$\text{then also } \frac{a+b}{b} = \frac{c+d}{d}, \text{ and } \frac{a-b}{b} = \frac{c-d}{d};$$

$$\text{therefore } \frac{\frac{a+b}{b}}{\frac{a-b}{b}} = \frac{\frac{c+d}{d}}{\frac{c-d}{d}}, \text{ and also } \frac{a+b}{a-b} = \frac{c+d}{c-d},$$

$$\text{or } a+b : a-b :: c+d : c-d.$$

Ex æquali in proportione directâ.

412. "If there be three or any number of magnitudes of the same kind, and as many others, which taken two and two in the same order, have the same ratio: then (*ex æquo* or *ex æquali in proportione directâ*), the first shall be to the last of the first set of magnitndes as the first is to the last of the second set of magnitudes."

Let a, b, c be the first, and a', b', c' the second set of magnitudes; then, if

$$a : b :: a' : b', \text{ and } b : c :: b' : c',$$

we shall also have

$$a : c :: a' : c'.$$

$$\text{For if } \frac{a}{b} = \frac{a'}{b'}, \text{ and } \frac{b}{c} = \frac{b'}{c'},$$

$$\text{then also } \frac{a}{b} \times \frac{b}{c} = \frac{a'}{b'} \times \frac{b'}{c'};$$

$$\text{and therefore } \frac{a}{c} = \frac{a'}{c'},$$

$$\text{or } a : c :: a' : c'.$$

A similar demonstration will apply when there are more than three magnitudes in each set.

Ex æquali in proportione perturbatâ.

413. "If there be any number of magnitudes and as many others, which, taken two and two in a *cross* order, have the same ratio; namely, if the first be to the second of the first set of magnitudes, as the last but one to the last of the second set of magnitudes, and the second to the third of the first set of magnitudes, as the last but two to the last but one of the second set of magnitudes: then (*ex æquo perturbato* or *ex æquali in proportione perturbatâ*), the first of the first set of magnitudes shall be to the last as the first of the second set of magnitudes is to the last."

Let a, b, c be the first, and a', b', c' the second set of magnitudes; and let

$$a : b :: b' : c',$$

$$\text{and} \quad b : c :: a' : b',$$

$$\text{then} \quad a : c :: a' : c'.$$

$$\text{For if} \quad \frac{a}{b} = \frac{b'}{c'} \quad \text{and} \quad \frac{b}{c} = \frac{a'}{b'},$$

$$\text{then also} \quad \frac{a}{b} \times \frac{b}{c} = \frac{b'}{c'} \times \frac{a'}{b'}; \text{ and therefore, } \frac{a}{c} = \frac{a'}{c'},$$

$$\text{or} \quad a : c :: a' : c'.$$

A similar demonstration will apply when there are more than three magnitudes in each set.

Duplicate ratio in Geometry.

414. "If there be three magnitudes in continued proportion, the first shall have to the last the *duplicate* ratio of the first to the second."

Let $a : b :: b : c$, then also $a^2 : b^2 :: a : c$.

$$\text{For if} \quad \frac{a}{b} = \frac{b}{c},$$

then also $\frac{a}{b} \times \frac{a}{b} = \frac{b}{c} \times \frac{a}{b}$ or $\frac{a^2}{b^2} = \frac{a}{c}$;

$$\text{or } \frac{a^2}{b^2} = \frac{a}{c};$$

therefore $a^2 : b^2 :: a : c$.

In Geometry, where there is no independent definition of ratio, it is the enunciation of this proposition which forms the definition of duplicate ratios.

Triplicate ratio in Geometry.

415. "If there be four magnitudes in continued proportion, the first shall have to the fourth the duplicate ratio of the first to the second."

Let $a : b :: b : c :: c : d$, then $a^3 : b^3 :: a : d$.

$$\text{For if } \frac{a}{b} = \frac{b}{c} = \frac{c}{d},$$

then $\frac{a^2}{b^2} = \frac{a}{c}$, and $\frac{a^2}{b^2} \times \frac{a}{b} = \frac{a}{c} \times \frac{c}{d}$, or $\frac{a^3}{b^3} = \frac{a}{d}$;

therefore $a^3 : b^3 :: a : d$.

It is the enunciation of this proposition which forms the definition of triplicate ratio in Geometry.

Composition and resolution of proportions.

416. "If there be four magnitudes, a, b, c, d, which are proportionals, and four others, a', b', c', d', which are proportionals also, then their corresponding products or quotients,

$$aa',\ bb',\ cc',\ dd',$$

$$\text{or } \frac{a}{a'},\ \frac{b}{b'},\ \frac{c}{c'},\ \frac{d}{d'},$$

shall also be proportionals."

For if $\frac{a}{b} = \frac{c}{d}$, and $\frac{a'}{b'} = \frac{c'}{d'}$,

$$\text{then also}\quad \frac{a}{b}\times\frac{a'}{b'}=\frac{c}{d}\times\frac{c'}{d'},$$

$$\text{or}\quad aa' : bb' :: cc' : dd'.$$

$$\text{Again, if}\quad \frac{a}{b}=\frac{c}{d},\quad\text{and}\quad \frac{a'}{b'}=\frac{c'}{d'},$$

$$\text{then}\quad \frac{a}{b}\times\frac{b'}{a'}=\frac{c}{d}\times\frac{d'}{c'},\quad\text{or}\quad \frac{a}{a'}\times\frac{b'}{b}=\frac{c}{c'}\times\frac{d'}{d};$$

$$\text{and therefore}\quad \frac{\left(\frac{a}{a'}\right)}{\left(\frac{b}{b'}\right)}=\frac{\left(\frac{c}{c'}\right)}{\left(\frac{d}{d'}\right)},$$

$$\text{or}\quad \frac{a}{a'} : \frac{b}{b'} :: \frac{c}{c'} : \frac{d}{d'}.$$

417. "If four quantities, a, b, c, d, be proportionals, then a^n, b^n, c^n, d^n, shall also be proportionals."

$$\text{For if}\quad \frac{a}{b}=\frac{c}{d},$$

$$\text{then also}\quad \frac{a^n}{b^n}=\frac{c^n}{d^n},$$

$$\text{or}\quad a^n : b^n :: c^n : d^n.$$

418. "If there be any number of ratios which are equal to each other, then as one antecedent is to its consequent, so shall all the antecedents together be to all the consequents together."

$$\text{Let}\quad a : b :: c : d :: e : f,$$

$$\text{then also,}\quad a : b :: a+c+e : b+d+f.$$

For if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$,

then we have $\frac{a}{b} = \frac{a}{b}$, or $ab = ba$

$\frac{a}{b} = \frac{c}{d}$, or $ad = bc$

$\frac{a}{b} = \frac{e}{f}$, or $af = be$,

and consequently

$$ab + ad + af = ba + bc + be,$$

or, $$a(b + d + f) = b(a + c + e),$$

and therefore $$\frac{a}{b} = \frac{a + c + e}{b + d + f},$$

or, $$a : b :: a + c + e : b + d + f.$$

The same demonstration may easily be extended to any number of ratios which are equal to each other.

Self evident consequences of the algebraical definition of proportion are not always self evident consequences of the geometrical definition.

419. There are many propositions in the fifth book of Euclid, where hypotheses are made concerning ratios as greater and less than, as well as equal to, each other, and the circumstances under which they are so, which are either self evident, or so nearly self evident, consequences of the algebraical definition of ratio, that it will not be necessary to notice them in this place, as distinct propositions to be demonstrated: the case however is very different in a system of Geometry, where the want of a definition of ratio, as disconnected with the definition of proportion, makes it necessary to consider such properties of ratios, as much the objects of demonstration, as any of the properties of proportions or proportionals: thus when it is said that "of unequal magnitudes, the greater has a greater ratio to the same than the less has," it is required to bring this proposition within the operation of the definition of quantities

which are not proportionals, and to shew that the conditions are such as coincide with the circumstances under which the first of four magnitudes is defined to have to the second a greater ratio than the third to the fourth, which is in this case identical with the second: if however we represent algebraically the same three magnitudes by a, b, and c, where a is greater than c, then the first ratio is represented by $\frac{a}{b}$, and the second by $\frac{c}{b}$: and it is a necessary consequence of the meaning attached to the term *fraction*, which is synonymous with that of ratio, that the first fraction under such circumstances is greater than the second: no formal demonstration could add to the evidence of such a proposition.

The same observation would apply to the following proposition: "if there be six magnitudes, and if the first bears to the second a greater ratio than the third to the fourth, but the third does not bear to the fourth a less ratio than the fifth to the sixth: then the first bears to the second a greater ratio than the fifth to the sixth."

If the six magnitudes be severally denoted by a, b, c, d, e, f, then the three ratios which are the objects of the proposition are $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ respectively; if $\frac{a}{b}$ be greater than $\frac{c}{d}$, and $\frac{c}{d}$ be not less than $\frac{e}{f}$, then we infer immediately that $\frac{a}{b}$ is greater than $\frac{e}{f}$, and no demonstration can add to the evidence of such a conclusion: in the absence however of any geometrical mode of defining, independently of each other, the values of these ratios, it becomes requisite to shew that there are some equimultiples of the first and fifth, and also of the second and sixth, where the multiple of the first is greater than that of the second, but the multiple of the fifth not greater than that of the sixth: a conclusion by no means self evident, but requiring the authority of a formal demonstration.

CHAP. XII.

GENERAL THEORY OF SIMPLE ROOTS, WITH THE PRINCIPLES OF THE APPLICATION OF ALGEBRA TO GEOMETRY.

General meaning of the terms *root* and *power*.

420. WE have before explained the meaning of the term *root* in its ordinary acceptation, as the inverse of the term *power*, (Art. 13.): and have also investigated and exemplified, at considerable length, the rules for the determination of the *roots* from their corresponding *powers*, both for numbers and for simple and compound algebraical expressions (Chap. VIII.); there are other consequences of those operations, which are due to the fundamental assumptions of Algebra, which are amongst the most important in the whole science of symbols, which we shall now proceed to consider: it will form however a useful, and in some respects a necessary, introduction to this subject, to consider and determine the extent of the meaning which can be properly given to the term *root* in Arithmetic, and in a system of arithmetical algebra, properly so called.

Meaning of the term *root* in Arithmetic.

421. The term *root*, when applied to numbers or numerical quantities, denotes as we have seen, the number or numerical quantity, whether admitting of finite determination or not, which used as a factor in multiplication as often as its denomination indicates, produces the required number, whether fractional or not: thus the square or quadratic root of 25 is 5: for $5 \times 5 = 25$: the square root of $\frac{5329}{361}$ is $\frac{73}{19}$; for

$$\frac{73}{19} \times \frac{73}{19} = \frac{5329}{361};$$

the cubic or cube root of 1.030301 is 1.01; for $1.01 \times 1.01 \times 1.01$ produces 1.030301; the square root of 2 is denoted symbolically by $\sqrt{2}$, there being no finite arithmetical quantity which is equal to it: but the general rule for the extraction of the square root in numbers, will give us an approximate determination of its value, which is the more accurate, the further the operation is continued.

Its meaning in arithmetical algebra.

422. In a system of arithmetical algebra, which rejects the independent use of the signs + and −, and in which the values of the symbols are limited as far as this rejection makes it necessary, the term *root* will strictly coincide with its meaning in common arithmetic: it will denote the numerical quantity (though it may be symbolically represented) which used as a factor in multiplication as often as its denomination indicates, will produce the required expression: as *quantity* alone therefore is considered in such roots, there can be one root only corresponding to one power: and as no *affection* of quantity is capable of being considered in the powers, no affection of quantity can be considered in the roots.

Examples.

423. Thus the square root of a^2 is a and a only: the second root $-a$ belongs exclusively to a system of symbolical algebra: the square root of $a^2 - 2ab + b^2$ is $a - b$ or $b - a$, according as a is greater than b, or a less than b: the cube root of a^3 is a and a only: there are two other algebraical roots which we shall afterwards have occasion to notice: the cube root of

$$a^3 - 3a^2b + 3ab^2 - b^3$$

is $a - b$, when a is greater than b: if a be less than b, neither the cube nor its root, can be considered as having an arithmetical existence: it is hardly necessary however to extend these observations farther, as we have already sufficiently considered the limitations in the use and values of symbols, which a system of arithmetical algebra renders necessary. (Chap. III.)

Extended usage of the term arithmetical root.

424. The symbols in arithmetical algebra are equally comprehensive with numbers, and therefore may express lines or areas or any other species of quantity which numbers are capable of representing: in all cases therefore, when such quantities admit of powers which are

capable of interpretation, there will be a corresponding root, which may be called the arithmetical root, as belonging to this system of arithmetical algebra, though the quantities which they represent may not be so: thus if a^2 be an area, a is the line upon which an equal square may be described: if a^3 be a solid, a is the edge of an equal cube, and similarly in other cases: this extended use of the term *arithmetical root*, therefore, is limited by the independent use of the signs + and − only, and not by the nature of the quantities represented.

The term *root* is always the inverse of the term *power*.

425. In symbolical as well as in arithmetical algebra, we may consider the term *root* as the inverse of the term *power*, and conversely: for if the n^{th} *power* (or what is so called) of a be denoted by a^n, its n^{th} *root* may be denoted by $a^{\frac{1}{n}}$: and the general principle of indices will lead us to the equations $(a^n)^{\frac{1}{n}} = (a^{\frac{1}{n}})^n = a$: it follows therefore that the n^{th} root of the n^{th} power of a is equal to the n^{th} power of the n^{th} root of a, and the result is in both cases the original symbol or quantity a: in other words, the operation denoted by the n^{th} power of a, *undoes* the effect of the operation denoted by the n^{th} root of a, and, conversely, whatever the value of n may be, and whatever may be the particular nature and meaning of the operations which they severally denote: and it is for this reason that we conclude generally that those operations are respectively the *inverse* of each other.

Its meaning for particular values of the index.

426. When however we pass from general to particular values of this index, the same principle will at once lead us to the interpretation of the meaning of the operation which it denotes, whether the subjects of such operation be symbols, or numbers, or any quantities whatever which they may represent: thus, the square root of $+a$ or $-a$ will be the symbolical quantity which multiplied into itself, will produce $+a$ or $-a$: the n^{th} root of $+a$ or $-a$ (where n is a whole number) will be that symbolical quantity which employed as a factor

n times, will produce $+a$ or $-a$: and similarly in all other cases, whatever be the symbols or algebraical expressions, whose square or other roots may require to be determined or expressed.

Conditions which symbolical roots must satisfy.

427. In a system of symbolical algebra, the expressions $+a^2$ and $-a^2$, or, a^2 and $-a^2$, express different affections of the same quantity, which, in a system of arithmetical agebra, would be equally denoted by a^2: and there is only one arithmetical root a, which equally belongs to a^2, $+a^2$ and $-a^2$; for it is dependent upon the magnitude only of a^2 and not upon the particular affections, which may be denoted by the signs $+$ and $-$ attached to it: but in considering the algebraical roots of a^2 and $-a^2$, we must have regard only to the conditions which such roots must satisfy; and there will of course be as many roots, as there are symbolical expressions, which, in conformity with the general rules and assumptions of Algebra, will equally produce the required result: thus all expressions are equally algebraical roots of a^2 or of $-a^2$, which multiplied into themselves, produce a^2 or $-a^2$: and similarly in all other casses.

Two algebraical square roots of quantities affected with the sign $+$.

428. It is upon this principle, that we consider a and $-a$ as equally algebraical square roots of a^2: for $a \times a = a^2$ and also $-a \times -a = a^2$: or in other words a and $-a$, equally answer the algebraical conditions which the square root of a^2 is required to satisfy: in a similar manner, if we denote the arithmetical square root of a by $\sqrt{a}$, the two algebraical square roots of a will be severally denoted by $+\sqrt{a}$ and $-\sqrt{a}$, the first of which coincides with the arithmetical root: again, the two algebraical square roots of $a^2 - 2ab + b^2$ are equally $a - b$ and $b - a$, or, $(a - b)$ and $-(a - b)$, one of which only, under its most simple form, will coincide with the arithmetical root.

429. If we now proceed to consider the algebraical square roots of $-a^2$, it will immediately appear that such roots cannot be denoted by any simple algebraical symbol

like a or b, affected with the ordinary signs $+$ or $-$: for the general rule to which these signs are subject, makes it necessary that the square of any symbol affected with either one or other of these signs, must be affected with the sign $+$ only, and consequently such a result can never be identical with $-a^2$. If we are confined, therefore, to the use of those signs, $+$ and $-$, as indicating the only affections of quantity which are capable of being considered, we must conclude that there are no square roots of $-a^2$, or that the language of symbols is incompetent to express them.

Conditions to be satisfied by the square roots of quantities affected by the sign $-$.

430. It is true, that under such circumstances we might denote the square root of $-a^2$ by $\sqrt{-a^2}$, by merely prefixing to $-a^2$ the symbol of operation, $\sqrt{}$: and also that

In what case would the square root of $-a^2$ be termed imaginary.

$$\sqrt{-a^2} \times \sqrt{-a^2} = -a^2,$$

$$\text{and} \quad -\sqrt{-a^2} \times -\sqrt{-a^2} = -a^2,$$

would be identical equations, which are perfectly independent of the actual performance of the operation denoted by $\sqrt{}$, whether practicable or not: if there was no mode of denoting the result of this operation, the operation might be properly termed *impossible*, and the result *imaginary*: but such imaginary results would become equally real with the square roots of a^2, if signs can be assumed which are not inconsistent with the existing signs $+$ and $-$, and which satisfy all the algebraical conditions which the case requires.

431. If we consider, therefore, a as the arithmetical square root of $-a^2$, and assume, in the first place, i to represent the *imaginary* or peculiar sign, which must be attached to it in order to convert it into the proper algebraical root, we shall have

Symbolical determination of the sign $\sqrt{-1}$.

$$\sqrt{-a^2} = i \, . \, a;$$

and therefore

$$-a^2 = (i)^2 a^2,$$

treating the sign i as an ordinary symbol: but $-a^2$ is likewise equal to $(-1)a^2$, and therefore

$$-a^2 = (i)^2 a^2 = (-1)a^2,$$

$$\text{and} \quad (i)^2 = -1.$$

If we still continue to consider $(i)^2$ and -1 as subject to the same laws as ordinary symbols, we shall get

$$i = \sqrt{-1};$$

and therefore

$$\sqrt{-a^2} = i \,.\, a = \sqrt{-1} \,.\, a, \quad \text{or} \quad a\sqrt{-1}.$$

It thus appears that we may use $\sqrt{-1}$ as a sign of affection, whether admitting of interpretation or not, equally with the signs + and −, and also in perfect accordance with its ordinary symbolical character.

Quantities affected with the sign $\sqrt{-1}$, susceptible also of the signs + and −.

432. The square root of $-a^2$ is equally

$$+a\sqrt{-1} \quad \text{and} \quad -a\sqrt{-1},$$

for the same reason that the square root of a^2 is equally $+a$ and $-a$; for

$$-a\sqrt{-1} \times -a\sqrt{-1} = -a^2,$$

equally with

$$a\sqrt{-1} \times a\sqrt{-1}:$$

it follows, therefore, that $a\sqrt{-1}$ is susceptible of the signs + and −, and therefore of those affections which they may denote, which are proper to the quantity a modified by the sign $\sqrt{-1}$; a fact which will be found to be important, when we come to consider the principles of its interpretation.

433. We shewed on a former occasion, that the assumption of the independent use of the signs + and − was essential to the unlimited values of the symbols of

Algebra, however connected with each other, and also explained the advantages which resulted from making those signs of affection identical with the signs of the operations of Addition and Subtraction: the assumption of the sign $\sqrt{-1}$ is equally necessary for nearly the same reasons: for unless we are at liberty to consider the square roots of quantities affected with the sign $-$, we must necessarily suppose a greater than b, in such expressions as $\sqrt{(a^2-b^2)}$, a restriction which is contrary to the *assumed* generality of representation of the symbols of Algebra: if we suppose, therefore, a to be less than b, or that $a^2=b^2+c^2$, this expression will become $\sqrt{-c^2}$, and the consideration of such quantities will become a necessary consequence of this fundamental assumption of Algebra.

The use of the sign $\sqrt{-1}$ is necessary to the assumed generality of algebraical symbols.

434. We may denote a^2 by $a^2 \times 1$, as well as $-a^2$ by $a^2 \times -1$: and consequently we may denote $\sqrt{a^2}$ by $a\sqrt{1}$, in the same manner as we denote $\sqrt{-a^2}$ by $a\sqrt{-1}$. Under such circumstances, we may consider $\sqrt{1}$ and $\sqrt{-1}$, as the recipients of the peculiar affections which may or may not distinguish the arithmetical from the algebraical square root; for $\sqrt{1}$ is symbolically identical with $+1$ or -1, and therefore $a\sqrt{1}$ is symbolically identical with $+a$ and $-a$: in the same manner, $\sqrt{-1}$ being equally susceptible of the two signs $+$ and $-$, it follows that $a\sqrt{-1}$ includes within itself the double value $+a\sqrt{-1}$ and $-a\sqrt{-1}$.

$\sqrt{1}$ and $\sqrt{-1}$ the recipients of the affections which distinguish the arithmetical from the algebraical square root.

435. The same principle of representation may be extended to the n^{th} roots of a^n, or $-a^n$; for a^n is identical with $a^n \times 1$, and $-a^n$ is also identical with $a^n \times (-1)$: it follows, therefore, that

The same observation extended to $\sqrt[n]{1}$ and $\sqrt[n]{-1}$.

$$\sqrt[n]{a^n} \quad \text{or} \quad (a^n)^{\frac{1}{n}} = a\sqrt[n]{1} \quad \text{or} \quad a(1)^{\frac{1}{n}}:$$

and also

$$\sqrt[n]{-a^n} \quad \text{or} \quad (-a^n)^{\frac{1}{n}} = a\sqrt[n]{-1} \quad \text{or} \quad a(-1)^{\frac{1}{n}}.$$

It would thus appear, that whatever may be the peculiar affections which must modify the common arithmetical root of a^n and $-a^n$, in order to form the corresponding algebraical roots, they must be symbolized by the similar roots of 1 and -1; and, if capable of determination, they may be determined from them.

In all cases, indeed, it is convenient for the purposes of a general theory, to consider 1 and -1 as the recipients of all the peculiar affections which the signs of Algebra may be made to express: thus,

$$\sqrt[3]{1} \text{ and } \sqrt[3]{-1}, \quad \sqrt[4]{1} \text{ and } \sqrt[4]{-1},$$
$$\sqrt[n]{1} \text{ and } \sqrt[n]{-1},$$

may be considered as signs or symbols expressing affections and not magnitude, equally with $\sqrt{1}$ and $\sqrt{-1}$, and as forming the symbolical connection between the corresponding arithmetical and algebraical roots of different and corresponding denominations: it will form one of the principal objects of the investigations which follow, to ascertain whether the affections thus symbolized are multiple or simple, and whether they are susceptible or not of symbolical transformation into others which are equivalent.

Examples of algebraical operations connected with the sign $\sqrt{-1}$.

436. Before we proceed to the interpretation of the sign $\sqrt{-1}$, under whatever circumstances it may present itself, and the very important consequences which will be found to follow from it, it may be proper to give a few examples of the reduction and transformation of algebraical expressions in which this sign occurs.

(1) $\sqrt{-49} = 7\sqrt{-1}$. In this and in other cases, we shall consider the sign $\sqrt{-1}$ as bearing the

double sign + or −, when their separate consideration is not required.

(2) $\sqrt{-20} = 2\sqrt{5}\sqrt{-1}$.

(3) $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) = 2a$.

(4) $(a + b\sqrt{-1}) - (a - b\sqrt{-1}) = 2b\sqrt{-1}$.

(5) $(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = a + c$
$+ (b + d)\sqrt{-1}$.

(6) $(a + b\sqrt{-1}) - (c + d\sqrt{-1}) = a - c$
$+ (b - d)\sqrt{-1}$.

(7) $a\sqrt{-b} = a\sqrt{b}\sqrt{-1}$.

(8) $a\sqrt{-b} \times \alpha\sqrt{-\beta} = -a\alpha\sqrt{b\beta}$;

for $a\sqrt{-b} = a\sqrt{b}\sqrt{-1}$, and $\alpha\sqrt{-\beta} = \alpha\sqrt{\beta}\sqrt{-1}$; and therefore

$$a\sqrt{-b} \times \alpha\sqrt{-\beta} = a\alpha\sqrt{b\beta}\,.\,(-1) = -a\alpha\sqrt{b\beta}.$$

The quantity $\sqrt{b\beta}$ bears the double sign + or −, and consequently the product we are considering may be affected with either of the signs + or −; the result which we have put down is such as will arise, in case $\sqrt{b}$ and $\sqrt{\beta}$ were both affected with the same sign, whether + or −.

(9) $\sqrt{(-a^2 + 2ab - b^2)} = (a - b)\sqrt{-1}$.

(10) $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2$.

It thus appears, that $a^2 + b^2$ is equally resolvible into factors with $a^2 - b^2$.

(11) $(a + b\sqrt{-1})^2 = a^2 + 2ab\sqrt{-1} - b^2$.

(12) $(a+b\sqrt{-1})\,(c+d\sqrt{-1}) = ac-bd$
$+(ad+bc)\sqrt{-1}$.

(13) $(a-b\sqrt{-1})\,(c-d\sqrt{-1}) = ac-bd$
$-(ad+bc)\sqrt{-1}$.

(14) $(a+b\sqrt{-1}+c\sqrt{-1})\,(a-b\sqrt{-1}-c\sqrt{-1})$
$=a^2+(b+c)^2$.

(15) $(a+b\sqrt{-1}+c\sqrt{-1})(a+b\sqrt{-1}-c\sqrt{1-})$
$=(a+b\sqrt{-1})^2+c^2=a^2-b^2+c^2+2ab\sqrt{-1}$.

(16) $(a+b\sqrt{-1}+c\sqrt{-1})\,(a+b\sqrt{-1}-c\sqrt{-1})$
$\times\,(a-b\sqrt{-1}+c\sqrt{-1})\,(a-b\sqrt{-1}-c\sqrt{-1})$
$=a^4+b^4+c^4-2a^2b^2-2a^2c^2+2b^2c^2$.

(17) $(\sqrt{-1})^2=-1$.

(18) $(\sqrt{-1})^3=-\sqrt{-1}$.

(19) $(\sqrt{-1})^4=1$.

(20) $(\sqrt{-1})^5=\sqrt{-1}$. The same values occur after every fourth term of the series

$(\sqrt{-1})^6$, $(\sqrt{-1})^7$, $(\sqrt{-1})^8$, $(\sqrt{-1})^9$,

and so on, continued as far as we please.

(21) $(1+\sqrt{-1})^2=2\sqrt{-1}$.

(22) $(1-\sqrt{-1})^2=-2\sqrt{-1}$.

(23) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^3=1$.

(24) $$\left(\frac{-1-\sqrt{-3}}{2}\right)^3 = 1.$$

The two expressions,

$$\frac{-1+\sqrt{-3}}{2}, \text{ and } \frac{-1-\sqrt{-3}}{2},$$

answer the conditions which are required to be satisfied by the cube roots of 1.

(25) $$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{-1-\sqrt{-3}}{2}.$$

In a similar manner it will be found, that

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{-1+\sqrt{-3}}{2}.$$

(26) $$\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^4 = -1 = \left(\frac{1-\sqrt{-1}}{\sqrt{2}}\right)^4.$$

(27) $$\left(\frac{-1+\sqrt{-1}}{\sqrt{2}}\right)^4 = -1 = \left(\frac{-1-\sqrt{-1}}{\sqrt{2}}\right)^4.$$

(28) $$\frac{a\sqrt{-1}}{b\sqrt{-1}} = \frac{a}{b}.$$

(29) $$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = \frac{a^2+b^2}{(a-b\sqrt{-1})^2} = \frac{(a+b\sqrt{-1})^2}{a^2+b^2}$$

$$= \frac{a^2-b^2}{a^2+b^2} + \frac{2ab\sqrt{-1}}{a^2+b^2}.$$

(30) $$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}} = \frac{ac+bd-(ad-bc)\sqrt{-1}}{c^2+d^2}.$$

(31) $$\frac{a-b\sqrt{-1}}{c-d\sqrt{-1}} = \frac{ac+bd+(ad-bc)\sqrt{-1}}{c^2+d^2}.$$

(32) $$\frac{1}{a+b\sqrt{-1}} = \frac{a-b\sqrt{-1}}{a^2+b^2}$$

(33) $$\frac{1}{a+b\sqrt{-1}} + \frac{1}{a-b\sqrt{-1}} = \frac{2a}{a^2+b^2}.$$

(34) $$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} + \frac{a-b\sqrt{-1}}{a+b\sqrt{-1}} = \frac{2(a^2-b^2)}{a^2+b^2}.$$

(35) $$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} - \frac{a-b\sqrt{-1}}{a+b\sqrt{-1}} = \frac{4ab\sqrt{-1}}{a^2+b^2}.$$

(36) $$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}} + \frac{a-b\sqrt{-1}}{c-d\sqrt{-1}} = 2 \cdot \frac{ac+bd}{c^2+d^2}.$$

(37) $$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}} - \frac{a-b\sqrt{-1}}{c-d\sqrt{-1}}$$
$$= 2 \cdot \frac{(bc-ad)\sqrt{-1}}{c^2+d^2}.$$

(38) $$\sqrt{(a+b\sqrt{-1})} + \sqrt{(a-b\sqrt{-1})}$$
$$= \sqrt{\{2a + 2\sqrt{(a^2+b^2)}\}}.$$ (Ex. 62. Art. 181.)

(39) $$\sqrt[4]{(a+b\sqrt{-1})} + \sqrt[4]{(a-b\sqrt{-1})}$$
$$= \sqrt{\{\sqrt{\{2a+2\sqrt{(a^2+b^2)}\}} + 2\sqrt[4]{(a^2+b^2)}\}}.$$
(Ex. 64. Art. 183.)

(40) $$(a-b\sqrt{-1})(a^2+ab\sqrt{-1}-b^2)$$
$$= a^3+b^3\sqrt{-1}.$$

(41) $$(a-b\sqrt{-1})+$$
$$(a^4+a^3b\sqrt{-1}-a^2b^2-ab^3\sqrt{-1}+b^4)$$
$$= a^5-b^5\sqrt{-1}.$$

(42) $$\frac{a^2+b^2}{a+b\sqrt{-1}}=a-b\sqrt{-1}.$$

(43) $$\frac{a^3-b^3\sqrt{-1}}{a+b\sqrt{-1}}=a^2-ab\sqrt{-1}-b^2.$$

(44) $$\frac{a^4-b^4}{a+b\sqrt{-1}}=a^3-a^2b\sqrt{-1}-ab^2+b^3\sqrt{-1}.$$

(45) $$\frac{a^2-b^2}{\sqrt{a}+\sqrt{b}.\sqrt{-1}}$$

$$=a^{\frac{3}{2}}-ab^{\frac{1}{2}}\sqrt{-1}-a^{\frac{1}{2}}b+b^{\frac{3}{2}}\sqrt{-1}.$$

(46) $$(a+b\sqrt{-1})^m=a^m+ma^{m-1}b\sqrt{-1}$$

$$-\frac{m(m-1)}{1.2}a^{m-2}b^2-\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}b^3\sqrt{-1}$$

$$+\frac{m(m-1)(m-2)(m-3)}{1\;.\;2\;.\;3\;.\;4}a^{m-4}b^4+\&c.$$

(47) $$(a+b\sqrt{-1})^m+(a-b\sqrt{-1})^m$$

$$=2\left\{a^m-\frac{m(m-1)}{1.2}a^{m-2}b^2\right.$$

$$\left.+\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}a^{m-4}b^4+\&c.\right\}.$$

(48) $$(a+b\sqrt{-1})^m-(a-b\sqrt{-1})^m$$

$$=2\sqrt{-1}\left\{ma^{m-1}b-\frac{m(m-1)(m-2)(m-3)}{1\;.\;2\;.\;3}a^{m-3}b^3\right.$$

$$\left.+\frac{m(m-1)\ldots.(m-5)}{1.2\ldots\ldots5}a^{m-5}b^5-\&c.\right\}.$$

$$(49)\quad \sqrt{(a+b\sqrt{-1})}$$

$$= \sqrt{\left\{\frac{a}{2} + \frac{1}{2}\sqrt{(a^2+b^2)}\right\}}$$

$$+ \sqrt{\left\{-\frac{a}{2} + \frac{1}{2}\sqrt{(a^2+b^2)}\right\}}\sqrt{-1}.$$

The identity of these expressions may be ascertained by squaring both sides.

$$(50)\quad \sqrt{\{a - b\sqrt{-1}\}}$$

$$= \sqrt{\left\{\frac{a}{2} + \frac{1}{2}\sqrt{(a^2+b^2)}\right\}}$$

$$- \sqrt{\left\{-\frac{a}{2} + \frac{1}{2}\sqrt{(a^2+b^2)}\right\}}\sqrt{-1}.$$

Principles of interpretation of the sign $\sqrt{-1}$.

437. There are certain general principles which must be attended to, in the interpretation of the meaning of the sign $\sqrt{-1}$, when attached to a symbol with a specific value, which we shall now proceed to consider.

Meaning of the term *affectation*.

If we suppose the sign $\sqrt{-1}$ to designate a property or affection of quantity, dependent both upon the specific nature of the quantity itself and other general algebraical conditions which it may be required to satisfy, then we may likewise consider the attaching this sign to a symbol which designates any given quantity, as equivalent to imposing that peculiar affection upon it, which is appropriate to them both: thus if a designated a line drawn in a given direction, and if we supposed or should otherwise determine that the sign $\sqrt{-1}$ in such a case designated the movement or transfer of that line through a given angle, then $a\sqrt{-1}$ would represent the line in question in its new position: the *affectation* (if we may be allowed to use this term in such a sense) of this second line again with the same sign $\sqrt{-1}$, which is equivalent to

$$a\sqrt{-1}\sqrt{-1} \text{ or to } -a,$$

would designate a line making the same angle with $a\sqrt{-1}$, that the line designated by $a\sqrt{-1}$, makes with that designated by a: under such circumstances, it is very easy to conceive the successive *affectation* of the line a by the same sign $\sqrt{-1}$, which will be symbolically represented by

$$a\sqrt{-1},\ \ a\sqrt{-1}\sqrt{-1},\ \ a\sqrt{-1}\sqrt{-1}\sqrt{-1}, \text{ \&c.}$$

or by

$$a\sqrt{-1},\ \ a(\sqrt{-1})^2,\ \ a(\sqrt{-1})^3,\ \ a(\sqrt{-1})^4, \text{ \&c.}$$

and which will designate a series of lines making with each other, the same angle which the line denoted by $a\sqrt{-1}$ makes with the primitive line a.

438. Again, the symbolical quantities

Multiple affectations.

$$a(\sqrt{-1})^2,\ \ a(\sqrt{-1})^3,\ \ a(\sqrt{-1})^4,$$
$$a(\sqrt{-1})^5,\ \ a(\sqrt{-1})^6, \text{ \&c.}$$

which represent the result of the *double*, *triple*, *quadruple*, *quintuple*, *sextuple*, &c. *affectation* of the line a by the sign $\sqrt{-1}$, are likewise equivalent to

$$-a,\ \ -a\sqrt{-1},\ \ a,\ \ a\sqrt{-1},\ \ -a, \text{ \&c.}$$

where the same symbolical quantities recur for every *fourth affectation*: it will follow therefore that the lines represented by

$$a(\sqrt{-1})^2,\ \ a(\sqrt{-1})^3,\ \ a(\sqrt{-1})^4,$$
$$a(\sqrt{-1})^5,\ \ a(\sqrt{-1})^6,$$

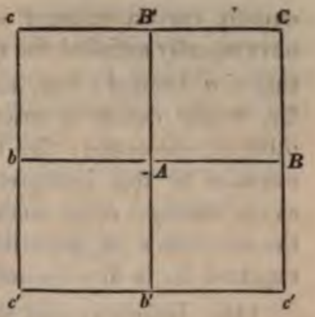
Algebra are necessary, the interpretations which we give to them, have no essential connection with them: they form in all cases a subject of investigation which is secondary to the deduction of the results themselves, and are admissible only so far as they are consistent with them.

A necessary connection between the interpretation of a general result, and others which are subordinate to it.

443. But though there exists no essential connection between the symbolical results and their interpretation, yet there may exist a necessary and mathematical connection between the interpretations themselves: for the interpretation of the more general result, must include the interpretation of the other results which are subordinate to it: thus, the interpretation of the succession of symbolical results $a, a\sqrt{-1}, a(\sqrt{-1})^2, a(\sqrt{-1})^3$, &c. necessarily includes that of $+a$ and $-a$: if by any other means therefore we have determined the interpretation of $+a$ and $-a$, when a has a specific value, then its coincidence with the interpretation derived from the more general case is essential to the correctness of one at least of the two: under such circumstances, it is the authority of the less general case, which most commonly prevails, inasmuch as it generally precedes the interpretation of the more general case in the order of investigation, and is more immediately and more essentially connected with the first principles of the science.

Tests of the accuracy of interpretations.

444. Assuming therefore the correctness of the interpretation of the less general case, it is by an inductive process of reasoning only, that we pass from it to the interpretation of the more general case, and the existence of one does not determine, in the mathematical sense of the term, the existence of the other: but it is the necessary connection which exists between the interpretation of the more general result and those which are subordinate to it, which makes it so important to examine and ascertain the latter, as tests of the *unity* as well as correctness of the interpretation of the former: of this kind is the following, as applicable to our interpretation of the sign $\sqrt{-1}$, when applied to symbols designating lines.



If upon the equal lines AB, AB', Ab, Ab' which are at right angles to each other, we describe a series of squares

$$ABCB',\ AB'cb,$$
$$Abc'b',\ Ab'c'B,$$

it would follow from principles, which we have considered before (Art. 104), that if the first and third were affected with the sign $+$, the second and fourth must necessarily be affected with the sign $-$: now the first and third are constructed upon the lines AB and Ab, which may be designated by the symbols $+a$ and $-a$: and they are similarly described upon them, that is, towards the same parts, from right to left: in a similar manner, the second and fourth are the squares similarly described upon the lines AB' and Ab': now the first and third of these squares are symbolically represented by a^2, which is the square of $+a$ or $-a$: whilst the second and fourth are symbolically represented by $-a^2$, which is the square of $+a\sqrt{-1}$ and $-a\sqrt{-1}$: it follows, therefore, that if AB and Ab are denoted by $+a$ and $-a$, that AB' and Ab' may be correctly denoted by $+a\sqrt{-1}$ and $-a\sqrt{-1}$; for such a mode of denoting these lines, or such an interpretation of the symbolical quantities $+a\sqrt{-1}$ and $-a\sqrt{-1}$, fully satisfies the conditions which are required to be fulfilled: and if we take for granted the correctness of the interpretation of $+a$ and $-a$, and of $+a^2$ and $-a^2$, it is the only interpretaion of $+a\sqrt{-1}$ and $-a\sqrt{-1}$, which can do so.

Causes of the occasional vagueness of interpretations.

445. Interpretations must in many cases partake in some degree of the vagueness of the language in which they are expressed: thus we have said that the *affectation* of a, when denoting a line, with the sign $\sqrt{-1}$, was equivalent to its transfer through an angle of 90^0: but it would have been

equally correct to have said, or, in other words, it would have equally satisfied the symbolical conditions to have said, that if a denoted a line, $a\sqrt{-1}$, $a(\sqrt{-1})^2$, $a(\sqrt{-1})^3$, &c. would denote a series of lines at right angles to each other in succession: the idea of transfer is in no respect essential to this interpretation and is merely introduced as one amongst other modes of exhibiting or of producing the succession of quantities which the symbolical results required to be interpreted, may be made to represent.

This interpretation of $\sqrt{-1}$ applicable to lines representing forces or velocities.

446. Before we quit the consideration of the interpretation of the sign $\sqrt{-1}$ when applied to lines, it may be proper to remark that it will equally apply to any other quantities which lines may represent, and which are capable of the same affections with lines: thus if a and $-a$ denote two equal forces in opposite directions, $a\sqrt{-1}$ and $-a\sqrt{-1}$ may likewise denote two other equal forces in opposite directions, and acting at right angles to the former; if a and $-a$ denoted velocities in opposite directions, $a\sqrt{-1}$ and $-a\sqrt{-1}$ may denote velocities in opposite directions, which are likewise at right angles to the former: and similarly in other cases of a similar kind.

Interpretation under other circumstances.

447. There are many cases however of quantities, which cannot be represented, unless symbolically, by lines, which are susceptible of affections denoted by + and −, which are appropriate to their specific nature: thus if a represented property possessed, $-a$ may represent the same property owed: under such circumstances what is the meaning which may be attached to $a\sqrt{-1}$ and $-a\sqrt{-1}$?

Property and debt.

If we consider the succession of quantities

$$a,\ a\sqrt{-1},\ a(\sqrt{-1})^2,\ a(\sqrt{-1})^3,$$

or,

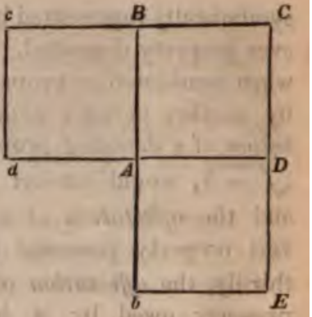
$$a,\ a\sqrt{-1},\ -a,\ -a\sqrt{-1},$$

and if the first represents property possessed, and the third property owed, the second can neither represent property possessed nor owed, under the same circumstances or by the same person, inasmuch as in such a case it would be symbolically represented by a or $-a$: it may represent however property deposited, which admits of similar relations when considered as property possessed and property owed by another person; under such circumstances, the *affectation* of a denoting property possessed by A by the sign $\sqrt{-1}$, would convert it into property possessed by B: and the *affectation* of $a\sqrt{-1}$ by $\sqrt{-1}$, would convert property possessed by B into property owed by A: thirdly, the *affectation* of $-a$ by $\sqrt{-1}$ would convert property owed by A into property owed by B: and fourthly, the *affectation* of $-a\sqrt{-1}$ by $\sqrt{-1}$ would convert property owed by B into property possessed by A: the repetition of the process of *affectation* by the sign $\sqrt{-1}$, would reproduce continually the same succession of transfers of property from A to B, and of conversions of property possessed into debt, and of debt into property possessed, which is required to correspond to the succession of the same symbolical results.

In this case, the interpretation of the sign $\sqrt{-}$ which we have given, satisfies the symbolical conditions, and also coincides with the interpretation of the meaning of the signs + and −, which is otherwise established: we cannot give it the additional authority of the coincidence of this interpretation with the interpretation of the meanings of the quantities corresponding to a^2 and $-a^2$, for those quantities in the case under consideration admit of no interpretation.

448. If a denoted a sum gained by A, $-a$ might denote the same sum lost by A: in this case the affectation of a by $\sqrt{-1}$, would transfer the gain from A to B: the affectation of $a\sqrt{-1}$ by $\sqrt{-1}$ would convert the gain of B, into a loss by A; the affectation of $-a$ by $\sqrt{-1}$ would transfer the loss from A to B: and the affectation of $-a\sqrt{-1}$ by $\sqrt{-1}$, would again transfer the loss by B, to a gain by A: we then obtain a repetition of the same changes in the same manner as in the other cases which we have considered. Loss and gain.

Its interpretation when applied to symbolical expressions representing rectangular areas.



449. If we supposed ab to represent the area $ABCD$, then $-ab$ would represent the area $ABcd$ or $ADEb$ and conversely: in this case $ab\sqrt{-1}$ might be interpreted to represent the equal rectangle described upon AB or AD, at right angles to $ABCD$, inasmuch as such an hypothesis would satisfy all the required symbolical conditions: for if the first *affectation* of ab or of $ABCD$ by $\sqrt{-1}$ moved the rectangular area $ABCD$ round AB through an angle of 90^0, the second *affectation* of ab by $\sqrt{-1}$, would move it through an additional angle of 90^0, and make it coincide with $ABcd$: and again, if the first affectation of ab or $ABCD$ by $\sqrt{-1}$ moved the rectangle $ABCD$ round AD through an angle of 90^0, the second affectation of ab by $\sqrt{-1}$ would move it through an additional angle of 90^0 and would make it coincide with $ADEb$; the result corresponding in both cases to the interpretation we have given, is represented by $-ab$, which is identical with the symbolical result $ab(\sqrt{-1})^2$; again, the triple affectation of ab or of the rectangle $ABCD$ with the sign $\sqrt{-1}$, would move it round AB or AD, through three right angles and the symbolical result, or $-ab\sqrt{-1}$ would therefore represent a rectangle whose plane is perpendicular to the planes of $ABCD$, $ABcd$ or $ADEb$, and occupying the same position therefore, with respect to the rectangle represented by $ab\sqrt{-1}$, that the rectangle represented by $-ab$ bears to the rectangle represented by ab; whilst the fourth affectation of ab by $\sqrt{-1}$, would move the corresponding rectangle through four right angles and make it coincide with the primitive rectangle $ABCD$ represented by ab, whether it moved it round AB or AD; it is not necessary to notice the repetitions of these

operations, which would produce a repetition of the same symbolical results corresponding to the same positions of the rectangles: and would thus shew that the interpretation which we have given, fully satisfies all the required symbolical conditions.

Different rectangles which may be represented by ab and $-ab$, $ab\sqrt{-1}$ and $-ab\sqrt{-1}$.

450. The rectangle $ABCD$, whose sides are $AD = a$ and $AB = b$, is represented by ab, whilst the rectangle $ABcd$, contained by $Ad = -a$ and $AB = b$, is represented by $-ab$: in this instance, as we have shewn in its proper place, (Art. 102.) the interpretation of the meaning of $+ab$ and $-ab$, whilst b remained the same, both in sign and magnitude, would be *determined* by that of $+a$ and $-a$, and would also equally correspond to all rectangles which can be similarly constructed upon lines which $+a$ and $-a$ may represent; they will therefore represent all rectangles which are parallel to each other, and therefore also perpendicular to the same plane. In this case, we have represented the second side AB of the rectangle by b, used as a symbol in arithmetical Algebra, neither possessing nor admitting any sign of affection which might distinguish its position as different from that occupied by the lines $+a$ and $-a$: if we still consider b as an arithmetical symbol, AB may continue to be denoted by b, whether in a position which is parallel to its first position or making any angle with it, and the rectangle formed by it with any line represented by $+a$ or $-a$, and consequently all such rectangles which are perpendicular to the same plane, would, under such circumstances, be equally represented by ab or $-ab$; but if we assign to b and to a, the same algebraical character, and therefore take into consideration their positions with respect to each other, as far at least as our signs are competent to express them, then we must examine the successive values of

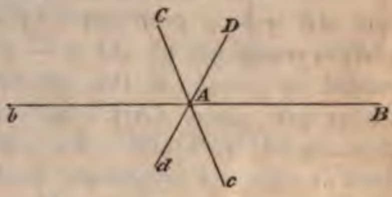
$$b,\ b\sqrt{-1},\ -b,\ -b\sqrt{-1}$$

equally with those of

$$a,\ a\sqrt{-1},\ -a,\ -a\sqrt{-1},$$

and determine the positions of the resulting rectangles which correspond to the several symbolical results,

$$ab,\ -ab,\ ab\sqrt{-1},\ \text{and}\ -ab\sqrt{-1}.$$



Thus the lines designated by a and $-a$, representing AB and Ab, form one line BAb, whilst the lines designated by b and $-b$, representing AC and Ac, form another line at right angles to it: again, the lines designated by $b\sqrt{-1}$ and $-b\sqrt{-1}$, representing AD and Ad, form a third line DAd, at right angles to each of the other lines BAb and CAc, and also at right angles to the plane of the paper, in the figure which we have drawn: now the line AB forms four rectangles with the lines AC and Ac, AD and Ad, the two first of which correspond to ab and $-ab$, and the two others to $ab\sqrt{-1}$ and $-ab\sqrt{-1}$: and in the same manner, the line Ab forms four other rectangles with the same four lines AC and Ac, AD and Ad, which correspond to $-ab$, ab, $-ab\sqrt{-1}$, and $ab\sqrt{-1}$ respectively: it follows, therefore, that the four rectangles in the plane passing through BAb and DAd, are affected with the sign $\sqrt{-1}$, whilst the four rectangles in the plane, passing through BAb and CAc, which is at right angles to the former, are not affected by the sign $\sqrt{-1}$, but by the signs + and − only.

We might represent AB and Ab by $a\sqrt{-1}$ and $-a\sqrt{-1}$, AC and Ac by b and $-b$, and, therefore, AD and Ad by $b\sqrt{-1}$ and $-b\sqrt{-1}$; in this case the four rectangles in the plane of BAb and CAc would correspond to $ab\sqrt{-1}$ and $-ab\sqrt{-1}$, whilst the four rectangles in the plane at right angles to it, passing through BAb and DAd, would correspond to ab and $-ab$: or we might represent AB and Ab by

$$a\sqrt{-1} \text{ and } -a\sqrt{-1},$$

AC and Ac by $b\sqrt{-1}$ and $-b\sqrt{-1}$, and therefore, AD and Ad by b and $-b$: in this case, the four rectangles in the plane passing through BAb and CAc, would correspond to ab and $-ab$, and the other four rectangles in the plane passing through BAb and DAd, would correspond to $ab\sqrt{-1}$ and $-ab\sqrt{-1}$; or we might represent AB and Ab, by a and $-a$, AC and Ac, by $b\sqrt{-1}$ and $-b\sqrt{-1}$ and therefore, AD and Ad by b and $-b$: in this case, the four rectangles in the plane passing through BAb and CAc, would correspond to $ab\sqrt{-1}$ and $-ab\sqrt{-1}$, whilst the other four rectangles in the plane passing through BAb and DAd, would correspond to ab and $-ab$: in whatever manner, therefore, the representation of the lines containing those rectangles is varied, consistently with the interpretation which we have given of the signs $+$, $-$, $\sqrt{-1}$, $-\sqrt{-1}$, when applied to lines, we arrive at the same general conclusion, "that whilst the rectangles in one plane are distinguished by expressions affected by the signs $+$ and $-$, the rectangles in the plane at right angles to it are affected by the signs $\sqrt{-1}$ and $-\sqrt{-1}$.

The sign $\sqrt{-1}$ does not properly present itself in the representation of parallelopipedons about the same point.

451. A very little consideration will shew that the signs $\sqrt{-1}$ and $-\sqrt{-1}$, will not only admit of no interpretation, when applied to parallelopipedons, but also that they cannot properly present themselves in the representation of such quantities: for if one of the edges of a parallelopipedon be denoted by a or $-a$, the two other edges at right angles to the former, if considered with reference to the first, would be denoted by

$$b\sqrt{-1} \text{ and } c\sqrt{-1},$$

$$\text{or by } -b\sqrt{-1} \text{ and } -c\sqrt{-1},$$

$$\text{or by } b\sqrt{-1} \text{ and } -c\sqrt{-1},$$

$$\text{or by } -b\sqrt{-1} \text{ and } -c\sqrt{-1}:$$

in all such cases, the parallelopipedon itself would be denoted by $+abc$ or $-abc$, and therefore its representation would be independent of the signs $\sqrt{-1}$ and $-\sqrt{-1}$; if however, we should agree to represent one of the edges by a symbol used arithmetically, and consider the two other edges *only* with reference to each other, then they would be denoted by

$$b \text{ and } c\sqrt{-1}, \text{ or by } -b \text{ and } c\sqrt{-1},$$

$$\text{or by } b \text{ and } -c\sqrt{-1}, \text{ or by } -b \text{ and } -c\sqrt{-1};$$

upon such an hypothesis all the parallelopipedons which can be formed round the same point and axis, would be equally affected by the sign $\sqrt{-1}$, which would not therefore indicate any property of one parallelopipedon as distinguished from another.

The signs $1, \sqrt{-1}, -1, -\sqrt{-1}$ included in $(1)^{\frac{1}{4}}$.

452. The successive signs $1, \sqrt{-1}, -1, -\sqrt{-1}$, to which our interpretations have hitherto been confined, are all of them included in the single sign or symbolical expression $(1)^{\frac{1}{4}}$;

$$\text{for } (1)^{\frac{1}{2}} = 1 \text{ or } -1,$$

and therefore

$$(1)^{\frac{1}{4}} = (1)^{\frac{1}{2}} \text{ or } (-1)^{\frac{1}{2}},$$

and is equal therefore to

$$1 \text{ or } -1 \quad \sqrt{-1} \text{ or } -\sqrt{-1}:$$

it follows therefore that these several signs may be considered as the four symbolical roots of 1, inasmuch as they are symbolical expressions whose fourth powers are severally equal to 1; the four roots of a^4 (a being the arithmetical root) are therefore

$$a, \ -a, \ a\sqrt{-1}, \ -a\sqrt{-1},$$

and are identical with the square roots of a^2 and $-a^2$; in considering however the peculiar affections designated by $(1)^{\frac{1}{4}}$ or its equivalent signs, when attached to a symbol with a specific value, it is not necessary to consider them as the results of the extraction of the fourth root of a^4 or even of the square root of a^2 and $-a^2$; inasmuch as those affections are determined from the signs themselves, and the general algebraical conditions which they are required to satisfy, without any reference to the mode of their derivation.

There are n symbolical values of $a(1)^{\frac{1}{n}}$.

453. We have shewn that there are four symbolical roots of 1, or rather four symbolical values of $(1)^{\frac{1}{4}}$ and no more; we shall afterwards shew that there are n symbolical roots of 1 or n symbolical values of $(1)^{\frac{1}{n}}$ and no more; or in other words, that there are n such symbolical values, whose n^{th} powers, formed according to the general laws of Algebra, are equal to 1. If we commence by *assuming* their existence generally (and in some cases we have shewn that they do exist), it will be easy to shew that they must possess properties, when viewed in connection with each other, which will materially aid us in the interpretation of the affections which they may designate, when applied to symbols with specific values; for they cannot express magnitude, but affections of magnitude only, and it is in this respect alone, that the values of $a(1)^{\frac{1}{n}}$ can be distinguished from that of a, or the algebraical distinguished from the arithmetical root.

One of which is the arithmetical root.

454. In the first place, 1 is in all cases one of the values of $(1)^{\frac{1}{n}}$; for the n^{th} power of 1 is equal to 1, which is the only symbolical condition which such a value is required to satisfy; it follows, therefore, that one of the values of $a(1)^{\frac{1}{n}}$ is always a, or in other words, that one

of the (n) algebraical roots of a^n coincides with the arithmetical root.

Remarkable relation of the symbolical roots.

455. If we designate the other values of

$$(1)^{\frac{1}{n}} \text{ by } a_1,\ a_2, \ldots\ldots a_{n-1},$$

it may be shewn that if there be any one of these values which is different from 1, and also from all those which correspond to inferior values of n, then all those values are different from each other, and may be represented likewise by the terms of the series

$$1,\ a_1,\ a_1^2, \ldots\ldots a_1^{n-1}.$$

For if a_1 be a value of $(1)^{\frac{1}{n}}$, then we must have

$$a_1^n = 1,\quad a_1^{2n} = 1,\quad a_1^{3n} = 1, \ldots\ldots a_1^{(n-1)n} = 1,$$

and therefore also

$$a_1^n = 1,\quad (a_1^2)^n = 1,\quad (a_1^3)^n = 1, \ldots\ldots (a_1^{(n-1)})^n = 1.$$

It follows therefore that a_1, a_1^2, $a_1^3, \ldots\ldots a_1^{n-1}$ are values of $(1)^{\frac{1}{n}}$, since they severally satisfy the required symbolical conditions: and if a_1 be such a value of $(1)^{\frac{1}{n}}$, that no power of it less than the n^{th} is equal to 1, they will likewise be different from each other: thus a_1^2 is necessarily, under such circumstances, different from a_1; for if they were identical, or $a_1^2 = a_1$, then $a_1 = 1$; also a_1^3 must be different from a_1^2 or a_1, for otherwise if $a_1^3 = a_1^2$ or $a_1^3 = a_1$, we should either have $a_1 = 1$, or $a_1^2 = 1$; a_1^4 must be different from a_1^3 or a_1^2 or a_1, for if $a_1^4 = a_1^3$ or $a_1^4 = a_1^2$ or $a_1^4 = a_1$, then we should have $a_1 = 1$, or $a_1^2 = 1$, or $a_1^3 = 1$; and generally a_1^{n-1} must be different from

$$a_1^{n-2}, \text{ or } a_1^{n-3}, \ldots\ldots \text{or } a_1,$$

for otherwise if

$$a_1^{n-1} = a_1^{n-2}, \text{ or } a_1^{n-1} = a_1^{n-3}$$
$$\text{or } a_1^{n-1} = a_1^{n-4}, \ldots\ldots \text{or } a_1^{n-1} = a_1,$$

we should have

$$a_1 = 1, \text{ or } a_1^2 = 1, \text{ or } a_1^3 = 1, \ldots\ldots \text{or } a_1^{n-2} = 1:$$

it follows therefore that the terms of the series

$$1,\ a_1,\ a_1^2,\ a_1^3, \ldots\ldots a_1^{n-1}$$

are not only severally values of $(1)^{\frac{1}{n}}$, but likewise under the circumstances indicated in the hypothesis which we have made, they are all of them different from each other.

The extension of the same series to such terms as

$$a_1^n,\ a_1^{n+1},\ a_1^{n+2}, \text{ \&c.},$$

would still produce values of $(1)^{\frac{1}{n}}$, but they would be identical with the successive terms of the series

$$1,\ a_1,\ a_1^2, \ldots\ldots a_1^{n-1};$$

for

$$a_1^n = 1,\ a_1^{n+1} = a_1^n \times a_1 = a_1,\ a_1^{n+2} = a_1^n \times a_1^2 = a_1^2,$$

and so on; the same series of n different values of $(1)^{\frac{1}{n}}$, will therefore perpetually recur, however far this series may be extended.

There can be no more than n symbolical values of $a\,(1)^{\frac{1}{n}}$.

456. We have thus got n hypothetical different values of $(1)^{\frac{1}{n}}$, and it is very easy to shew that there can be no more; for if we suppose $x = (1)^{\frac{1}{n}}$, then $x^n = 1$, and therefore, $x^n - 1 = 0$; now if $x = 1$, then $x - 1$ is a factor of $x^n - 1$, for otherwise $x^n - 1$ could not be identically equal to zero; if $x = a_1$, then $x - a_1$ is a factor, for the same reason; if $x = a_2$, then $x - a_2$ is also a factor and so on, a factor in all cases corresponding to every value of x; it follows therefore, that

$$x^n - 1 = (x - 1)\ (x - a_1)\ (x - a_2) \ldots (x - a_{n-1}),$$

and as the number of such factors cannot exceed n, it follows that there cannot be more than n values of x or of $(1)^{\frac{1}{n}}$, whether they are different or not; it follows, therefore, that the n values of $(1)^{\frac{1}{n}}$ which, upon a particular hypothesis, we have shewn would exist and be different from each other, are the only values, which under the same circumstances $(1)^{\frac{1}{n}}$ could possess.

Important variation in the form of the symbolical values of $(1)^{\frac{1}{n}}$.

457. The series of hypothetical values of $(1)^{\frac{1}{n}}$ which we have been considering, will easily admit of an important variation of form; for it is obvious, that

$$a_1^{n-1} = \frac{a_1^n}{a_1} = \frac{1}{a_1} = a_1^{-1}; \quad a_1^{n-2} = \frac{1}{a_1^2} = a_1^{-2};$$

$$a_1^{n-3} = \frac{1}{a_1^3} = a_1^{-3}; \quad \ldots\ldots a_1^{n-r} = \frac{1}{a_1^r} = a_1^{-r};$$

and the terms of the series therefore,

$$1,\ a_1,\ a_1^2,\ \ldots\ a_1^{n-2},\ a_1^{n-1},$$

if n be an odd number, may be arranged as follows:

$$1;\quad a_1,\ a_1^{-1};\quad a_1^2,\ a_1^{-2};\quad a_1^3,\ a_1^{-3};$$
$$\ldots\ldots a_1^{\left(\frac{n-1}{2}\right)},\quad a_1^{-\left(\frac{n-1}{2}\right)}.$$

If n be an even number, the middle term of the original series will be $a_1^{\frac{n}{2}}$, which is equal to $\frac{1}{a_1^{\frac{n}{2}}}$, and therefore equal to -1: for under such circumstances it must be equal to 1 or -1, and it cannot, by the hypothesis which we have made, be equal to 1. In this case, therefore, the series may be arranged as follows:

$$1, -1;\quad a_1,\ a_1^{-1};\quad a_1^2,\ a_1^{-2};\quad \ldots a_1^{\left(\frac{n}{2}-1\right)},\ a_1^{-\left(\frac{n}{2}-1\right)}.$$

The preceding properties of the symbolical values of $a(1)^{\frac{1}{n}}$ essential to their interpretation.

458. The preceding investigation of the properties of the values of $(1)^{\frac{1}{n}}$, or of the n roots of 1, will form the proper introduction to the interpretation of their meaning, when attached to symbols with specific values: it will be

found, indeed, that there are not many quantities which symbols may represent, which admit of a succession of affections which will correspond to the succession of their roots: in Geometry, however, we shall find both lines and areas, by the changes of their position, capable of exhibiting the complete succession of changes which are required to correspond to the symbolical conditions which those values are required to fulfil.

Interpretation of the values of $a(1)^{\frac{1}{n}}$ when a is a line.

459. If we suppose a circle divided into n equal parts, and the radii to be drawn from its centre to the several points of division; then, if a denote the radius in its primitive position,

$$a(1)^{\frac{1}{n}}, \quad a(1)^{\frac{2}{n}}, \quad a(1)^{\frac{3}{n}}, \dots\dots a(1)^{\frac{n-1}{n}},$$

$$\text{or} \quad a \cdot a_1, \quad a \cdot a_1^2, \quad a_1 \cdot a^3 \dots\dots a \cdot a_1^{n-1},$$

will denote the successive radii, reckoned in the same direction from the primitive line.

For if we consider the *affectation* of a by a_1, as equivalent to its transfer through an angle equal to $\frac{360^0}{n}$, or $\frac{1}{n}$th part of the circumference, then the successive *affectations* of a by the same sign or symbol a_1, corresponding successively to

$$aa_1^2, \quad aa_1^3, \quad aa_1^{n-1}, \quad aa_1^n,$$

will be equivalent to its successive transfers through the same angle; and the lines thus designated, therefore, make angles of

$$\frac{2}{n} \times 360, \quad \frac{3}{n} \times 360, \dots \frac{(n-1)}{n} \times 360^\circ,$$

and the last of

$$\frac{n}{n} \times 360^\circ, \text{ or } 360^\circ,$$

and coincides, therefore, with the primitive line: and since $a \cdot a_1^n = a$, such an interpretation of a_1 when thus applied, satisfies the required symbolical condition.

Again, the n^{th} affectation of a by a_1^2, transfers the line designated by a through

$$\frac{2n}{n} \times 360, \text{ or } 2 \times 360^\circ:$$

the n^{th} affectation of a by a_1^3, transfers it through

$$\frac{3n}{n} \times 360, \text{ or } 3 \times 360^\circ:$$

and the n^{th} affectation of a by a_1^{n-1}, transfers it through

$$\frac{(n-1)\,n}{n} \times 360^\circ, \text{ or } (n-1) \times 360^\circ.$$

In every case, therefore, the n^{th} affectation of a by any one of these $(n-1)$ roots of 1, which are different from 1, the common symbolical result of which is a, brings the primitive line to its primitive position, and the interpretation satisfies therefore the symbolical condition which in every case is required to be fulfilled.

The lines denoted by $a(1)^{\frac{1}{n}}$ may be grouped into pairs making equal angles with the line a.

460. The lines, which the values of $a\,(1)^{\frac{1}{n}}$ (different from a) designate, may be grouped into pairs, making equal angles with the primitive line: thus

$$a a_1 \text{ and } a \cdot a_1^{n-1}, \text{ or } a \cdot a_1^{-1},$$

represent lines making angles of

$$\frac{360^0}{n} \text{ and } \frac{n-1}{n} \times 360^0, \text{ or } \frac{-360^0}{n} \text{ with } a;$$

$$a \cdot a_1^2 \text{ and } a \cdot a_1^{n-2}, \text{ or } a \cdot a_1^{-2},$$

represent lines making angles of

$$\frac{2}{n} \times 360^\circ \text{ and } \left(\frac{n-2}{n}\right) \times 360^\circ, \text{ or } \frac{-2}{n} \times 360^\circ \text{ with } a;$$

$$a \cdot a_1^r \text{ and } a \cdot a_1^{n-r}, \text{ or } a \cdot a_1^{-r},$$

represent lines making angles of

$$\frac{r}{n} \times 360^\circ \text{ and } \left(\frac{n-r}{n}\right) \times 360^\circ, \text{ or } \frac{-r}{n} \times 360^\circ \text{ with } a.$$

So far, therefore, as the determination of the lines corresponding to $a\,(1)^{\frac{1}{n}}$ is concerned, it is indifferent in what direction we proceed round the circle, whether from right to left or from left to right, inasmuch as the same series of lines must result in both cases.

If n be an even number, $\frac{n}{2} \cdot \frac{360}{n} = 180^\circ$, and the line corresponding to $a \cdot a_1^{\frac{n}{2}}$ or $-a$ (for $a_1^{\frac{n}{2}} = -1$), is opposite to the primitive line, and therefore affected with the sign $-$. Under the same circumstances, likewise, the lines corresponding to

$$a \cdot a_1 \text{ and } a \cdot a_1^{\frac{n}{2}+1}, \text{ or } a \cdot a_1^{\frac{n}{2}} \times a_1, \text{ or } -a \cdot a_1;$$

$$\text{to } a \cdot a_1^2 \text{ and } a \cdot a_1^{\frac{n}{2}+2}, \text{ or } -a \cdot a_1^2;$$

$$\text{to } a \cdot a_1^r \text{ and } a \cdot a_1^{\frac{n}{2}+r}, \text{ or } -a \cdot a_1^r,$$

which are severally represented by the same symbolical expressions, affected with the signs $+$ and $-$, must correspond to lines which are opposite to each other, in conformity with the interpretation which we have given of those signs under such circumstances.

All multiples of 4 right angles may be rejected.

461. In considering the affectations of lines with different values of $(1)^{\frac{1}{n}}$, we express the angles of transfer by multiples of the angle corresponding to the first affectation, whether with a_1, a_1^2, a_1^3, &c. or a_1^{n-1}. Under such circumstances, such multiples of the angles of transfer may exceed 4 right angles or a multiple of 4 right angles: but as the angle of position of the affected line is alone considered, without any reference to the mode in which

it is transferred to it, we may reject, therefore, either 4 right angles or any multiple of 4 right angles which the expressed angle of transfer may contain, and the remainder, which is less than 4 right angles, will be the angle required.

Assumed symbolical representation of 4 right-angles.

462. Inasmuch as angles of 4 right angles and their multiples are of very frequent occurrence, not only in the theory of these roots of 1, but in many other most important theories which are connected with them, it becomes convenient to adopt some conventional symbol, by which they may always be expressed. It has been usual to make use of 2π for this purpose:* thus, a line making an angle of θ with a primitive line, coincides with a line making angles of

$$(2\pi+\theta), \text{ or } (4\pi+\theta), \text{ or } (6\pi+\theta), \ldots \text{ or } (2n\pi+\theta),$$

rejecting the multiples of 2π; a line making an angle of $\pi+\theta$, or $180^0+\theta$, with a primitive line, coincides with a line making angles of

$$3\pi+\theta, \text{ or } 5\pi+\theta, \ldots \text{ or } (2m+1)\pi+\theta,$$

and we may reject in this case as before, the multiples of 2π; a line making an angle of $-\theta$ with a primitive line, coincides with a line making an angle of

$$(2\pi-\theta), \text{ or } (4\pi-\theta), \text{ or } (2m\pi-\theta),$$

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* The ratios of different angles are the same as the ratios of the arcs of the same circle which subtend them at its centre; and it is in this sense that arcs are said to be the measures of angles: in order to determine therefore, the ratio which any angle bears to a right angle, we must determine the ratio which the corresponding arc of the circle bears to a quadrantal arc, and conversely: for these four magnitudes form the terms of a proportion.

An angle may be symbolically represented by the ratio of the arc subtending it to the quadrant of the same circle, or by the ratio of the arc to the radius of the circle: for such a quantity is the same for different circles, if the angle be the same, and if the angle varies, it varies as the angle: if we suppose, therefore, the radius of the circle to be 1 under all circumstances, then the angle will be symbolically represented by the corresponding arc, or conversely: it is upon this hypothesis, that we represent indifferently by the same symbol 2π, both 4 right angles and the circumference of a circle whose radius is 1.

or we may reject, as in all other cases, 2π and its multiples.

Supplement and complement.

463. The angle $\pi-\theta$ is called the *supplement* of θ, or conversely, one being equal to what the other wants of 180°; and θ is also called the *complement* of $\frac{\pi}{2}-\theta$, one being equal to what the other wants of 90° or a right angle.

Second mode of representing the values of $a(1)^{\frac{1}{n}}$.

464. The different values of $a(1)^{\frac{1}{n}}$ as we have shewn, are essentially connected with the angles formed with the corresponding lines and the primitive line, and it would be convenient, therefore, if they could be expressed by equivalent forms which involved that angle symbolically: such expressions are ϵ^{θ} and $\epsilon^{-\theta}$, corresponding to a_1 and a_1^{-1}; for they possess the proper inverse relation to each other, their product being $=1$, equally with that of a_1 and a_1^{-1}: and they involve symbolically likewise, the angles θ and $-\theta$, which the corresponding lines make with the primitive line: it will follow also if $a_1=\epsilon^{\theta}$ and $a_1^{-1}=\epsilon^{-\theta}$, that

$$a_1^2=\epsilon^{2\theta} \text{ and } a_1^{-2}=\epsilon^{-2\theta},\ a_1^3=\epsilon^{3\theta} \text{ and } a_1^{-3}=\epsilon^{-3\theta},$$
$$a_1^{n-1}=\epsilon^{(n-1)\theta} \text{ and } a_1^{-(n-1)}=\epsilon^{-(n-1)\theta},$$

and consequently the series of values of

$$a_1^2,\ a_1^{-2};\quad a_1^3,\ a_1^{-3};\ldots a_1^{n-1} \text{ and } a_1^{-(n-1)},$$

are all powers of the same quantity, whose exponents express the several angles which the corresponding lines make with the primitive line: it follows also since

$$a_1=\epsilon^{\theta}, \text{ that } a_1^{n}=\epsilon^{n\theta}=1:$$

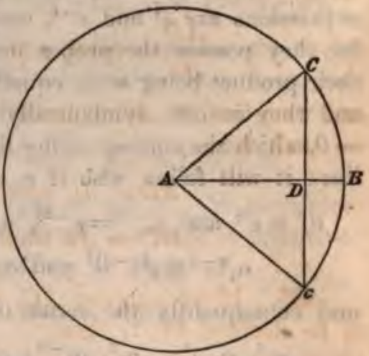
but $n\theta$ is equal to 2π, since

$$\theta=\frac{360}{n}=\frac{2\pi}{n}:$$

it appears, therefore, that $\epsilon^{2\pi}=1$, an equation which will enable us hereafter to determine the quantity, whether symbolical or not, to which ϵ is equivalent: so far however as the consistent representation of the values of $a(1)^{\frac{1}{n}}$ are concerned, it is merely necessary to suppose, that the equation $\epsilon^{2\pi} = 1$ is true, whatever the particular value of ϵ may be.

Definitions of the sines and cosines of angles.

465. We have thus determined two symbolical modes of representing the lines, corresponding to $a(1)^{\frac{1}{n}}$, which are equivalent to each other, though different in form: it remains to investigate a third mode of exhibiting them, which is algebraically equivalent to the former, though connected with the geometrical properties of such lines, when considered with reference to the primitive line.



In the plane of this paper let us describe a circle, whose radius is AB; and let us draw the radii AC and Ac making equal angles BAC and BAc with the primitive line; and let us join Cc cutting AB in D: since $AC = Ac$ and AD is common to the two triangles ADC and ADc, and the angle BAC is equal to the angle BAc, it follows that $CD = cD$, and the angle $ADC = ADc$, and therefore to a right angle. Now if the angle BAC* be given, the ratios of AC to AD and of AC to CD are given: and in every case there is a necessary connection between the angle BAC and the two ratios of AC to AD, and of AC to DC, and it will be the object of subsequent investigations and calculations to

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[*] The sides, or the ratio of the sides of a triangle, determine its angles, and it is by such means only, that angles are *given* or determined in Geometry.

assign the succession of values of one or both of these ratios, which correspond to the succession of values of the angles, and conversely.

In order to recal this essential connection, it has been usual to term the ratio $\frac{AD}{AC}$ the *cosine* of the angle BAC, and the other ratio $\frac{DC}{AC}$ the *sine* of the same angle: or if we call the angle BAC, θ, then $\frac{AD}{AC}$ is called the cosine of θ, and $\frac{DC}{AC}$ the sine of θ: and for greater brevity, the first is written cos θ, and the second sin θ: thus

$$\frac{AD}{AC} = \cos\theta, \text{ and } \frac{DC}{AC} = \sin\theta;$$

and therefore, if AC expressed arithmetically or $AB = a$, we shall have

$$AD = AC \cos\theta = a \cos\theta,$$
$$DC = AC \sin\theta = a \sin\theta.$$

466. If we suppose AC, or the radius of the circle to be denoted by 1, then

When the radius of the circle is 1.

$$AD = \cos\theta, \text{ and } DC = \sin\theta,$$

or the two sides of the right-angled triangle, whose hypothenuse is 1, become the cosine and sine of the angle at its base.

467. Let us now consider the fundamental properties of the cosine and sine of an angle, which result from the preceding definition of their meaning.

Their fundamental properties.

In the first place,

$$\cos\theta = \cos(-\theta)\ldots\ldots\ldots\ldots(a).$$

For

$$AD = AC \cos BAC = a \cos \theta;$$

and the same line AD is likewise equal to

$$Ac \cos BAc = a . \cos(-\theta), \text{ (for } BAc = -\theta);$$

consequently

$$AD = a \cos \theta = a \cos(-\theta);$$

and therefore $\cos \theta = \cos(-\theta)$.

In the second place,

$$\sin \theta = -\sin(-\theta) \ldots\ldots\ldots\ldots (\beta).$$

For

$$DC = AC \sin . BAC = a \sin \theta;$$

and the line Dc, is also equal to

$$Ac \sin BAc = a . \sin(-\theta);$$

but DC and Dc being equal and opposite to each other, are affected with different or opposite signs: consequently

$$DC = -Dc;$$

and therefore,

$$a . \sin \theta = -a . \sin(-\theta), \text{ or, } \sin \theta = -\sin(-\theta).$$

In the third place,

$$\cos^2 \theta + \sin^2 \theta = 1 \ldots\ldots\ldots\ldots (\gamma).$$

For

$$\frac{AD}{AC} = \cos \theta, \text{ and } \frac{DC}{AC} = \sin \theta;$$

and therefore,

$$\frac{AD^2}{AC^2} + \frac{DC^2}{AC^2} = (\cos \theta)^2 + (\sin \theta)^2 = \cos^2 \theta + \sin^2 \theta;$$

$$\text{but} \quad AD^2 + DC^2 = AC^2;$$

and therefore,

$$\frac{AD^2}{AC^2} + \frac{DC^2}{AC^2} = 1;$$

consequently,

$$\cos^2\theta + \sin^2\theta = 1.$$

Symbolical definition of $\cos\theta$ and $\sin\theta$, when viewed without reference to their geometrical definition.

468. The three properties of the cosine and sine of an angle, which are given in the preceding article, are the necessary consequences of their geometrical definition expressed by means of symbolical language: if they are viewed however without reference to that definition, they will form, when taken together, the algebraical or symbolical definition of the expressions designated by $\cos\theta$ and $\sin\theta$: thus θ is the common symbol, whatever be its meaning or representation, with which the symbolical quantities $\cos\theta$ and $\sin\theta$ are connected: the equation

$$\cos^2\theta + \sin^2\theta = 1,$$

expresses the common condition to which they are subject and enables us to express one of them, in terms of the other: whilst the equations

$$\cos\theta = \cos(-\theta) \text{ and } \sin\theta = -\sin(-\theta),$$

which are independent of each other, and of the former equation, distinguish one of these expressions from the other, so that they are no longer convertible at pleasure, as they would be, in case the equation $\cos^2\theta + \sin^2\theta = 1$, was the only condition by which they were limited.

When viewed with reference to their geometrical definition.

When viewed in connection however with the geometrical definition, we may either consider those equations as the necessary symbolical consequences of it, or we may consider that definition as the geometrical interpretation of the symbolical equations: in this case θ and $-\theta$ become the representation of two equal angles on different sides of the primitive radius of the circle: $\cos\theta$ and $\sin\theta$ are the symbolical expressions of the ratios borne by the hypothenuse to each of the sides of the right-angled triangle: $\cos\theta = \cos(-\theta)$ or $a\cos\theta = a\cos(-\theta)$, indicates that the two right-angled triangles corresponding to the angles θ and $-\theta$, have a common side: whilst

$$\sin\theta = -\sin(-\theta), \text{ or } a\sin\theta = -a\sin(-\theta),$$

shews that the second sides of those right-angled triangles are equal, but are drawn in directions which are opposite to each other.

We may use indifferently the geometrical or symbolical definition of sin θ and cos θ.

469. Having thus established the principles of interpretation, which connect the symbolical definition with the corresponding geometrical quantities, and having shewn that the symbolical equations which might be considered as constituting the symbolical definition of $\cos\theta$ and $\sin\theta$, are necessary consequences of the geometrical definition, we shall afterwards be at liberty to make use of one or the other indifferently as may best suit our purpose: if we make use of the geometrical definition as our point of departure, we must deduce our consequences from it geometrically, merely clothing them in the corresponding symbolical language: but if we wish to make use of those symbolical equations as our point of departure, we must seek for other symbolical expressions which are equivalent to $\cos\theta$ and $\sin\theta$, and which are within the operation of the ordinary rules and laws of Algebra: for unless we can discover such equivalent expressions for $\cos\theta$ and $\sin\theta$, we must leave the results of their different combinations in a form no further reduced, than if we had replaced $\cos\theta$ and $\sin\theta$ by simple symbols x and y, connected by the equation $x^2+y^2=1$, and with the further condition that y and not x would change its sign from $+$ to $-$, upon the substitution of $-\theta$ in the place of θ.

Research of equivalent algebraical values of cos θ and sin θ.

470. For this purpose, let us examine the composition of the fundamental equation

$$\cos^2\theta + \sin^2\theta = 1,$$

and endeavour to discover from its factors or otherwise, equivalent expressions for $\cos\theta$ and $\sin\theta$, which are essentially algebraical.

In the first place,

$$\cos^2\theta + \sin^2\theta = (\cos\theta + \sqrt{-1}\sin\theta)\times$$
$$(\cos\theta - \sqrt{-1}\sin\theta) = 1,$$

and consequently the factors

$$\cos\theta+\sqrt{-1}\sin\theta \text{ and } \cos\theta-\sqrt{-1}\sin\theta,$$

possess the same inverse relation to each other, which is possessed by ϵ^{θ} and $\epsilon^{-\theta}$, the signs of affection of lines making angles θ and $-\theta$ with the primitive line: again, the factor $\cos\theta+\sqrt{-1}\sin\theta$ becomes $\cos\theta-\sqrt{-1}\sin\theta$, upon the substitution of $-\theta$ for θ, and conversely, another property possessed in common with the same quantities ϵ^{θ} and $\epsilon^{-\theta}$: let us assume therefore, in the first instance as an hypothesis,

$$\cos\theta+\sqrt{-1}\sin\theta=\epsilon^{\theta},$$

$$\text{and}\quad \cos\theta-\sqrt{-1}\sin\theta=\epsilon^{-\theta},$$

from which we shall get, by additon and subtraction,

$$2\cos\theta=\epsilon^{\theta}+\epsilon^{-\theta} \text{ and } 2\sqrt{-1}\sin\theta=\epsilon^{\theta}-\epsilon^{-\theta},$$

and consequently,

$$\cos\theta=\frac{\epsilon^{\theta}+\epsilon^{-\theta}}{2} \text{ and } \sin\theta=\frac{\epsilon^{\theta}-\epsilon^{-\theta}}{2\sqrt{-1}}.$$

The process followed in the deduction of these assumed equivalent expressions for $\cos\theta$ and $\sin\theta$ would shew that they would satisfy the three fundamental equations

$$\cos^2\theta+\sin^2\theta=1,\ \cos\theta=\cos(-\theta) \text{ and } \sin\theta=\sin(-\theta):$$

but, if we should consider that process merely as a guide to the assumptions to be made of algebraical expressions which might be equivalent to $\cos\theta$ and $\sin\theta$, it would readily follow that

$$\left(\frac{\epsilon^{\theta}+\epsilon^{-\theta}}{2}\right)^2+\left(\frac{\epsilon^{\theta}-\epsilon^{-\theta}}{2\sqrt{-1}}\right)^2=\frac{\epsilon^{2\theta}+2+\epsilon^{-2\theta}-\epsilon^{2\theta}+2-\epsilon^{-2\theta}}{4}$$

$$=\frac{4}{4}=1,$$

$$\text{and also that } \frac{\epsilon^{\theta}+\epsilon^{-\theta}}{2}$$

would remain unaltered, upon the substitution of $-\theta$ for θ, whilst the corresponding expression for $\sin\theta$ would become

$$\frac{\epsilon^{-\theta} - \epsilon^{\theta}}{2\sqrt{-1}} = -\left(\frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2\sqrt{-1}}\right)$$

under the same circumstances: it would hence appear that the assumed expressions

$$\frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2} \text{ and } \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2\sqrt{-1}}$$

satisfy the three fundamental equations; and are therefore, in every respect equivalent to $\cos\theta$ and $\sin\theta$.

Their great importance.

471. These expressions will be found to lead to the most important consequences, not merely with reference to the symbolical determination of all the values of $(1)^{\frac{1}{n}}$ and the completion of their theory, but likewise with reference to the exhibition of the relations of the sines and cosines of different angles, which form a series of propositions constituting the science of Goniometry, to which the name of Trigonometry is more generally given: as it is by means of the propositions of this science, that we are enabled to bring Geometry within the operation of the laws of Algebra, we shall commence by illustrating the use of these formulæ in the investigation of such of the propositions in this science as may be considered the most fundamental and important.

The values of cos 0 and sin 0.

472. Prop. The cosine of an angle equal to zero is 1, and its sine is zero.

For
$$\cos\theta = \frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2};$$

and therefore, $\cos 0 = \dfrac{1+1}{2} = 1$:

$$\text{and } \sin\theta = \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2\sqrt{-1}},$$

and therefore, $\sin 0 = \dfrac{1-1}{2\sqrt{-1}} = 0.$

473. Prop. The cosine of an angle equal to 180^0 or π is -1 and its sine is zero.

The values $\cos\pi$ and $\sin\pi$.

For

$$\cos\pi = \frac{\epsilon^{\pi} + \epsilon^{-\pi}}{2} = \frac{-1-1}{2} = -1 : \text{ for } \epsilon^{2\pi} = 1,$$

(Art. 464.) and therefore, $\epsilon^{\pi} = (1)^{\frac{1}{2}} = 1$ or -1; and since ϵ^{π} cannot be 1, it must therefore be -1;

also, $$\sin\pi = \frac{\epsilon^{\pi} - \epsilon^{-\pi}}{2\sqrt{-1}} = \frac{-1+1}{2} = 0.$$

474. Prop. Inasmuch as

The values of the cosines and sines of multiples of π.

$$\epsilon^{\pi} = -1,\ \epsilon^{2\pi} = 1,\ \epsilon^{3\pi} = (-1)^3 = -1,$$
$$\epsilon^{4\pi} = 1,\ \epsilon^{5\pi} = -1,$$

and so on, it will follow immediately that

$$\begin{array}{lll}
\cos 2\pi = 1 & \text{and} & \sin 2\pi = 0, \\
\cos 3\pi = -1 & \text{and} & \sin 3\pi = 0, \\
\cos 4\pi = 1 & \text{and} & \sin 4\pi = 0, \\
\cos 5\pi = -1 & \text{and} & \sin 5\pi = 0, \\
\cos 6\pi = 1 & \text{and} & \sin 6\pi = 0, \\
\cdots\cdots & & \cdots\cdots \\
\cos 2n\pi = 1 & \text{and} & \sin 2n\pi = 0, \\
\cos(2n+1)\pi = -1 & \text{and} & \sin(2n+1)\pi = 0.
\end{array}$$

In other words, the cosines of all even multiples of π or 180^0 are equal to 1, the cosines of all odd multiples of π are equal to -1, and the sines of all multiples of π whether odd or even are equal to *zero*.

475. Prop. The cosine of an angle equal to 90^0 is zero, and its sine is equal to 1.

The values of $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$.

For
$$\cos\frac{\pi}{2} = \frac{\epsilon^{\frac{\pi}{2}} + \epsilon^{-\frac{\pi}{2}}}{2} = \frac{\sqrt{-1} + \dfrac{1}{\sqrt{-1}}}{2} = 0:$$

for $\epsilon^{\frac{\pi}{2}} = (\epsilon^{\pi})^{\frac{1}{2}} = (-1)^{\frac{1}{2}}$ and $\dfrac{1}{\sqrt{-1}} = -\sqrt{-1}$:

$$\text{also, } \sin\frac{\pi}{2} = \frac{\epsilon^{\frac{\pi}{2}} - \epsilon^{-\frac{\pi}{2}}}{2\sqrt{-1}} = \frac{\sqrt{-1} - \dfrac{1}{\sqrt{-1}}}{2\sqrt{-1}} = 1.$$

The values of the cosines and sines of multiples of $\frac{\pi}{2}$:

476. Inasmuch as

$$\epsilon^{\frac{3\pi}{2}} = (\sqrt{-1})^3 = -\sqrt{-1},$$

$$\epsilon^{\frac{5\pi}{2}} = \sqrt{-1}, \quad \epsilon^{\frac{7\pi}{2}} = -\sqrt{-1},$$

and so on, it will follow that

$$\cos\frac{3\pi}{2} = 0 \quad \text{and} \quad \sin\frac{3\pi}{2} = -1,$$

$$\cos\frac{5\pi}{2} = 0 \quad \text{and} \quad \sin\frac{5\pi}{2} = 1,$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\cos\frac{(2n+1)\pi}{2} = 0 \quad \text{and} \quad \sin\frac{(2n+1)\pi}{2} = -1,$$

$$\cos\frac{(4n+1)\pi}{2} = 0 \quad \text{and} \quad \sin\frac{(4n+1)\pi}{2} = 1.$$

In other words, the cosines of all odd multiples of 90^0, or of a quadrant are equal to *zero*, whilst the corresponding sines are alternately 1 and − 1, according as the multiple of the quadrant diminished by 1, is divisible by 4 or by 2 only, or is *pariter*, or *impariter par*.

The cosine of an angle is equal to the sine of its complement and conversely.

477. PROP. The cosine of $\left(\frac{\pi}{2} - \theta\right)$ is equal to the sine of θ, and conversely.

For $\cos\left(\frac{\pi}{2} - \theta\right) = \dfrac{\epsilon^{\left(\frac{\pi}{2} - \theta\right)} + \epsilon^{-\left(\frac{\pi}{2} - \theta\right)}}{2}$

$$= \frac{\epsilon^{\frac{\pi}{2}} \cdot \epsilon^{-\theta} + \epsilon^{-\frac{\pi}{2}} \cdot \epsilon^{\theta}}{2} = \frac{\sqrt{-1} \cdot (\epsilon^{-\theta} - \epsilon^{\theta})}{2}$$

$$= \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2\sqrt{-1}} = \sin\theta.$$

Also, $$\sin\left(\frac{\pi}{2}-\theta\right)=\frac{\epsilon^{\left(\frac{\pi}{2}-\theta\right)}-\epsilon^{-\left(\frac{\pi}{2}-\theta\right)}}{2\sqrt{-1}}$$

$$=\frac{\epsilon^{\frac{\pi}{2}}\cdot\epsilon^{-\theta}-\epsilon^{-\frac{\pi}{2}}\cdot\epsilon^{\theta}}{2\sqrt{-1}}=\frac{\sqrt{-1}\cdot(\epsilon^{-\theta}+\epsilon^{\theta})}{2\sqrt{-1}}$$

$$=\frac{\epsilon^{\theta}+\epsilon^{-\theta}}{2}=\cos\theta.$$

In other words, the cosine of an angle is equal to the sine of its *complement* and conversely.

$\operatorname{Sin}(\pi-\theta)=\sin\theta$ and $\cos(\pi-\theta)=-\cos\theta$.

478. PROP. The cosine of $(\pi-\theta)$ is equal to the cosine of θ, affected with the sign $-$; whilst the sine of $(\pi-\theta)$ is equal to the sine of θ.

$$\cos(\pi-\theta)=\frac{\epsilon^{(\pi-\theta)}+\epsilon^{-(\pi-\theta)}}{2}=\frac{\epsilon^{\pi}\cdot\epsilon^{-\theta}+\epsilon^{-\pi}\cdot\epsilon^{\theta}}{2}$$

$$=\frac{-\epsilon^{-\theta}-\epsilon^{\theta}}{2}=-\cos\theta:\ \text{for } \epsilon^{\pi}=\epsilon^{-\pi}=-1.$$

Also,

$$\sin(\pi-\theta)=\frac{\epsilon^{(\pi-\theta)}-\epsilon^{-(\pi-\theta)}}{2\sqrt{-1}}$$

$$=\frac{\epsilon^{\pi}\cdot\epsilon^{-\theta}-\epsilon^{-\pi}\cdot\epsilon^{\theta}}{2\sqrt{-1}}=\frac{\epsilon^{\theta}-\epsilon^{-\theta}}{2\sqrt{-1}}=\sin\theta.$$

In other words, the sine of an arc is equal to the sine of its *supplement*, but the cosine of an arc is equal to the cosine of its *supplement* affected with the sign $-$.

479. There are many other propositions, such as

Other propositions admitting of similar proof.

$$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta \text{ and } \sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta,$$

$$\sin(\pi+\theta)=-\sin\theta \text{ and } \cos(\pi+\theta)=-\cos\theta,$$

and others of a similar kind, which are very easily deducible in a manner similar to those above given: but they likewise follow more readily and perhaps more uniformly from the following general propositions, which are fundamental in the Arithmetic of sines.

Expressions for the cosines and sines of the sum and difference of two arcs in terms of the sines and cosines of the simple arcs.

480. Prop. To express the cosines and sines of

$$(\theta + \theta') \text{ and } (\theta - \theta')$$

in terms of the cosines and sines of θ and θ'.

Inasmuch as

$$\epsilon^{\theta} = \cos\theta + \sqrt{-1}\sin\theta$$

for all values of θ, it will follow that

$$\epsilon^{\theta'} = \cos\theta' + \sqrt{-1}\,.\sin\theta',$$

and also that

$$\epsilon^{(\theta+\theta')} = \cos(\theta+\theta') + \sqrt{-1}\ \sin(\theta+\theta'):$$

but

$$\epsilon^{(\theta+\theta)} = \epsilon^{\theta}\times\epsilon^{\theta'} = (\cos\theta + \sqrt{-1}\sin\theta)(\cos\theta' + \sqrt{-1}\sin\theta')$$
$$= \cos\theta\cos\theta' - \sin\theta\sin\theta' + \sqrt{-1}\,\{\sin\theta\cos\theta' + \cos\theta\sin\theta'\}:$$

consequently,

$$\cos(\theta+\theta') + \sqrt{-1}\,.\sin(\theta+\theta')$$
$$= \cos\theta\cos\theta' - \sin\theta\sin\theta' +$$
$$\sqrt{-1}\,\{\sin\theta\cos\theta' + \cos\theta\sin\theta'\} \ \ldots\ldots\ldots\ldots\ (\alpha).$$

In a similar manner, if we commence with

$$\epsilon^{-\theta} = \cos\theta - \sqrt{-1}\,.\sin\theta,$$
$$\text{and} \qquad \epsilon^{-\theta'} = \cos\theta' - \sqrt{-1}\,.\sin\theta',$$

we shall find,

$$\epsilon^{-(\theta+\theta')} = \cos(\theta+\theta') - \sqrt{-1}\,.\sin(\theta+\theta')$$
$$= \cos\theta\cos\theta' - \sin\theta\sin\theta' -$$
$$\sqrt{-1}\,\{\sin\theta\cos\theta' + \cos\theta\sin\theta'\} \ \ldots\ldots\ldots\ldots\ldots\ (\beta).$$

If we add (algebraically) the equations (α) and (β) together and divide by 2, we shall get

$$\cos(\theta + \theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta' \ldots\ldots\ldots (\gamma):$$

and if we subtract (algebraically), the equation (β) from (α) and divide the result by $2\sqrt{-1}$, we shall get

$$\sin(\theta + \theta') = \sin\theta\cos\theta' + \sin\theta\cos\theta' \ldots\ldots\ldots (\delta).$$

If in equations (γ) and (δ), we put $-\theta'$ in the place of θ', we shall get

$$\cos(\theta - \theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta' \ldots\ldots\ldots (\epsilon),$$

$$\sin(\theta - \theta') = \sin\theta\cos\theta' - \cos\theta\sin\theta' \ldots\ldots\ldots (\eta).$$

The preceding propositions exemplified.

481. We will exemplify the use of these formulæ, by their application to the determination of some equivalent forms.

(1) $\text{Cos}(\pi + \theta) = \cos\pi \,.\, \cos\theta - \sin\pi \,.\, \sin\theta = -\cos\theta$:

for $\cos\pi = -1$ and $\sin\pi = 0$ (Art. 473.):

this results from the formula (γ).

(2) $\text{Sin}(\pi + \theta) = \sin\pi \,.\, \cos\theta + \cos\pi \,.\, \sin\theta = -\sin\theta$:

for $\sin\pi = 0$ and $\cos\pi = -1$:

this results from the formula (δ).

(3) $$\text{Cos}\left(\frac{3\pi}{2} - \theta\right) = \cos\frac{3\pi}{2} \,.\, \cos\theta + \sin\frac{3\pi}{2} \,.\, \sin\theta$$

$= -\sin\theta$: for $\cos\frac{3\pi}{2} = 0$ and $\sin\frac{3\pi}{2} = -1$ (Art. 476.);

this results from the formula (ϵ).

(4) $$\text{Sin}\left(\frac{3\pi}{2} - \theta\right) = \sin\frac{3\pi}{2}\cos\theta - \cos\frac{3\pi}{2}\sin\theta$$

$= -\cos\theta$: for $\sin\frac{3\pi}{2} = -1$ and $\cos\frac{3\pi}{2} = 0$;

this results from the formula (η).

$$(\delta)\quad \cos 2\theta = \cos(\theta + \theta) = \cos^2\theta - \sin^2\theta;$$

this results from making $\theta' = \theta$, in the formula (γ).

The value of $\cos 2\theta$ enables us to calculate the values of $\cos\theta$ and $\sin\theta$.

482. Inasmuch as

$$\cos^2\theta + \sin^2\theta = 1, \text{ and } \cos^2\theta - \sin^2\theta = \cos 2\theta,$$

it follows, (by adding and subtracting), that

$$2\cos^2\theta = 1 + \cos 2\theta$$

$$2\sin^2\theta = 1 - \cos 2\theta.$$

These are two important subordinate formulæ, and enable us to calculate the arithmetical values of

$$\cos\theta \text{ and } \sin\theta,$$

from the known arithmetical value of $\cos 2\theta$.

Thus, if

$$\cos 2\theta = 0, \text{ or } 2\theta = 90^\circ = \frac{\pi}{2},$$

$$\text{then, } \cos^2\theta = \tfrac{1}{2}$$

$$\text{and } \cos\theta = \cos 45^\circ = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}.$$

The calculation of the sine and cosine of 1′.

483. If we should suppose $\cos 2\theta = \frac{1}{2}$, a value of the cosine which geometrical considerations would at once shew to correspond to an angle of 60°*, we should find

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* Or thus: if we suppose $\cos 2\theta = \frac{1}{2}$, and therefore,

$$\sin 2\theta = \sqrt{\left(1 - \tfrac{1}{4}\right)} = \frac{\sqrt{3}}{2},$$

then we have

$$\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta = \tfrac{1}{4} - \tfrac{3}{4} = -\tfrac{1}{2}$$

and the corresponding value of the sine, or,

$$\sin 4\theta = \frac{\sqrt{3}}{2} = \sin 2\theta = \sin(\pi - 2\theta):$$

it follows therefore, that $4\theta = \pi - 2\theta$, and consequently,

$$6\theta = \pi, \text{ or } 2\theta = \frac{\pi}{3} = 60^\circ;$$

this gives the *least* value of 2θ, which answers the required conditions.

$$2\cos^2 30 = 1 + \cos 60 = 1 + \tfrac{1}{2},$$

$$\text{and} \quad \cos 30 = \sqrt{\left(\frac{1+\frac{1}{2}}{2}\right)} = \frac{\sqrt{3}}{2} = c_1\,.$$

In a similar manner

$$\cos\frac{30^\circ}{2} = \cos 15^\circ = \sqrt{\left(\frac{1+c_1}{2}\right)} = c_2,$$

$$\cos\frac{30^\circ}{2^2} = \cos 7^\circ.30' = \sqrt{\left(\frac{1+c_2}{2}\right)} = c_3,$$

$$\cos\frac{30^\circ}{2^3} = \cos 3^\circ.45' = \sqrt{\left(\frac{1+c_3}{2}\right)} = c_4,$$

and so on, by continued bisections of the angles, until we come to

$$\cos\frac{30^\circ}{2^{11}} = \cos .52''.44'''\frac{1}{16} = \sqrt{\left(\frac{1+c_{11}}{2}\right)} = .9999999674,$$

and the corresponding value of the sine,

$$\text{or,} \quad \sin\frac{30^0}{2^{11}} = \sqrt{(1-c_{11}{}^2)} = .000255625.$$

But

$$\frac{30^0}{2^{11}} = \frac{30 \times 60'}{2^5 \times 2^6} = \frac{225}{256} \times 1';$$

and inasmuch as the sines of very small angles may be considered as bearing to each other very nearly the same ratio with the angles themselves*, we may suppose without sensible error that

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* The proposition whose truth is here assumed, is a very simple consequence of the series which will be found hereafter to express the sine of θ, in terms of θ and its powers; in the absence however of this mode of verification, it will be sufficient for practical purposes, to observe that

$$\sin\frac{30^0}{2^9} = .0010224959,$$

$$\sin\frac{30^0}{2^{10}} = .0005112482,$$

$$\sin\frac{30^0}{2^{11}} = .0002556254,$$

$$\frac{\sin\frac{30^0}{2^{11}}}{\sin 1'} = \frac{225}{256};$$

and therefore,

$$\sin 1' = \frac{256}{225}\cdot\sin\frac{30^0}{2^{11}} = \frac{256}{225}\times .000225625$$

$$= .0002908882;$$

and therefore,

$$\cos 1' = .9999999577.$$

Process and formulæ requisite for calculating the sines and cosines of all angles up to 90°.

484. The knowledge of the sine and cosine of 1′ of a degree, will form the basis of our calculation of the sines and cosines of all the angles differing from each other by 1′, between an angle of 1′ and an angle of 90°.

For this purpose we shall require the aid of the formulæ

$$\sin(n+1)\theta = 2\cos\theta\sin n\theta - \sin(n-1)\theta\ldots\ldots(\alpha),$$

$$\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta\ldots\ldots(\beta),$$

which may be investigated as follows.

$$\sin(\theta'+\theta) = \sin\theta'\cos\theta + \cos\theta'\sin\theta,$$

$$\sin(\theta'-\theta) = \sin\theta'\cos\theta - \cos\theta'\sin\theta;$$

consequently, by addition we get,

$$\sin(\theta'+\theta) + \sin(\theta'-\theta) = 2\cos\theta\sin\theta';$$

and therefore,

$$\sin(\theta'+\theta) = 2\cos\theta\sin\theta' - \sin(\theta'-\theta);$$

if we now replace θ' by $n\theta$, we get,

$$\sin(n+1)\theta = 2\cos\theta\sin n\theta - \sin(n-1)\theta.$$

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and the ratio of the first sine to the second and of the second to the third, is very nearly that of 2 to 1, which is the ratio of the first angle to the second, and of the second to the third.

In a similar manner,

$$\cos(\theta' + \theta) = \cos\theta' \cos\theta - \sin\theta' \sin\theta,$$
$$\cos(\theta' - \theta) = \cos\theta' \cos\theta + \sin\theta' \sin\theta;$$

and therefore,

$$\cos(\theta' + \theta) = 2\cos\theta \cos\theta' - \cos(\theta' - \theta),$$

which gives, replacing θ' by $n\theta$,

$$\cos(n+1)\theta = 2\cos\theta \cos n\theta - \cos(n-1)\theta.$$

If we now consider $\theta = 1'$, and substitute successively the natural numbers 1, 2, 3, 4, &c. for n, we shall get

$$\sin 2' = 2\cos 1' \sin 1' \ldots\ldots\ldots = .0005817764,$$
$$\sin 3' = 2\cos 1' \sin 2' - \sin 1' = .0008726645,$$
$$\sin 4' = 2\cos 1' \sin 3' - \sin 2' = .0011635526,$$
$$\sin 5' = 2\cos 1' \sin 4' - \sin 3' = .0014544406,$$

……………………………………

$$\cos 2' = 2\cos 1' \cos 1' - \cos 0 = .9999998308,$$
$$\cos 3' = 2\cos 1' \cos 2' - \cos 1' = .9999996192,$$
$$\cos 4' = 2\cos 1' \cos 3' - \cos 2' = .9999993231,$$
$$\cos 5' = 2\cos 1 \cos 4' - \cos 3' = .9999989423.$$

It is not necessary to extend the double *canon* of sines and cosines beyond 45°.

485. In this manner we might proceed as far as the sine and cosine of 45°, after which the same values will reappear, which have been previously determined: for $\sin(45^\circ + A) = \cos(45 - A)$ and $\cos(45^\circ + A) = \sin(45^\circ - A)$, inasmuch as the angles $45^\circ + A$ and $45^\circ - A$ are complemental to each other: the sines therefore of the angles in the series ascending from 45° will be equal severally to the cosines of the *complemental* angles in the descending series which have already been determined; and similarly the cosines of the angles in the series ascending from 45° will be equal to the sines of the *complemental* angles in the descending series, which have already been determined: it thus becomes necessary to construct the *canon* or *table* of the sines and cosines as far as 45° and no further.

486. The preceding process enables us to calculate the actual arithmetical values of the sines and cosines of all angles (differing by 1′ only) as far as 90^0, beyond which their values recur in an inverse order:* it is the practicability of effecting such determinations which is important for our theory, as it will enable us to assign absolutely the different symbolical values of $(1)^{\frac{1}{n}}$: before we resume the consideration of this subject, however, we shall notice some other propositions which are of considerable importance in the science of Goniometry, which has been the chief object of our investigations in the immediately preceding Articles.

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* We have not attempted, in the preceding construction of the *canon* of sines and cosines, to point out various artifices by which the calculations may be shortened and their accuracy verified: one of the most important of these is supplied by the formula

$$\sin(30+\theta) = \cos\theta - \sin(30-\theta),$$

by which the sines of the angles from 30° to 60°, may be calculated by the mere subtraction of the sines of $(30^0-\theta)$ where θ may have every value from 1′ to 30°, from the cosine of θ: and since

$$\sin(30^0+\theta) = \cos(60-\theta),$$

the determination of the series of sines of the angles from 30° to 60°, will give us the corresponding series of cosines in an inverse order.

In the formulæ, which we have given, the sines and cosines of all angles from 1′ upwards, are calculated independently of each other, from ascertained values of the sine and cosine of 1′, and consequently the calculated values of the sine and cosine of any angle θ may serve to verify not only the correctness of their values, but likewise of the values of the sines and cosines of all other angles upon which they are dependent: for the equation

$$\cos^2\theta + \sin^2\theta = 1,$$

must be satisfied by the calculated values of $\cos\theta$ and $\sin\theta$, otherwise they are not correct: thus if we take the values of the sine and cosine of 5′, which are given in Art. 484, we shall find

$$\begin{aligned} \cos^2 5' &= .99999788460111872929, \\ \sin^2 5' &= .00000211539745892836, \\ \text{and } \cos^2 5 + \sin^2 5 &= .99999999999857765765; \end{aligned}$$

a quantity which differs from 1 by a quantity less than .000000000002, a discrepancy which arises from the influence of terms in the calculated values of $\cos 5'$ and $\sin 5'$, which are necessarily omitted, as being beyond the 10th place of decimals.

487. Inasmuch as θ and θ' are equal respectively to the sum and difference of

Sum and difference of the sines or cosines of two angles.

$$\frac{\theta+\theta'}{2} \text{ and } \frac{\theta-\theta'}{2},$$

it follows that

$$\sin\theta = \sin\left\{\frac{(\theta+\theta')}{2} + \frac{(\theta-\theta')}{2}\right\}$$

$$= \sin\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right) + \cos\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right)$$

$$\text{and } \sin\theta' = \cos\left\{\frac{(\theta+\theta')}{2} - \frac{(\theta-\theta')}{2}\right\}$$

$$= \sin\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right) - \cos\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right):$$

consequently, by adding and subtracting, we get

$$\sin\theta + \sin\theta' = 2\sin\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right);$$

$$\sin\theta - \sin\theta' = 2\cos\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right).$$

In a similar manner, it would appear that

$$\cos\theta + \cos\theta' = 2\cos\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right),$$

$$\cos\theta - \cos\theta' = -2\sin\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right).$$

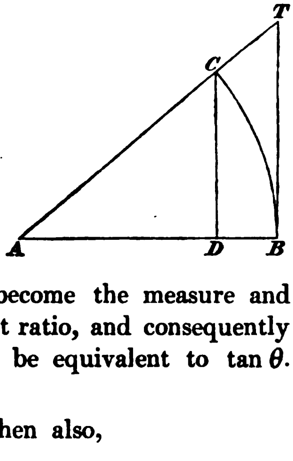
These are very important formulæ, inasmuch as they enable us to pass from the sum of the sines of two arcs or the sum of the cosines of two arcs, connected by the signs + or −, to equivalent *products* of cosines or sines of their sum or difference: a species of transformation which is very frequently required in the adaptation of formulæ to the purposes of calculation.

Definition of the tangent of an angle,

488. The ratio of the sine of θ to its cosine is denominated the *tangent* of θ, or $\tan\theta = \frac{\sin\theta}{\cos\theta}$: in Geometry, the tangent of the angle BAC would denote the ratio of DC to AD, or of the perpendicular to the base of the right-angled triangle DAC: it would follow therefore that

$$DC = AD \tan BAC = AD \tan\theta:$$

and if we should suppose $AB = 1$, or the radius of the circle to be 1, and if we should draw the tangent BT to the circle at B, meeting AC produced in T, then we should have $\frac{BT}{AB} = \frac{DC}{AD}$; and if $AB = 1$, then also it would follow that $BT = \frac{DC}{AD}$, or the geometrical tangent BT would become the measure and representative of the value of that ratio, and consequently under such circumstances, would be equivalent to $\tan\theta$.



$$\text{If} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{then also,}$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \frac{\cos\theta}{\sin\theta} = \cot\theta:$$

and of the cotangent.

that is, under such circumstances, the sine becomes the cosine of θ, and conversely, and we complete the analogy of nomenclature and representation, by denominating the tangent of the complement of θ the *cotangent* of θ and denoting it by $\cot\theta$.

Expression for the tangent of the sum and difference of two angles.

489. Again,

$$\tan(\theta+\theta') = \frac{\sin(\theta+\theta')}{\cos(\theta+\theta')} = \frac{\sin\theta\cos\theta' + \cos\theta\sin\theta'}{\cos\theta\cos\theta' - \sin\theta\sin\theta'},$$

$$= \frac{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\theta'}{\cos\theta'}}{1 - \dfrac{\sin\theta\cos\theta'}{\cos\theta\cos\theta'}};$$

by dividing the numerator and denominator by $\cos\theta\cos\theta'$: if we now replace $\dfrac{\sin\theta}{\cos\theta}$ and $\dfrac{\sin\theta'}{\cos\theta'}$, by $\tan\theta$ and $\tan\theta'$, we shall get

$$\tan(\theta+\theta') = \frac{\tan\theta + \tan\theta'}{1 - \tan\theta\tan\theta'}.$$

In a similar manner, we shall find, putting $-\theta'$ in the place of θ

$$\tan(\theta-\theta') = \frac{\tan\theta - \tan\theta'}{1 + \tan\theta\tan\theta'}.$$

Particular values of the tangent.

490. By the aid of these formulæ and the *assumed* value of the tangent, we shall easily deduce or verify the following results;

$\tan 0 = 0$, $\tan\frac{\pi}{4}$ or $\tan 45^0 = 1$,

$\tan\frac{\pi}{6}$ or $\tan 30 = \frac{1}{\sqrt{3}}$, $\tan\frac{2\pi}{6}$ or $\tan 60 = \sqrt{3}$,

$\tan\frac{\pi}{2} = \infty$, $\tan 2\pi = 0$,

$\tan\frac{3\pi}{2} = -\infty$, $\tan 4\pi = 0$,

$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$, $\tan(\pi-\theta) = -\tan\theta$,

$$\tan(\pi+\theta)=\tan\theta, \qquad \tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta,$$

$$\tan(2\pi-\theta)=-\tan\theta, \quad \tan(2\pi+\theta)=\tan\theta,$$

$$\tan(2n\pi+\theta)=\tan\theta, \quad \tan\{(2n+1)\pi+\theta\}=\tan\theta.$$

The secant and cosecant of an angle.

491. The reciprocal of the ratio expressed by the cosine of θ is called the *secant* of θ: or the secant of θ or $\sec\theta=\frac{1}{\cos\theta}$: if we now replace θ by $\frac{\pi}{2}-\theta$, we get

$$\sec\left(\frac{\pi}{2}-\theta\right)=\frac{1}{\cos\left(\frac{\pi}{2}-\theta\right)}=\frac{1}{\sin\theta},$$

which is denominated the *cosecant* of θ or the secant of the complement of θ, and is written thus, $(\text{cosec}\,\theta)$. In Geometry, the secant of BAC (Fig. in Art. 488.) would express the ratio $\frac{AC}{AD}$, a ratio which is equal to $\frac{AT}{AB}$: and if AB or the radius of the circle be 1, then AT will represent the ratio which is denominated the secant.

The versed sine of an angle.

492. The *versed sine* of θ, written vers. θ, is used to express $1-\cos\theta$: it is the algebraical difference of 1 and the ratio $\frac{AD}{AC}$, or of $\frac{AB}{AC}$ and $\frac{AD}{AC}$; and is equal therefore to $\frac{BD}{AC}$: if AB or AC be 1, it is clearly represented by BD.

The terms tangent, cotangent, secant, cosecant, and versed sine, have a geometrical origin.

493. The terms tangent and cotangent, secant and cosecant, and versed sine, have not only a geometrical origin, but their introduction is properly due to a view of the primary objects of this science, which is essentially geometrical, as representing or determining the relations of lines described in or about a circle: under such circumstances, the investigation of theorems connected with the relations of those lines, the changes of their algebraical signs when represented symbolically, and the

determination of their values corresponding to arcs which are sub-multiples of a circumference as well as of the sums or multiples of other arcs, would or rather might be independent of the sines or cosines of such arcs or of their relations to each other: but in an algebraical view of this science, such as we have given, the tangent and cotangent, the secant and cosecant, and the versed sine, can only be considered as furnishing the means of expressing under a varied or abbreviated form, results which are otherwise expressed by means of cosines and sines, and whose algebraical signs and arithmetical values are entirely dependent upon them: in other words, the use of such terms and the substitution of such equivalent modes of expression for quantities otherwise expressed by means of sines and cosines, is in no respect essential to the science of Goniometry, or to any of its applications.

494. We shall now resume the consideration of the values of $(1)^{\frac{1}{n}}$, with a view to their symbolical determination, and for this purpose, we shall commence with the following most important proposition.

Demoivre's formula.

Since $\epsilon^{\theta} = \cos\theta + \sqrt{-1}\sin\theta$,

if we replace θ by $n\theta$, we shall get

$$\epsilon^{n\theta} = \cos n\theta + \sqrt{-1} \, . \sin n\theta;$$

but $\epsilon^{n\theta} = (\epsilon^{\theta})^n = (\cos\theta + \sqrt{-1} \, . \sin\theta)^n$:

consequently,

$$\cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^n \dots (\alpha).$$

Also, since

$$\epsilon^{-\theta} = \cos\theta - \sqrt{-1}\sin\theta;$$

it also follows, in a similar manner, that

$$\cos n\theta - \sqrt{-1}\sin n\theta = (\cos\theta - \sqrt{-1}\sin\theta)^n \dots (\beta).$$

There are n different values of θ, when the value of $n\theta$ is inferred from the value of its sine or cosine.

495. If in these equations (α) and (β), the value of θ be absolutely assigned, they will admit of no variation of form, and it will be merely necessary that the value of the angle θ on one side should be $\frac{1}{n}$th part of the value of the angle ($n\theta$) on the other. But if the value of the angle $n\theta$ is to be *inferred* from the known value of its cosine or sine, then it will be found that there are n different values of θ on the second side of these equations, which equally answer the required conditions.

For

$$\cos n\theta = \cos(2\pi + n\theta) = \cos(4\pi + n\theta)$$
$$= \cos(2n\pi + n\theta),$$

and

$$\sin n\theta = \sin(2\pi + n\theta) = \sin(4\pi + n\theta)$$
$$= \sin(2n\pi + n\theta);$$

consequently, as the value of the angle on the second side of these several equations, would, in every case, be $\frac{1}{n}$th part of its value on the first, we shall get

$$\cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\,.\,\sin\theta)^n,$$

$$\cos(2\pi + n\theta) + \sqrt{-1}\sin(2\pi + n\theta)$$
$$= \left\{\cos\left(\frac{2\pi}{n} + \theta\right) + \sqrt{-1}\sin\left(\frac{2\pi}{n} + \theta\right)\right\}^n,$$

$$\cos(4\pi + n\theta) + \sqrt{-1}\sin(4\pi + n\theta)$$
$$= \left\{\cos\left(\frac{4\pi}{n} + \theta\right) + \sqrt{-1}\sin\left(\frac{4\pi}{n} + \theta\right)\right\}^n,$$

$$\cos\{(2n-2)\pi + n\theta\} + \sqrt{-1}\sin\{(2n-2)\pi + n\theta\}$$
$$= \left\{\cos\left(\frac{(2n-2)\pi}{n} + \theta\right) + \sqrt{-1}\sin\left(\frac{(2n-2)\pi}{n} + \theta\right)\right\}^n,$$

$$\cos(2n\pi + n\theta) + \sqrt{-1}\sin(2n\pi + n\theta)$$
$$= \{\cos(2\pi + \theta) + \sqrt{-1}\sin(2\pi + \theta) = \cos\theta + \sqrt{-1}\sin\theta,$$

$$\cos\{(2n+2)\pi+n\theta\}+\sqrt{-1}\sin\{(2n+2)\pi+n\theta\}$$

$$=\left\{\cos\left(2\pi+\frac{2\pi}{n}+\theta\right)+\sqrt{-1}\sin\left(2\pi+\frac{2\pi}{n}+\theta\right)\right\}^n$$

$$=\left\{\cos\left(\frac{2\pi}{n}+\theta\right)+\sqrt{-1}\sin\left(\frac{2\pi}{n}+\theta\right)\right\}^n,$$

and so on, the series of such equations being continued, as far as we choose.

Only n different values of θ, which produce different values of the sine and cosine.

496. Now, the cosines and sines of the several angles differing from each other by 2π, or by multiples of 2π or four right angles, are all identical with each other; and inasmuch as the values of these angles are to be *inferred* from the values of their sines and cosines, it follows that as they all of them equally answer the required conditions, there is no reason for the selection of one of them in preference to any other: but the corresponding angles on the second side of the several equations, being $\frac{1}{n}$th part only of those on the first, will differ from each other, when taken in the order of the series, by $\frac{2\pi}{n}$ only, forming the series

$$\theta,\ \frac{2\pi}{n}+\theta,\ \frac{4\pi}{n}+\theta,\ldots\ldots\frac{(2n-2)\pi}{n}+\theta,$$

$$2\pi+\theta,\ 2\pi+\frac{2\pi}{n}+\theta,\ 2\pi+\frac{4\pi}{n}+\theta,$$

and so on; the cosines and sines, one or both of them, of the n first of which differ or may differ from each other, inasmuch as they form a series of angles which are or may be less than 2π: but the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, $(n+r)^{\text{th}}$ terms of the series, being equal to

$$2\pi+\theta,\ 2\pi+\frac{2\pi}{n}+\theta,\ldots.2\pi+\frac{2(r-1)\pi}{n}+\theta,$$

will have the same cosines and sines with the angles

$$\theta,\ \frac{2\pi}{n}+\theta, \ldots\ldots \frac{2(r-1)\pi}{n}+\theta,$$

or with the first n terms of the original series: it follows therefore, that there are only n values of the angles on the second side of the equations, whose cosines and sines are or may be different from each other.

It appears, therefore, that whilst the first side of the equation

$$\cos(2r\pi+n\theta)+\sqrt{-1}\sin(2r\pi+n\theta)$$

$$=\left\{\cos\left(\frac{2r\pi}{n}+\theta\right)+\sqrt{-1}\sin\left(\frac{2r\pi}{n}+\theta\right)\right\}^n,$$

admits but of one value, when r is any one of the terms of the series

$$0,\ 1,\ 2,\ 3,\ 4, \ldots\ldots\ldots r,$$

that there are n and no more different values, corresponding to the n first terms of the same series, of

$$\cos\left(\frac{2r\pi}{n}+\theta\right)+\sqrt{-1}\sin\left(\frac{2r\pi}{n}+\theta\right),$$

by which the conditions of the equation are satisfied; it follows, therefore, that any one of these values may be considered as one of the n symbolical roots of

$$\cos n\theta+\sqrt{-1}\sin n\theta,$$

or of the equivalent quantity

$$\cos(2r\pi+n\theta)+\sqrt{-1}\sin(2r\pi+n\theta).$$

The angle $\frac{(2n-2)\pi}{n}+\theta$ is the supplement of $\frac{2\pi}{n}-\theta$ to 2π or four right angles, and it may be replaced therefore by $-\left(\frac{2\pi}{n}-\theta\right)$: in a similar manner, the angle

$\dfrac{(2n-4)\pi}{n}+\theta$ may be replaced by $-\left(\dfrac{4\pi}{n}-\theta\right)$ and so on, if n be an even number, as far as

$$\frac{(2n-n)\pi}{n}+\theta \text{ or } \pi+\theta,$$

which may be replaced by $-(\pi-\theta)$: but if n be an odd number, then the last of these angles which is greater than π, is

$$\frac{\left(2n-\dfrac{2n-1}{2}\right)}{n}\pi+\theta \text{ or } \pi+\frac{\pi}{2n}+\theta,$$

which may be replaced by $-\left(\pi-\dfrac{\pi}{2n}-\theta\right)$: by such substitutions our attention will be confined to the sines and cosines of angles which are less than 180°.

497. If we suppose $n\theta$ equal to 0, or to 2π, or to 4π, or to $2n\pi$, then in all such cases, we shall have

The n symbolical values of $(1)^{\frac{1}{n}}$.

$$\cos n\theta = 1 \text{ and } \sin n\theta = 0;$$

$$\text{and therefore, } \cos n\theta + \sqrt{-1}\,.\,n\theta = 1:$$

under such circumstances, the n different values of

$$(\cos n\theta + \sqrt{-1}\sin n\theta)^{\frac{1}{n}} \text{ or of } (1)^{\frac{1}{n}}$$

become

$$\cos 0+\sqrt{-1}\sin 0,\ \cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n},$$

$$\cos\frac{4\pi}{n}+\sqrt{-1}\sin\frac{4\pi}{n},\ldots$$

$$\cos\frac{(2n-2)\pi}{n}+\sqrt{-1}\,.\sin\frac{(2n-2)\pi}{n};$$

beyond which values, the same series of values recurs in the same order: these are the n symbolical roots of 1, or values of $(1)^{\frac{1}{n}}$, the determination of which has been the primary object of the preceding investigations.

When n is even.

If n be an even number, we may replace

$$\frac{(2n-2)\pi}{n} \text{ by } -\frac{2\pi}{n}, \quad \frac{(2n-4)\pi}{n} \text{ by } -\frac{4\pi}{n},$$

proceeding in this manner as far as $\frac{(2n-n)\pi}{n}$ or π, whose cosine and sine is identical with that of $-\pi$: under such circumstances, the n roots of 1, will be represented by

$$1, \cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n} \text{ and } \cos\frac{2\pi}{n} - \sqrt{-1}\sin\frac{2\pi}{n},$$

$$\cos\frac{4\pi}{n} + \sqrt{-1}\sin\frac{4\pi}{n} \text{ and } \cos\frac{4\pi}{n} - \sqrt{-1}\sin\frac{4\pi}{n},$$

$$\cos\frac{(n-2)\pi}{n} + \sqrt{-1}\sin\frac{(n-2)\pi}{n}$$

$$\text{and } \cos\frac{(n-2)\pi}{n} - \sqrt{-1}\sin\frac{(n-2)\pi}{n},$$

and lastly $\cos\pi$ or -1.

When n is odd.

But if n be an odd number, the series of symbolical roots of 1, will terminate with

$$\cos\frac{(n-2)\pi}{n} + \sqrt{-1}\,.\sin\frac{(n-2)\pi}{n}$$

$$\text{and } \cos\frac{(n-2)\pi}{n} - \sqrt{-1}\sin\frac{(n-2)\pi}{n},$$

there being no root equal to $\cos\pi$ or -1.

Examples.

498. Thus, let it be required to assign the cube roots of 1. They are symbolically represented by

$$1, \cos\frac{2\pi}{3} + \sqrt{-1}\sin\frac{2\pi}{3}$$

$$\text{and } \cos\frac{2\pi}{3} - \sqrt{-1}\,.\sin\frac{2\pi}{3}: \text{ but } \cos\frac{2\pi}{3} = -\cos\frac{\pi}{3} = -\tfrac{1}{2}$$

$$\text{and } \sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}:$$

the roots required, therefore, are

$$1, \quad \frac{-1+\sqrt{3}.\sqrt{-1}}{2}, \text{ and } \frac{-1-\sqrt{3}.\sqrt{-1}}{2}.$$

The four values of $(1)^{\frac{1}{4}}$.

Again, let it be required to assign the four roots of 1, or the values of $(1)^{\frac{1}{4}}$.

They are symbolically represented by

$$1, \ \cos\frac{2\pi}{4}+\sqrt{-1}\sin\frac{2\pi}{4},$$

$$\cos\frac{2\pi}{4}-\sqrt{-1}\sin\frac{2\pi}{4} \text{ and } -1:$$

$$\text{but } \cos\frac{2\pi}{4}=\cos\frac{\pi}{2}=0 \text{ and } \sin\frac{2\pi}{4}=\sin\frac{\pi}{2}=1:$$

the roots, therefore, are

$$1, \quad \sqrt{-1}, \quad -\sqrt{-1}, \quad -1,$$

coinciding with those which have already been otherwise determined. (Art. 452.)

The five values of $(1)^{\frac{1}{5}}$.

Let it be required to assign the five roots of 1, or the values of $(1)^{\frac{1}{5}}$.

They are symbolically represented by

$$1, \ \cos\frac{2\pi}{5}+\sqrt{-1}\sin\frac{2\pi}{5}, \ \cos\frac{2\pi}{5}-\sqrt{-1}\sin\frac{2\pi}{5},$$

$$\cos\frac{\pi}{5}+\sqrt{-1}\sin\frac{\pi}{5} \text{ and } \cos\frac{\pi}{5}-\sqrt{-1}\sin\frac{\pi}{5}.$$

Now, by reference to a table of natural sines and cosines, we shall find

$$\cos\frac{\pi}{5}=\cos 36^{\circ}=.8090170, \quad \sin 36^{\circ}=.5877853,$$

$$\cos\frac{2\pi}{5}=\cos 72^{\circ}=.3090170, \quad \sin 72^{\circ}=.9510565:$$

the five roots of 1 required, therefore, are

$$1,\ .309017+.9510565\sqrt{-1},\ .309017-.9510565\sqrt{-1},$$
$$.809017+.5877853\sqrt{-1},\ .809017-.5877853\sqrt{-1}.$$

The values of $(-1)^{\frac{1}{n}}$ included amongst those of $(1)^{\frac{1}{2n}}$.

499. The roots of -1, or the values of $(-1)^{\frac{1}{n}}$ are included amongst the $2n$ roots of 1, or amongst the values of $(1)^{\frac{1}{2n}}$: for $x^{2n}-1=(x^n-1)(x^n+1)$; and, therefore, the values which satisfy the equation $x^n+1=0$ or $x^n=-1$, must satisfy likewise the equation $x^{2n}-1=0$ or $x^{2n}=1$; or, in other words, the values of $(-1)^{\frac{1}{n}}$ are likewise values of $(1)^{\frac{1}{2n}}$.

The same conclusion may be made to follow from the equation

$$\cos n\theta+\sqrt{-1}\sin n\theta=(\cos\theta+\sqrt{-1}\sin\theta)^n:$$

for if we make

$$n\theta=\pi,\ \text{or}\ 3\pi,\ \text{or}\ 5\pi,\ \text{or}\ (2n+1)\pi,$$
$$\text{then,}\quad \cos n\theta=-1\ \text{and}\ \sin n\theta=0;$$

and, therefore,

$$-1=(\cos\theta+\sqrt{-1}\sin\theta)^n,$$
$$\text{where}\quad \theta=\frac{\pi}{n},\ \text{or}\ \frac{3\pi}{n},\ \text{or}\ \frac{5\pi}{n},\ \&c.$$

as far as $\dfrac{(2n-1)}{n}\pi$, after which the same values recur in the same order; and it is obvious that those values are identical with the alternate values of θ, beginning with the second, which satisfy the equation $\cos 2n\theta+\sqrt{-1}\sin 2n\theta=1=(\cos\theta+\sqrt{-1}\sin\theta)^{2n}$, which are

$$0,\ \frac{2\pi}{2n},\ \frac{4\pi}{2n},\ \frac{6\pi}{2n},\ \frac{8\pi}{2n},\ \ldots\ldots\frac{(4n-2)\pi}{2n}.$$

The values of $(\sqrt{-1})^{\frac{1}{n}}$ included amongst those of $(1)^{\frac{1}{4n}}$.

500. In a similar manner, if we supposed

$$\cos 4n\theta+\sqrt{-1}\sin 4n\theta=\sqrt{-1}$$
$$=(\cos\theta+\sqrt{-1}\sin\theta)^{4n};$$

and, consequently,

$$4n\theta = \frac{\pi}{2}, \text{ or } 5 \,.\, \frac{\pi}{2}, \text{ or } 9 \,.\, \frac{\pi}{2}, \text{ as far as } (4n-3)\frac{\pi}{2},$$

then the values of $(\sqrt{-1})^{\frac{1}{n}}$ would be every fourth term, beginning with the second, of the successive values of $(1)^{\frac{1}{4n}}$, which are

$$0, \frac{2\pi}{4n}, \frac{4\pi}{4n}, \frac{6\pi}{4n}, \frac{8\pi}{4n}, \frac{10\pi}{4n}, \text{ \&c}:$$

in other words, the n roots of $\sqrt{-1}$ or the values of $(\sqrt{-1})^{\frac{1}{n}}$, are included amongst those of $(1)^{\frac{1}{4n}}$.

In the same manner, the values of $(-\sqrt{-1})^{\frac{1}{n}}$ will coincide with every fourth of the values of $(1)^{\frac{1}{4n}}$, beginning with the fourth: and generally the values of the n^{th} root of any one of the m roots of 1, different from 1, will be found amongst the series of those forming the values of $(1)^{\frac{1}{mn}}$.

General results of the preceding theory.

501. It thus appears that the n roots of 1 or the values of $(1)^{\frac{1}{n}}$ are included in the formula

$$\cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n},$$

where r may have any value in the series $0, 1, 2, 3 \ldots\ldots r$: it likewise appears that there are only n such values which are different from each other, corresponding to the n first terms of this series, the succeeding terms of the series reproducing the same succession of values in the same order: and, lastly, that the arithmetical values of

$$\cos\frac{2r\pi}{n} \text{ and } \sin\frac{2r\pi}{n},$$

which enter into the symbolical expressions of these roots, admit of accurate or of approximate calculation.

In a similar manner, it appears that the n roots of -1, (if it is wished to consider them apart from the $2n$ roots of 1) are included in the formula

$$\cos\left(\frac{2r+1}{n}\right)\pi + \sqrt{-1}\ \sin\left(\frac{2r+1}{n}\right)\pi,$$

to which the same remarks are applicable which were made in the last Article: it is not necessary to adapt a formula for the expression of the n roots of $\sqrt{-1}$ or $-\sqrt{-1}$, or of any other roots of 1, which are of less frequent occurrence than those of 1 or -1.

All the roots of 1 reducible to the form $\alpha+\beta\sqrt{-1}$.

502. All the roots of 1, and therefore the symbolical representations of the affections which they designate, are reducible to the form

$$\alpha + \beta\sqrt{-1},$$

where $\alpha^2 + \beta^2 = 1$, and where α and β are limited to the values

$$\cos\frac{2r\pi}{n} \text{ and } \sin\frac{2r\pi}{n}:$$

and inasmuch as $(a^n)^{\frac{1}{n}} = a\,(1)^{\frac{1}{n}}$, where a is the arithmetical root of a^n, it follows that the n roots of a^n are expressed by the formula

$$a\left(\cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n}\right),$$

which is reducible to the form

$$a\,.\,(\alpha + \beta\sqrt{-1}),$$

$$\text{or } \alpha' + \beta'\sqrt{-1},$$

if $\alpha' = a\alpha$ and $\beta' = a\beta$.

Equivalent forms of $a+b\sqrt{-1}$.

503. If we commence with the form

$$a + b\sqrt{-1},$$

we shall easily reduce it to another form which is equivalent to it, namely,

$$\sqrt{a^2+b^2}\,.\left\{\frac{a}{\sqrt{a^2+b^2}} + \frac{b\sqrt{-1}}{\sqrt{a^2+b^2}}\right\}:$$

and if we now make

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \text{ and, therefore, } \sin\theta = \frac{b}{\sqrt{a^2+b^2}},$$

(which we are at liberty to do, inasmuch as

$$\cos^2\theta + \sin^2\theta = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = 1),$$

then we shall get

$$a + b\sqrt{-1} = \sqrt{(a^2+b^2)}\,\{\cos\theta + \sqrt{-1}\sin\theta\},$$

$$\text{or } = \sqrt{(a^2+b^2)}\,e^{\theta}:$$

for e^{θ} and $\cos\theta + \sqrt{-1}\sin\theta$, have been shewn to be equivalent symbolical expressions, whether used as signs, or otherwise.

Conversely, $\sqrt{(a^2+b^2)}\,e^{\theta}$,

or $\sqrt{(a^2+b^2)}\,(\cos\theta + \sqrt{-1}\sin\theta)$, is equivalent to

$$a + b\sqrt{-1},$$

if $a = \sqrt{(a^2+b^2)}\cos\theta$ and $b = \sqrt{(a^2+b^2)}\sin\theta$:

in one case, therefore, a and b are given, to determine

$$\sqrt{a^2+b^2},\ \cos\theta \text{ and, therefore, } \sin\theta:$$

in the other case,

$$\sqrt{a^2+b^2},\ \cos\theta \text{ and, therefore, } \sin\theta$$

are given, to determine a and b.

The geometrical signification of a and b in the formula $a+b\sqrt{-1}$.

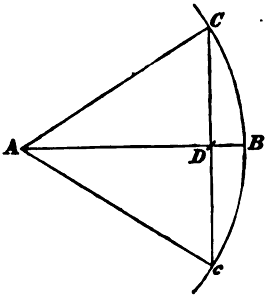
504. If with a radius

$$AB = \sqrt{(a^2+b^2)},$$

we describe an arc of a circle, and make the angle $BAC = \theta$,

$\left(\text{whose cosine} = \frac{a}{\sqrt{a^2+b^2}} \text{ and}\right.$

$\left.\text{sine} = \frac{b}{\sqrt{a^2+b^2}}\right)$,



then a and b in the formula,

$$a + b\sqrt{-1},$$

will express the sides AD and CD of the right-angled triangle, whose hypothenuse $AC = \sqrt{(a^2 + b^2)}$, considering AC as the arithmetical root of $a^2 + b^2$.

Again, $\sqrt{a^2 + b^2} . \epsilon^\theta$ expresses the line which is arithmetically equal to $AC = \sqrt{a^2 + b^2}$, and which makes an angle θ with the primitive line: and inasmuch as

$$\sqrt{(a^2 + b^2)}\,\epsilon^\theta = \sqrt{(a^2 + b^2)}\ (\cos\theta + \sqrt{-1}\sin\theta)$$
$$= a + b\sqrt{-1},$$

it follows that $a + b\sqrt{-1}$, where a and b express the actual arithmetical or geometrical values of the sides AD and DC of the right-angled triangle, will express its hypothenuse $AC = \sqrt{(a^2 + b^2)}$, making an angle θ with the base AD or with the primitive line.

Of a and b in the formula $a-b\sqrt{-1}$.

In a similar manner, we should find, if we make the angle $BAc = -\theta$, that

$$Ac = \sqrt{(a^2 + b^2)}\,\epsilon^{-\theta} = \sqrt{(a^2 + b^2)}\,(\cos\theta - \sqrt{-1}\sin\theta)$$
$$= a - b\sqrt{-1};$$

and, therefore, $a - b\sqrt{-1}$ would denote the hypothenuse $Ac = \sqrt{a^2 + b^2}$ of a right-angled triangle ADc, making an angle $-\theta$ $\left(\text{whose cosine} = \dfrac{a}{\sqrt{a^2 + b^2}}\right)$ with its base AD or with the primitive line.

Interpretation of the signs + and − when connecting the symbols a and $b\sqrt{-1}$ which designate lines.

505. We thus arrive at the very important interpretation of the meaning of the signs + and −, when connecting symbols designating lines, when one of them is affected with the sign $\sqrt{-1}$: their *sum* or *difference*, in the sense which is symbolized by the signs + and −, when thus used, would designate the hypothenuse of the right-angled triangle, whose sides in one case were a and $b\sqrt{-1}$, and in the other a and $-b\sqrt{-1}$,

where $b\sqrt{-1}$ and $-b\sqrt{-1}$ would designate lines drawn in opposite directions to each other.

The general algebraical representation of lines in position as well as magnitude.

506. By such means, lines are designated in all positions with respect to the primitive line: if they are given in length (a), and make a given angle θ with the primitive line or axis, whatever it may be, then they are designated by

$$a \cdot \epsilon^{\theta} \text{ or } a(\cos\theta + \sqrt{-1}\sin\theta),$$
$$\text{where } \epsilon^{\theta} \text{ and } \cos\theta + \sqrt{-1}\sin\theta,$$

are the equivalent algebraical signs of their position; if they are given, by means of the sides of right-angled triangles, of which the lines form the hypothenuses, and whose bases coincide with the primitive line or axis, then they are expressed by the form

$$a + b\sqrt{-1}.$$

The transition from one form to the other is immediate, by making in one case

$$a = a\cos\theta \text{ and } b = a\sin\theta;$$

and in the other,

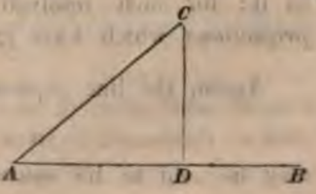
$$\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}},\ \sin\theta = \frac{b}{\sqrt{(a^2+b^2)}}$$
$$\text{and } a = \sqrt{(a^2+b^2)}.$$

Varied but equivalent interpretations of $a+b\sqrt{-1}$ when a and b are lines.

507. The formula

$$a + b\sqrt{-1},$$

where a and b designate lines, and therefore a and $b\sqrt{-1}$ designate lines at right angles to each other, will admit of other interpretations which are different in form though substantially equivalent to each other: thus, if $AD = a$, be taken on the primitive axis and $DC = b$ at right angles to it, then AD and DC are called the *co-ordinates* of the point C: under such circum-



stances we may consider $a + b\sqrt{-1}$ as designating the line AC which passes through the *origin* A of the co-ordinates and also through the point C, whose co-ordinates are a and b: or it may denote the distance from the origin of the point whose co-ordinates are a and b, *the co-ordination* of a and b designating their position with respect to each other.

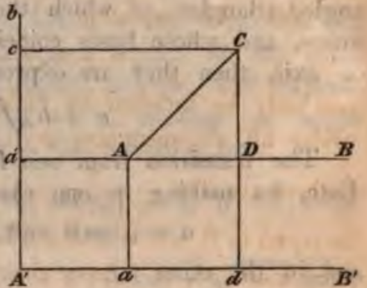
Again, the line designated by

$$a + b\sqrt{-1},$$

is one whose projection (by perpendiculars from its extremities) upon the primitive axis is a, and upon an axis at right angles to it is b: thus, if the primitive axis be $A'B'$ (parallel to AB), $A'b$ the axis at right angles to it; and if AC be the line represented by

$$a + b\sqrt{-1},$$

where $a = AD$ and $b = DC$, then ad is equal and parallel to AD, and, therefore, expressed by the same symbol a, and $a'c$ is equal and parallel to DC, and therefore expressed by the same symbol b.



In a similar manner, the line expressed by

$$a + b\sqrt{-1},$$

may be said to admit of *resolution* into the lines a and b, upon the primitive axis and the axis at right angles to it: for such resolved lines are identical with the *projections* which have just been considered.

Again, the line expressed by

$$a + b\sqrt{-1},$$

may be said to be *equivalent* to the line a upon the primitive axis, and to the line b upon the axis at right angles to it: or it may be said to include within itself, the expression of these two lines.

All these varied forms of expressing, or exhibiting the interpretation of

$$a + b\sqrt{-1},$$

though equivalent to each other, are more or less adapted to the varied nature of the quantities which the line in such a position may represent, or to the particular object or objects which the proposed investigation may be required to effect.

If $a+b\sqrt{-1}=0$, then $a=0$ and $b=0$.

508. The equation

$$a + b\sqrt{-1} = 0,$$

can only be satisfied by making $a = 0$, and $b = 0$: for under all circumstances $a + b\sqrt{-1} = \sqrt{(a^2 + b^2)}\ \epsilon^{\theta}$ $\left(\text{where } \cos\theta = \dfrac{a}{\sqrt{(a^2+b^2)}}\right)$: and inasmuch as ϵ^{θ} cannot be equal to 0, whilst a and b have any arithmetical values, zero included, it follows that if $a + b\sqrt{-1} = 0$, then also $\sqrt{(a^2 + b^2)} = 0$, which can only take place when $a = 0$ and $b = 0$.

If $a+b\sqrt{-1}=c+d\sqrt{-1}$ then $a=c$ and $b=d$.

509. Again, the equation

$$a + b\sqrt{-1} = c + d\sqrt{-1},$$

can only be satisfied by making $a = c$ and $b = d$: for under such circumstances, by *subtracting* $c + d\sqrt{-1}$ from both sides, we get

$$a - c + (b - d)\sqrt{-1} = 0,$$

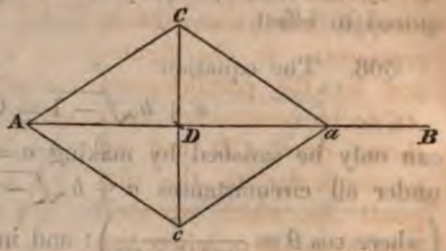
and, therefore, by the last case, $a-c=0$ and $b-d=0$, or $a = c$ and $b = d$.

The preceding conclusions consequences also of the geometrical interpretation of $a+b\sqrt{-}$

510. The conclusion deduced in both these cases is perfectly consistent with the interpretation which we have given of $a + b\sqrt{-1}$, considering it as representing the

hypothenuse of a right-angled triangle, whose base and perpendicular are a and b respectively: for if the hypothenuse be equal to zero, its sides must be equal to zero likewise. Again, the formulæ $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$ express two lines arithmetically equal to $\sqrt{(a^2+b^2)}$ and $\sqrt{(c^2+d^2)}$ respectively: if $a+b\sqrt{-1}=c+d\sqrt{-1}$, the lines corresponding must not only be equal in length, but also algebraically parallel to each other; for otherwise they would be affected with different signs, and therefore not algebraically equal or equivalent to each other: it will readily follow from thence that $a=c$ and $b=d$.

The algebraical sum and difference of two equal lines.



511. If the line AC be expressed by

$$a+b\sqrt{-1},$$

where $a=AD$ and $b=DC$, then the line Ac equal to AC, and making the angle DAc equal the angle DAC, is expressed by

$$a-b\sqrt{-1};$$

therefore, their algebraical *sum* is $2a$, and their algebraical *difference* $2b\sqrt{-1}$: now if we complete the *rhombus* $ACac$, the diagonal Aa coinciding with the primitive axis or parallel to it, is equal to $2AD$ or $2a$: and the other diagonal CDc at right angles to the primitive axis is equal to $2CD$ or $2b\sqrt{-1}$: it follows, therefore, that the algebraical sum of the equal lines AC and Ac when symbolically represented, is equal to that diagonal of the rhombus formed upon them, which they include: and that their algebraical difference under the same circumstances, is equal to the second diagonal, which is at right angles to the first.

It is obvious, that if we represented the angle DAC by θ, DAc by $-\theta$, and, therefore,

$$AC \text{ by } \sqrt{a^2+b^2}\,(\cos\theta+\sqrt{-1}\sin\theta),$$

and

$$Ac \text{ by } \sqrt{a^2+b^2}\,(\cos\theta - \sqrt{-1}\sin\theta),$$

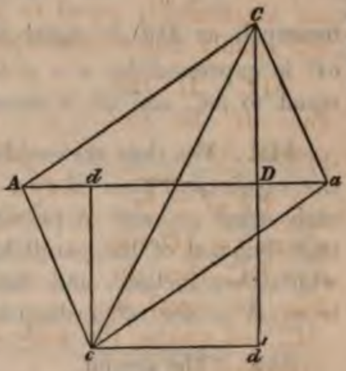
that their sum would be

$$2\sqrt{(a^2+b^2)}\cos\theta,$$

and that their difference would be $2\sqrt{(a^2+b^2)}\sqrt{-1}\sin\theta$; expressions which are respectively equivalent to $2a$ and $2b\sqrt{-1}$, or to the diagonals Aa and Cc respectively.

The algebraical sum of two unequal lines.

512. More generally, let AC and Ac represent two unequal lines in the positions represented in the figure, and let the parallelogram $ACac$ be constructed upon them: draw CD and cd perpendicular to the diagonal Aa, which we shall consider as the primitive line or axis, or parallel to it: then AC is denoted by



$$a + b\sqrt{-1},$$

where $AD = a$ and $DC = b$: similarly Ac is denoted by

$$c - d\sqrt{-1}$$

where $Ad = c$ and $dc = d$: their algebraical *sum*

$$= a + c + (b - d)\sqrt{-1},$$

and their algebraical *difference*

$$= a - c + (b + d)\sqrt{-1}.$$

It is obvious, however, since $Ac = Ca$, and the angle cAd is equal to the alternate angle CaD, and the right angle Adc to the right angle CDa, that $Ad = aD$ and $cd = CD$ or $b = d$: it follows, therefore, that

$$a + c + (b - d)\sqrt{-1} = a + c = AD + Da = Aa,$$

or to the diagonal of the parallelogram included by AC and Ac: and that

$$a - c + (b + d)\sqrt{-1} = a - c + 2b\sqrt{-1},$$

which is represented by Cc: for

$$a - c = AD - Ad = Dd = cd',$$

if cd' be drawn equal and parallel to Dd: and

$$2b = 2CD = Cd',$$

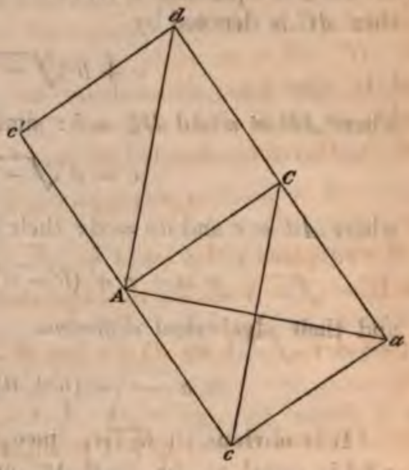
inasmuch as Dd' is equal and parallel to cd: therefore cC is expressed by $a - c + 2b\sqrt{-1}$, where $a - c$ is equal to cd', and $2b$ is equal to Cd'.

The sum and difference of two lines represented by the diagonals of the parallelogram constructed upon them.

513. We thus are enabled to conclude generally, that the *algebraical sum* of two lines making any angle with each other or with a primitive line or axis, is equal to that diagonal of the parallelogram constructed upon them which they include, and that their *algebraical difference* is equal to the other diagonal of the parallelogram.

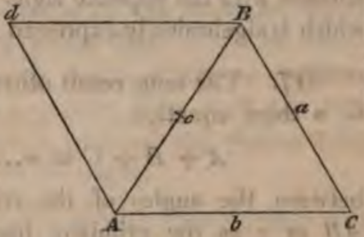
The second of these consequences otherwise deduced.

514. The second of these consequences may be deduced in a somewhat different manner: in speaking of the *sum* and *difference* of the lines AC and Ac, when these terms are used in their algebraical sense, the effect of the operation which is denoted by the word *difference*, is to change the sign of the second of these quantities from + to − or conversely: if we produce, therefore, cA to c', making $Ac' = Ac$, then Ac and Ac', however otherwise affected, will be likewise affected by the opposite signs + and −, and the *difference* of AC and Ac



will be equivalent to the *sum* of AC and Ac'; but the *sum* of AC and Ac' is Ad, which is the diagonal of the parallelogram $ACdc'$ which they *include* and which is equal and parallel to the second diagonal Cc of the original parallelogram $ACac$: it follows, therefore, that if the sum of AC and Ac be represented by Aa, their difference must be represented by Cc.

Relations between the sides and angles of triangles.



515. The relations between the sides and angles of triangles, as determined by means of their cosines or sines, is an immediate consequence of the preceding theory: for if we designate the *magnitudes* of the three sides BC, CA, AB of the triangle ABC by a, b, c, the opposite angles being severally denoted by A, B, C, then the *sum* of the sides AC and CB or AC and Ad (if Ad be equal and parallel to CB), is equal to the third side AB, which is the diagonal of the parallelogram $ACBd$ constructed upon AC and Ad: but AC makes an angle A with the primitive line AB, and is, therefore, denoted by $b\epsilon^A$: whilst CB or Ad makes an angle $BAd = B$ on the opposite side of AB, and is, therefore, denoted by $a\epsilon^{-B}$: consequently

$$b\epsilon^A + a\epsilon^{-B} = c;$$

or if we replace ϵ^A by $\cos A + \sqrt{-1}\sin A$, and ϵ^{-B} by $\cos B - \sqrt{-1}\sin B$, this equation becomes

$$b(\cos A + \sqrt{-1}\sin A) + a(\cos B - \sqrt{-1}\sin B) = c,$$

or

$$b\cos A + a\cos B - c + \sqrt{-1}\,(b\sin A - a\sin B) = 0 \ldots\ldots(1)$$

which is equivalent to the two equations (Art. 508.)

$$b\cos A + a\cos B - c = 0 \ldots\ldots\ldots\ldots\ldots\ldots(\alpha)$$
$$b\sin A - a\sin B = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

The sum of three sides of a triangle equal to zero.

516. It appears from the preceding investigation, that the *sum* of the two sides AC and CB of the triangle ABC is equal or equivalent to AB, and consequently that the *sum* of the three sides AC, CB and BA of the triangle, taken in that order, is equal to zero; for BA and AB express the same line reckoned in opposite directions, and, therefore, are expressed algebraically by the same symbol affected with the opposite signs + and − : it is this result which is algebraically expressed by the general equation (1).

The sum of the angles of a triangle equal to π.

517. The same result otherwise expressed, would lead to a third equation

$$A + B + C = \pi \dots\dots\dots\dots\dots (\gamma),$$

between the angles of the triangle; for if we consider AB or c as the primitive line or axis, and reckon the lines in the order AB, BC and CA, then BC will be expressed by $a\,\{\cos(\pi - B) + \sqrt{-1}\sin(\pi - B)\}$, inasmuch as BC makes an angle $\pi - B$ with AB produced: also CA will be expressed by

$$b\,\{\cos(2\pi - B - C) + \sqrt{-1}\sin(2\pi - B - C)\},$$

inasmuch as CA makes an angle equal to $\pi - B + \pi - C$, or $2\pi - B - C$ with the line AB: but as we have already seen that the sum of the three sides AB, BC, CA when taken in that order is equal to zero, it will follow that

$$c + a\cos(\pi - B) + b\cos(2\pi - B - C) = 0,$$

if we omit the expressions affected with the sign $\sqrt{-1}$, which form a distinct equation: this equation is identical with the equation (α) (Art. 515.)

$$c - a\cos B - b\cos A = 0:$$

and, therefore, $\cos(\pi - B) = -\cos B$, as has been otherwise shewn (Art. 478.); and also

$$\cos(2\pi - B - C) = -\cos A,$$

or

$$\cos(B + C) = -\cos A = \cos(\pi - A),$$

and, therefore, $B + C = \pi - A$[*],

$$\text{or } A + B + C = \pi \ldots\ldots\ldots\ldots (\gamma).$$

The relations of the sides and angles of triangles determined from the preceding equations.

518. This equation, combined with equations (α) and (β) will enable us, not merely to deduce those properties of triangles, which depend upon the relations of their sides and angles to each other, but likewise the conditions by which triangles are determined, and the formulæ which involve the rules for the calculation of the sides and angles of such triangles from the requisite data: it will be necessary, however, to modify the forms of these equations, or to deduce such other equations from them, as may express explicitly the angles in terms of the sides, or the ratios of the sides in terms of the angles.

The sides of triangles proportional to the sines of the opposite angles.

519. In the first place, the second equation (β), is at once reducible to the form

$$\frac{a}{b} = \frac{\sin A}{\sin B};$$

which expresses the following very important proposition: "That the ratio of any two sides of a triangle is the same as the ratio of the sines of the angles opposite to them."

The greater side of a triangle is opposite to the greater angle.

520. If we combine this proposition with that expressed by the equation

$$A + B + C = \pi \ldots\ldots\ldots\ldots\ldots\ldots (\gamma),$$

we shall be enabled to determine all the relations which can exist between the sides and opposite angles of a triangle: in this manner it may be shewn that "the greater side of every triangle is opposite to the greater angle, and conversely:" for if a be greater than b, then $\sin A$ is greater than $\sin B$: but inasmuch as the angles are to be inferred from their sines, there is no reason, as far as the proposition in the last Article is concerned, why any angles may not be taken in the place of A and B, whose sines are in the ratio of a and b: it appears, however, from the equation (γ), that the sum of the two angles A and B must be less than π; therefore if they are both of them less than $\frac{\pi}{2}$, then A must be greater than B, since $\sin A$ is greater than $\sin B$, and the sines increase with the angles, until they reach 90°: if A be greater than 90°, then B is less than 90°, or A is greater than B: but if A be less than 90°, B must be less than 90°, and, therefore, less than A: for if B be greater than 90°, its supplement $\pi - B$ must be less than 90°: and since $\sin(\pi - B) = \sin B$, and since $\sin B$ is less than $\sin A$, it will follow also that $\sin(\pi - B)$ is less than $\sin A$, and, therefore, under the circumstances which we have supposed, $\pi - B$ must be less than A, or π less than $A + B$, which is impossible: it follows, therefore, that B must in every case be less than A.

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[*] $-\cos A = \cos(\pi - A) = \cos(3\pi - A)$, &c.: but $\pi - A$ is the only angle to which $B + C$ can be equal, inasmuch as the sum of all the interior and exterior angles together of the triangle is equal to 3π.

Its converse.

521. The converse of the last proposition would assert, that the greater angle of every triangle is opposite to the greater side, and its truth is included in the demonstration just given, or is an immediate inference from it: for if A and B be both of them less than $\frac{\pi}{2}$, and A greater than B, then $\sin A$ is greater than $\sin B$, and, therefore, a is greater than b; if A be greater than $\frac{\pi}{2}$, then $\pi - A$ is also greater than B, and therefore $\sin A$ or $\sin(\pi - A)$ is greater than $\sin B$; and, therefore, also a is greater than b.

The angles at the base of an isosceles triangle are equal.

522. If $a = b$, and therefore $\sin A = \sin B$, it necessarily follows that $A = B$: for if A be not equal to B, it must equal $\pi - B$, which is impossible: it follows, therefore, that the angles at the base of an isosceles triangle are equal to each other and conversely.

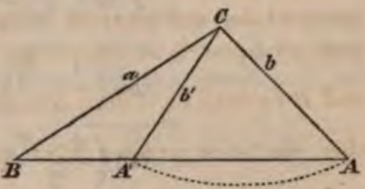
Two sides and the angle opposite to the greater of them determine a triangle.

523. If a and b are given, and also the angle A which is opposite to a, then when a is greater than b, since $\sin B = \frac{a}{b} \sin A$, it follows that $\sin B$, and, therefore,

the angle B may be calculated and determined, since B is necessarily less than $\frac{\pi}{2}$: in other words, the data abovementioned, under such circumstances, determine the triangle.

Two sides and the angle opposite to the less of them do not determine a triangle.

524. But if a and b, and the angle B be given, then the value of $\sin A$ is given, since $\sin A = \frac{a}{b} \sin B$: there are, however, two values of the angle A corresponding to the single value of $\sin A$, calculated from this equation, which are equally consistent with the data: one of these is less than $\frac{\pi}{2}$, and the other is its supplement, which are both of them greater than B, and both of them equally fulfil, therefore, the conditions which the data impose: in other words, there are two different triangles to which these data will equally correspond.



Thus, in the two triangles CBA and CBA', the greater side CB, and the angle B are common to both, and the side CA in one triangle is equal to the side CA' in the other: these two triangles, therefore, have the same data, but the angle A' which is opposite to the greater side CB in one triangle, is the supplement of the angle A, which is opposite to the same side in the other. But if the side CB, which is opposite to the angle to be determined, be not the greater of the two CB and CA, then the angle opposite to it cannot exceed $\frac{\pi}{2}$; and, consequently, there is only one triangle which can satisfy the required conditions.

Two sides and the angle which they include determine a triangle.

525. The next case which we shall have to consider, is that in which two sides and the included angle are given, and it is required to ascertain whether the remaining side

and angles can be from thence determined: in this case, the equation

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

involves the sines of two unknown angles A and B: but their sum $A+B$ is known, being equal to $\pi - C$, or to the supplement of the angle which is given: and

$$\text{since } \frac{a}{b} = \frac{\sin A}{\sin B},$$

it follows that (Art. 411).

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)}{2 \cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)}$$

$$= \frac{\tan \left(\frac{A+B}{2}\right)}{\tan \left(\frac{A-B}{2}\right)};$$

consequently,

$$\tan \left(\frac{A-B}{2}\right) = \frac{(a-b)}{(a+b)} \cdot \tan \left(\frac{A+B}{2}\right).$$

But $\frac{A+B}{2} = \frac{\pi - C}{2}$, and, therefore, $\tan \frac{A+B}{2} = \cot \frac{C}{2}$ is known from the canon of tangents: it appears, therefore, that $\tan \frac{A-B}{2}$ can be calculated, and, consequently, that the arc $\frac{A-B}{2}$ corresponding to it can be found from the same canon: it follows, therefore, that A which is $= \left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right)$, and B which is $= \left(\frac{A+B}{2}\right) - \left(\frac{A-B}{2}\right)$ can be determined: and the third side can be determined from the equation

$$\frac{c}{a} = \frac{\sin C}{\sin A}.$$

526. If two angles of a triangle are given, the third is given also, and the equations

If two angles of a triangle be given, its species is given.

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \frac{a}{c} = \frac{\sin A}{\sin C}, \quad \frac{b}{c} = \frac{\sin B}{\sin C},$$

will determine the ratios of the sides, but not their magnitudes: but if any one of the sides be given, the same equations will determine not merely the ratios but also the magnitudes of the remaining sides: in other words, they will determine the triangle.

527. If the three sides of a triangle a, b, c be given, the cosines of the angles, and therefore the angles themselves, may be determined in the following manner: if the equation

The three sides determine a triangle.

$$b \cos A + a \cos B - c = 0 \ldots\ldots (\alpha), \text{ (Art. 515)},$$

be put under the form

$$a \cos B = c - b \cos A,$$

and both its sides squared, we get

$$a^2 \cos^2 B = c^2 - 2bc \cos A + b^2 \cos^2 A;$$

or replacing $\cos^2 B$ and $\cos^2 A$ by $1 - \sin^2 B$ and $1 - \sin^2 A$ respectively, we get

$$a^2 - a^2 \sin^2 B = c^2 - 2bc \cos A + b^2 - b^2 \sin^2 A:$$

but $a^2 \sin^2 B = b^2 \sin^2 A$, by equation ($\beta$) (Art. 515);

$$\therefore\ a^2 = c^2 - 2bc \cos A + b^2,$$

$$\text{and } \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

In a similar manner, we should find

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

We are thus enabled to calculate the cosines of A, B and C by means of the three sides a, b, and c: or, in other words, the three sides determine the triangle.

Corresponding Propositions in Geometry.

528. The results which are given in the four last Articles include, or, by means of proper interpretations, may be made to include, those propositions in Geometry, in which triangles are compared to each other, with respect to their equality or similarity: thus it appears that two sides and an included angle, or three sides, or one side and two angles, determine a triangle, propositions which correspond to the 4th, 7th and 8th, and 26th Propositions in the first Book of Euclid: and that three angles, or one angle and the ratio of two sides including it, or the ratios of three sides, determine the species of a triangle, but not its magnitude; propositions which correspond to the 4th, 5th and 6th, Propositions of the Sixth Book of Euclid.

Connection of Algebra and Geometry.

529. The interpretations which are required to effect the transfer of the conclusions of Algebra to Geometry, in the case of the preceding propositions, are connected with general principles which it is important to examine and establish, in order to ascertain the extent and nature of the dependence of one science upon the other: for this purpose it will be necessary to enquire how far their first principles are in common, when the objects of investigation are the same: and also whether there are any propositions in Geometry, which are established by geometrical processes, which are incapable of algebraical investigation, or which are necessary as forming a connecting link between the two sciences.

530. Such an enquiry may be most naturally connected with the examination of the definitions of those quantities, whose properties and relations form the common objects of Geometry and Algebra, when the symbols of Algebra are employed to denote them: we shall therefore go through the most important of them in their proper order.

Examination of the definitions in Geometry.

531. "A geometrical point, line and surface, are distinguished from a physical point, line or surface, by the absence of physical length, breadth and thickness in the first, of physical breadth and thickness in the second, and of physical thickness in the third."

Definition of a geometrical point, line, or surface.

532. "A geometrical straight line is one which in all positions coincides with any other straight line which has two points in common with it, however far they may be extended."

Definition of a geometrical straight line.

It will follow as a necessary consequence of this definition, that two straight lines cannot enclose a space: for otherwise two straight lines would have two common points, which did not coincide with each other: it is also a necessary consequence of the same definition, that two straight lines cannot have a common segment: for if they have a part or segment in common, they have two points in common and, therefore, must coincide throughout their whole extent: these two properties of straight lines are necessary consequences of our definition of it, but are not necessary consequences of each other.

Its properties.

We have added as a further condition, that straight lines which have two points in common, must coincide in all positions: in other words, that they must continue to coincide with each other, when conceived to be moved round those points through any angle: for otherwise two arcs of circles which are described from the same centre, and which pass through two common points, would coincide throughout their whole extent, and would thus appear to fulfil the conditions imposed by the definition of a straight line: if we suppose one of them, however, to move round those points as an axis, they will be found to coincide in one position only, when they are in the same plane, with the centre of the circle on the same side of them.

Objections to other definitions.

533. It is no definition of a straight line to say that it lies *evenly* between its extreme points, unless the term *evenly* admits of a further definition which would assign some essential property of a straight line, which belonged to no other geometrical line: it is for this reason that Euclid has subsequently inserted among the Axioms, one of those essential properties, namely, that two straight lines cannot enclose a space: and has subsequently assumed, and from the consequences of that assumption, afterwards proved, that two straight lines cannot have a common segment.

It is a sufficient definition of a straight line to consider it as *the shortest distance between two points:* it requires, however, considerations of a nature somewhat refined and metaphysical, to infer from thence that there is only one such line, and, consequently, that all straight lines passing through two common points, must coincide with each other: and under any circumstances, inasmuch as it is necessary to refer to this consequence of the definition, and not to the definition itself, in the actual processes of Geometry, it is certainly more convenient, and in some respects more philosophical, to make this consequence the definition, and to leave the property which that definition assigns, to be determined as a consequence of it, if such a determination should be considered important for any of the applia tions of Geometry.

Definition of a plane geometrical surface.

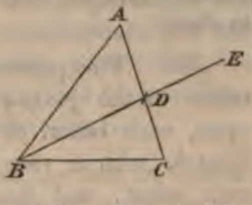
534. "A plane surface is one, in which if any two points whatsoever be taken, any point in the straight line which joins them, however far it may be produced, is likewise in that surface."

The properties of a plane surface are dependent, therefore, upon the properties of a straight line.

535. Two points determine the position of a straight line, and three points will determine the position of a plane

Three points determine a plane surface.

surface; for all plane surfaces, which pass through three common points, are coincident with each other, however far they may be extended: for let E be any point in a plane surface passing through A, B and C: then any or every point in the straight line BE is in that surface: and any and every point of the straight line AC or of AC produced, is in the same plane surface: consequently, the straight lines BE and AC, or BE and AC produced, have a common point D: but every plane surface which passes through A, B, and C, passes likewise through D, and, therefore, through E, which is in the same straight line with B and D: or, in other words, any point which is any one plane surface passing through three points, is likewise in every plane surface which passes through the same points; and, consequently, all such surfaces are coincident with each other.



Definition of a geometrical angle.

536. "A geometrical angle differs from a physical angle, in being contained by geometrical lines which have a common point, and those lines are supposed to be straight lines, unless it is otherwise expressed."

The idea conveyed by the term *angle*, like those expressed by the terms *point*, *line*, *surface*, is so simple and elementary, as not to admit of resolution into others, so that no description can express it more clearly than the term itself: we are compelled, therefore, in all those cases, to confine ourselves to divesting it of those physical qualities, which, being essentially variable and indefinite, are not the proper objects of a science of demonstration.

Definition of equality in Geometry.

537. "Geometrical magnitudes, whether angles, lines, or surfaces, are defined to be equal to each other, when they wholly coincide, or may be made by resolution or otherwise, to coincide with each other."

This test of equality or of identity, if that term be preferred, is essentially geometrical, and will be found to constitute a great distinction between the processes of reasoning followed in Geometry and Algebra: it is on this account that we shall examine its consequences somewhat in detail.

Its primary applications.

538. The primary or direct applications of this test, relate to the juxta-position to, or rather super-position upon, each other, of lines, angles, and bounded areas or figures, with a view to ascertain their coincidence with each other: and lines, angles or figures are said to be *given*, when they coincide respectively with an *assigned* line, angle, or figure.

Assumptions requisite in practical Geometry.

539. It is obvious, however, that such practical applications of lines, angles or figures to each other, cannot take place unless upon certain *hypotheses* or *assumptions:* first that a geometrical point can be determined: secondly, that a geometrical straight line can be drawn from one point to another, and produced at pleasure; and, thirdly, that geometrical lines or figures can be transferred, like physical or material lines or surfaces, from one point of space to another, so as to admit of super-position upon each other, with a view to ascertain their coincidence, or their non-coincidence.

Distinction between practical and physical Geometry.

540. The drawing of such lines, and the construction of such figures as are considered in Geometry, is hypothetical only, depending upon certain operations which are practically impossible: so far, therefore, as the investigation of the properties of such lines and figures is concerned, it is indifferent whether we assume them as actually existing, or regard them as the result of such hypothetical construction: in the latter case, it becomes necessary to trace their connection with the assumed fundamental operations, in the same manner as if such operations were practically possible.

Practical Geometry, properly so called, will be founded upon the hypothetical construction of figures which we have just been considering and will be subject to the same limitations. It is true that no physical line can approximate in its essential character to a geometrical line, and no physical figure to the corresponding geometrical figure: but the properties of such physical lines and figures will approximate more and more to the properties of the corresponding geometrical lines and figures, the greater the accuracy and *invariableness* of the fundamental practical operations may be.

The definition of equality most commonly used through the medium of its consequences.

541. Such direct applications of the test of equality will lead to conditions, which whenever ascertained or established, will supersede its immediate use: it will thus be found that all "triangles will be equal, by coincidence, which have two sides of one equal to two sides of the other, each to each, and the included angles equal:" "all angles will be equal, which are included between two sides which are respectively equal to each other, and also subtended by equal lines;" when these propositions are once demonstrated, the equality of triangles and of angles will be made to depend upon their satisfying the conditions which they involve: or, in other words, the reference will be made to the test of equality through the medium of such propositions.

It is a necessary consequence of our definition of equality, "that magnitudes which are equal to the same magnitude are equal to one another:" thus two lines, angles or figures which coincide with the same line, angle or figure, necessarily coincide with each other: again, it equally follows from the same definition, that "if equals be added to equals the wholes are equal;" that "if equals be taken from equals, the remainders are equal:" that "magnitudes which are the doubles of the same magnitude, are equal to one another;" and that "magnitudes which are the halves of the same magnitude, are equal to one another:" and it is through the medium of these propositions, that the test of equality is very frequently applied.

Applications of it, which are

542. In some cases, the application of this definition is partly direct and partly indirect, when a portion of two

partly direct and partly indirect.

figures can be made to coincide, and the equality of the parts in excess or defect may be inferred, from their satisfying the conditions of equality which are deduced from other propositions, whether axiomatic or not.

Applications of it by resolution of figures.

543. Lastly, and most generally, the equality of figures is inferred by resolving them into parts which may be made to coincide, or which satisfy severally the indirect conditions of equality: it is in this manner, that we are enabled to infer the equality of the square described upon the hypothenuse of a right-angled triangle, with the sum of the squares described upon its two sides, the most important of all propositions which regard the comparison of figures, with respect to their equality or inequality: and it is in this manner, that we are enabled to convert all kinds of figures, however numerous their sides or unsymmetrical their form, into equivalent triangles, parallelograms or squares.

No practical application of the test of equality in Algebra.

544. Though we may retain the same definition of equality, in Algebra, when its symbols are employed to represent geometrical magnitudes, yet there are no means of applying the test which it furnishes in the same manner in which it is applied in Geometry: we employ the same symbols to represent geometrical magnitudes which are equal in the geometrical sense of the word, and we infer the geometrical equality of the magnitudes which are represented by identical symbols, or by symbols which are equal to each other, or by the same combinations of them: but there are no algebraical means of applying this definition, and it is only by inference or interpretation that it is capable of being used in this science.

Example of the mode in which equality is inferred or determined in Algebra.

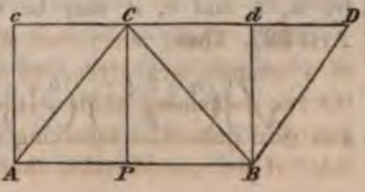
545. Thus, when two sides and the included angle of a triangle are given, we have found expressions for the two remaining angles, (through the medium of goniometrical quantities involving them) and also for the third side: if we should take, therefore, as in Geometry, two

triangles which have two sides of one equal respectively to two sides of the other, and the included angles equal, then the respective equality of the remaining angles in each triangle to each other, and also of the third sides in each, would be inferred from their being expressed under such circumstances, by identical symbols, or by identical combinations of them.

Having thus ascertained by algebraical means, the respective equality, under such circumstances, of all the sides and angles of two triangles, it would remain to determine or infer the equality of the triangles themselves: this may be determined by superposition as in Geometry: or it may be inferred as a consequence of the principle of sufficient reason, inasmuch as all the sides and angles, the only quantities which can distinguish one triangle from another, are equal to each other: or, lastly, by obtaining an algebraical expression for the area of one of the triangles in terms of two sides and the included angle, or in terms of the three sides, that is, in terms of any combination of sides and angles which are requisite to determine it; and by shewing that such expression is the same for the two triangles, which are from thence inferred to be equal to each other.

Expression for the area of a triangle in terms of two sides and the included angles.

546. We have before had occasion to consider the principles upon which the area of a rectangular parallelogram was represented by ab, when a and b represented its adjacent sides: (Art. 101.) it will require the aid of three propositions in Geometry, to connect this result with the algebraical representation of the area of a triangle.



Let ABC be a triangle: $ABCD$, a parallelogram upon the same base and between the same parallels: $ABdc$ a rectangle upon the same base and between the same parallels: the line CP is parallel to Ac, and is therefore perpendicular to AB.

In the first place, if AB and Ac be represented by c and β, the area of the rectangle $AcdB$ may be represented by $c\beta$.

In the second place, the area of the rectangle $AcdB$ is equal to that of the parallelogram $ACDB$, a geometrical proposition, which is a consequence of the geometrical definitions of equality and of parallel lines.

In the third place, the area of the parallelogram $ABDC$ is double of the area of the triangle ABC, a consequence likewise of the same definitions.

It follows, therefore, that the area of the triangle ABC may be represented by $\frac{c\beta}{2}$.

Again, Ac or β is equal to CP, a line which is at right angles to the base AB of the triangle: and we have proved by the aid of the 47th proposition of the first Book of Euclid (Art. 467.), that

$$\sin A = \frac{CP}{AC} = \frac{\beta}{b} \text{ (if } AC = b\text{)};$$

and, therefore, $\beta = b \sin A$: it follows, therefore, that the area of the triangle ABC, of which the two sides and the included angle are represented respectively by c, b and A is correctly represented by $\frac{bc \sin A}{2}$.

Expression for the area of a triangle in terms of its three sides.

547. If the three sides of a triangle be represented by a, b and c, it may be shewn from the formula in Art. 527, that

$$\sin A = 2\frac{\sqrt{\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)\right\}}}{bc}$$

and that, consequently, the area of the triangle may be represented by

$$\frac{bc\sin A}{2} \text{ or } \sqrt{\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)\right\}}.$$

It thus appears that the area of a triangle is determined, when two sides and an included angle are given, and also when three sides are given; and, consequently, in all cases where the data are sufficient to determine two sides and an included angle, or the three sides, of a triangle.

Transition from the representation of lines by symbols, to the representation of angles, areas and solids by means of them.

548. In the application of Algebra to Geometry, the symbols are not only in the first instance applied to denote geometrical lines, but we are required to express or determine by means of them, both angles, and areas, and solids: it is in the transition from the representation of one of these distinct species of quantity to that of the others, that we are obliged to resort to the aid of another science: for in Geometry, angles, lines, areas and solids are equally objects of the primary definitions, and equally objects of direct representation: and it is, therefore, by means of such essential connection between the lines which the symbols represent and the other objects of geometrical reasoning, which the definitions and propositions of Geometry supply, that we are enabled to establish likewise the connection between the symbols which denote lines and those other geometrical quantities, and thus to bring every species of quantity which is considered in one science equally under the dominion of the other.

Exemplified in the connection between the angles and sides of a right-angled triangle.

549. Thus, any angle of a triangle is determined by its three sides: if we further suppose one of the other angles of the triangle to be a right angle, (a supposition which in no respect affects the other angle to be determined, unless in making it necessary that it should be less than a right angle), the angle in question will be determined by any two sides: for in this case, if

c, b, a be the hypothenuse and two other sides of a right-angled triangle, then $c^2 = a^2 + b^2$, and any two sides will determine the third: under such circumstances also, the ratio or ratios of the sides will determine the angle equally with the sides themselves: for all the angles of triangles which are contained by sides which are parallel lines, are equal to each other, and they are independent therefore of the absolute magnitude of the sides themselves: it is for these reasons that we conclude that one or both of the angles of such a triangle will be determined by $\frac{a}{c}$ or $\frac{b}{c}$ or $\frac{a}{b}$: and in order to express the essential connection between the angle of such a triangle and any one of these ratios, we have agreed to denote the first by $\sin A$, the second by $\cos A$, the third by $\tan A$, where A is the angle opposite to the side a.

The investigations, given in a former part of this chapter, founded upon this fundamental proposition, have enabled us to calculate the values of the sine, cosine and tangent of the angles corresponding to their successive values, considered as parts of a right angle, and thus to complete the connection between the sides of such a triangle and its angles: and we have thus likewise been enabled to bring the representation of the position of lines under all circumstances, as well as of the lines themselves, under the dominion of symbols: when this symbolical connection is once established, it becomes the business of Algebra to follow out all its consequences, independently of any further aid from a subordinate science, than such as may be required for the proper interpretation of the results which are obtained.

Determination of areas by resolution.

550. The essential connection between the symbols which represent the adjacent sides of a rectangle, and the two sides and included angle or the three sides, of a triangle, and the symbolical representation of their areas, has already been examined: and inasmuch as all rectilinear

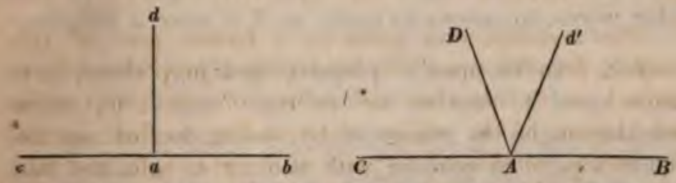
figures are resolvable into triangles, it is very easy to see in what manner their areas would generally admit of symbolical determination from the requisite data: it will be afterwards seen in what manner such areas may be represented in position as well as in magnitude, by which means also the properties of the solids bounded by given plane surfaces, may be brought under the dominion of Algebra, equally with those of the figures which are bounded by straight lines.

Definition of a right angle.

551. "One straight line meeting another, is said to be perpendicular to it, when it makes the adjacent angles equal to each other, and the angles thus formed are called right angles."

All right angles are equal to each other.

552. It will follow from this definition, combined with the definition of equality, that all right angles are equal to one another: for if *AD* and *ad* be two lines perpendicular (or at right angles) to *BC* and *bc*, and if the line *bc* be applied to the line *BC*, and the point *A* to the point *a*, then the line *ad* may be made to coincide with the line *AD*: for if not, let it occupy a different position, such as *Ad′*: then the angle *BAD* is equal to *DAC*; but *BAD* is greater than *BAd′*, and, therefore, *CAD* is greater than *BAd′*: much more then is *CAd′* which is greater than *CAD*, greater, therefore, than *BAd′*: but *BAd′* and *CAd′* are equal respectively to *bad* and *cad*, and, therefore, to each other, which is impossible: it follows, therefore, that *Ad′* must coincide with *AD*, and, therefore, the right angle *bad* or *cad* is equal to the right angle *BAD* or *CAD*: and similarly in all other cases.



Other consequences.

553. It is a necessary consequence of the preceding definition of a right angle, that the angles made by one

straight line with another on the same side of it are equal to two right angles, and also that all the angles made by one or more straight lines with another on both its sides round a common point, are together equal to four right angles.

Definition of a right angle in Algebra.

554. In Algebra, we begin by designating the sum of the two angles which any one straight line makes with another on the same side of it, by the same symbol π under all circumstances, and we consequently assume the sum of two such angles to be always the same: a right angle is the half of this sum, and is, therefore, always designated by $\frac{\pi}{2}$: in this, as well as in all other cases, in which the geometrical definition of equality is requisite in the establishment of a fundamental theorem in Geometry, it becomes necessary in the application of Algebra to that science, to make that theorem the point of departure, or, in other words, to assume its truth, as if it were a definition.

Meaning of angles equal to π or 0.

555. In Geometry, properly speaking, there is no angle equal to zero or to two right angles; but if we consider angles as *generated* by the motion of one line round a point in common with another or primitive line, then they will be said to make an angle equal to zero with each other, when they are in the same straight line, and reckoned in the same direction; and an angle of two right angles with each other when they are in the same straight line and in opposite directions: under such circumstances, the continuation of the angular motion, would generate angles greater than two right angles, such as we frequently have had occasion to consider: but it must be kept in mind, that the existence of such angles is due to this mode of generation, which is foreign to the practice, at least, if not to the spirit of Geometry*.

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* The proposition that arcs are the measures of angles, which is a remote consequence of the definitions of Geometry, corresponds to this mode of generating angles, and enables us to exhibit the measure, when the geometrical angle ceases to exist: thus an arc equal to a quadrant is the measure of a right angle, an arc equal to two quadrants is the measure of an angle of two right angles,

556. In Algebra, angles are used to determine the position of the lines which contain them with respect to each other, or to determine the sign by which the position of one line with respect to another, considered as a primitive line or axis, is represented: it is for this reason that angles equal to zero, and to two right angles enter essentially into the applications of Algebra to Geometry, as indicating that the two lines whose position with respect to each other is considered, are in the same straight line, coinciding with each other or placed in the same *direction* in one case, and in opposite directions, whether coinciding or not, in the other; that is, as indicating the two species of parallelism which are recognized in geometrical Algebra, but which are not distinguished from each other in Geometry: the first where the parallel lines are estimated or placed in the same direction, the second in which they are estimated or placed in opposite directions.

Their use and necessity.

557. "Lines in the same plane, which make equal angles towards the same parts with any line which meets them, are said to be parallel to each other."

Definition of parallel lines.

We have before had occasion to notice this definition of parallel lines, and the grounds of its adoption in a system of Geometry.

558. We have made use of this definition in Algebra, and it is the only definition of parallel lines which we can properly adopt in this science: for though it may be a necessary consequence of the common definition which is given of parallel lines, yet it is a geometrical consequence of it, and the use of the latter, therefore, in Algebra would remove us one step further from the first principles of Geometry; and as far as any applications of Algebra to Geometry are concerned, this consequence

This definition necessarily adopted in Algebra.

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angles, a circumference is the measure of an angle of four right angles, an arc greater than a circumference, such as an arc of five quadrants, is the measure of an angle of five right angles, which corresponds to the first case which we have noticed, when we reject a circumference from one, and four right angles from the other.

would in every respect occupy the place of the original algebraical definition.

Two species of parallelism in Algebra.

559. In Algebra, as far as our applications of it to Geometry have hitherto been considered, we are enabled to represent lines in magnitude and in position only, as far as their inclinations to each other are concerned and no farther: it is for this reason, that equal lines which are in the same plane, and make the same angle with a given line or axis, are considered as algebraically identical with each other, that is, are identical in their symbolical representation: this is the first and most complete species of parallelism, where the symbols which represent the lines are affected with the same algebraical sign.

There is another case where this identity of signs does not exist, which corresponds to lines which are equally parallel with those in the last case, according to the geometrical definition: it is that in which the lines are affected by signs which differ only by one of them being affected with the sign +, and the other with the sign − : thus a and $-a$, $a\epsilon^{\theta}$ and $a\epsilon^{\pi+\theta}$ or $-a\epsilon^{\theta}$, or $a(\cos\theta + \sqrt{-1}\sin\theta)$ and $a\,\{\cos(\pi+\theta) + \sqrt{-1}\sin(\pi+\theta)\}$ or

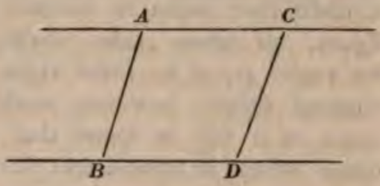
$$-a(\cos\theta + \sqrt{-1}\sin\theta),$$

would designate pairs of lines which are geometrically parallel to each other: for they apply to lines which are drawn in opposite directions, whether in the same straight line or not, a modification of position which is incapable of being exhibited, and, therefore, incapable of being considered in Geometry. Such lines, when algebraically considered, may be said to possess the second species of parallelism, of which we have spoken before. Art. 555.

The consideration of infinity not properly included in our notion of parallel lines.

560. The preceding view of the theory of parallel lines as it is considered in Algebra, regards the similarity or identity of their position with respect to each other only, and necessarily excludes all considerations of infinity: and inasmuch as straight lines are determined by any two points in them, and are, therefore, perfectly similar and symmetrical

in all their parts, it is clearly unnecessary to refer to the properties or relations of parts of them which are infinitely distant, in order to determine the properties of parallel lines as distinguished from any other: if, however, it is considered necessary to fix more precisely our notions of the relations of such lines, it would very readily follow as a consequence of our definition, that if any two points *A* and *B* are taken in two parallel lines, and also two other points *C* and *D* in them at any and equal distances respectively from *A* and *B*, then the points *A* and *B* and *C* and *D* will be always equidistant from each other: it will follow very readily from this proposition, that parallel lines can never meet, and also conversely (by means of one or two auxiliary propositions) that lines in the same plane which never meet are parallel to each other. The last mentioned property, however, can never be predicated of any two lines, unless as a consequence of some other proposition, which is or ought, therefore, to be the immediate object of our attention in the investigation of the relative position of the corresponding lines.



Different species of quadrilateral figures.

561. "A *trapezium* is any four-sided figure whatever."

"A *trapezoid* is a four-sided figure which has two of its sides parallel."

"A *parallelogram* is a four-sided figure contained by parallel lines."

"A *rectangle* is a parallelogram one of whose angles is a right angle."

"A *rhomb*[*] is a figure contained by four equal sides."

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[*] It has been proposed to abandon this limited usage of the term *rhomb*, and to replace the term *parallelogram* by it: for the property implied in the term parallelogram is not, properly speaking, peculiar to four-sided figures, inasmuch as regular hexagons, octagons, decagons, and other regular figures with an even number of sides, are contained by pairs of parallel lines.

"A *square* is a figure contained by four equal sides, and one of whose angles is also a right angle."

Limitation in the assumed conditions of such figures.

562. In the six different species of quadrilateral figures which are described (or defined) above, there is one condition introduced into the second, two into the third, three into the fourth, four into the fifth, and five into the sixth: why may we not proceed farther, and attach a distinctive name to another species of quadrilateral figure, all whose sides shall be equal, and two of its angles equal to *three* right angles? Such a quadrilateral figure, however, could not be formed, inasmuch as it will be found that we have assumed a condition which is inconsistent with the general properties of such figures.

The definitions of figures more or less dependent upon properties proved concerning them.

563. An enquiry is thus suggested of considerable importance, whether we should commence with defining or assuming the properties of figures to which we give distinctive names, and subsequently ascertain the possibility of their existence or not, or first determine the general properties of such figures, and infer from thence the extent to which we are justified in making assumptions respecting the relations or magnitudes of their sides and angles. It is the latter course of proceeding which is the most natural and philosophical, inasmuch as it will enable us to determine all the different species of figures which belong to a particular class, and the extent to which the defined properties of the figures may be said to determine them: under such circumstances also, the definitions of such figures will become the mere expressions of propositions respecting them, and not the foundation of the propositions themselves: for the properties of such figures are dependent upon their sides and angles, and are the proper and necessary consequences, therefore, of the definitions which respect them. We shall now proceed, therefore, to the algebraical investigation of the general properties of rectilinear figures, whether quadrilateral or not, with particular reference to the preceding enquiries.

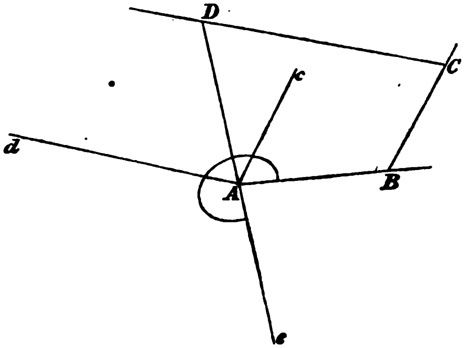
564. If we suppose four lines which are *arithmetically* represented by a, b, c, d, the three last of which make

angles θ, θ', θ'' respectively with a, considered as the primitive line or axis, then they will be *algebraically* represented by a, $b\,(\cos\theta + \sqrt{-1}\sin\theta)$,

$$c\,(\cos\theta' + \sqrt{-1}\sin\theta'),\ d\,(\cos\theta'' + \sqrt{-1}\sin\theta'')$$

Investigation of the conditions to be fulfilled in order that four lines may form a figure.

respectively: and the same expressions will equally represent such lines, whether they are drawn from the same point A, as in the figure, or are placed in any positions which are algebraically parallel to them: they will, therefore, represent them when they are so related to each other as to admit of forming a quadrilateral figure $ABCD$, where BC is equal and parallel to Ac, CD to Ad, and DA is equal to Ae, and in the same straight line with it: it remains to examine the conditions under which this may take place.



If we join AC, then we have

$$AC = AB + BC = a + b\cos\theta + \sqrt{-1}\sin\theta):$$

in a similar manner we shall find

$$AD = AC + CD = AB + BC + CD$$

$$= a + b\,(\cos\theta + \sqrt{-1}\sin\theta) + c\,(\cos\theta' + \sqrt{-1}\sin\theta'):$$

but

$$AD = -DA = -d\,(\cos\theta'' + \sqrt{-1}\sin\theta'');$$

and, therefore,

$$a+b\,(\cos\theta+\sqrt{-1}\sin\theta)+c\,(\cos\theta'+\sqrt{-1}\sin\theta')$$
$$=-\,d\,(\cos\theta''+\sqrt{-1}\sin\theta''),$$

or

$$a+b\,(\cos\theta+\sqrt{-1}\sin\theta)+c\,(\cos\theta'+\sqrt{-1}\sin\theta')$$
$$+\,d\,(\cos\theta''+\sqrt{-1}\sin\theta'')=0,$$

an equation which is resolvible into the two equations,

$$a+b\cos\theta+c\cos\theta'+d\cos\theta''=0\ldots\ldots\ldots\ldots(\alpha),$$
$$b\sin\theta+c\sin\theta'+d\sin\theta''=0\ldots\ldots\ldots\ldots(\beta),$$

which are the equations of condition in order that the four lines AB, Ac, Ad and Ae may admit of forming a quadrilateral figure.

Equations involving the exterior angles.

565. If we denote the four angles of the quadrilateral figure by A, B, C, D, and the corresponding exterior angles formed by producing the sides DA, AB, BC, CD in their respective directions, by A', B', C', D', then it is obvious that the angle $BAc=B'$, the angle $cAd=C'$, the angle $dAe=D'$ and $eAB=A'$; we may replace, therefore, the angles θ, θ' and θ'' respectively by B', $B'+C'$, and $B'+C'+D'$, when the equations (α) and (β) will become

$$a+b\cos B'+c\cos(B'+C')+d\cos(B'+C'+D')=0\ldots(\alpha'),$$
$$b\sin B'+c\sin(B'+C')+d\sin(B'+C'+D')=0\ldots(\beta').$$

Sum of the interior angles.

566. Inasmuch as

$$A+A'=\pi,\ B+B'=\pi,\ C+C'=\pi,\ D+D'=\pi,$$

and also $A'+B'+C'+D'=2\pi$, it will follow that

$$A+B+C+D=2\pi\ldots\ldots\ldots\ldots(\gamma),$$

a third equation of condition, which regards the four angles of the quadrilateral figure only.

567. Again, if we replace B' by $\pi - B$, $B' + C'$ by $2\pi - (B + C)$, and $B' + C' + D'$ by $3\pi - (B + C + D)$, the equations (α') and β') will become

Equations involving the interior angles.

$$a - b \cos B + c \cos (B + C) - d \cos (B + C + D) = 0 \quad (\alpha''),$$

$$b \sin B - c \sin (B + C) + d \sin (B + C + D) = 0 \quad (\beta'').$$

568. If we now suppose the quadrilateral figure to be a trapezoid, or two of its sides to be parallel, then $B + C = \pi$, and, therefore, $A + D = \pi$, when the preceding equations become

In case of a trapezoid.

$$a - b \cos B - c + d \cos D = 0 \ldots\ldots\ldots\ldots (\alpha_1),$$

$$b \sin B \qquad - d \sin D = 0 \ldots\ldots\ldots\ldots (\beta_1).$$

569. If we suppose the quadrilateral figure to become a parallelogram, then the sum of any two of its consecutive angles will be equal to π: consequently $A + B = \pi$, $B + C = \pi$, $C + D = \pi$, and $D + A = \pi$, or all the other angles may be considered as dependent on A: therefore, the equations (α'') and (β'') become

In case of a parallelogram.

$$a + b \cos A - c - d \cos A = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha_2),$$

$$b \sin A - d \sin A = 0 \quad \text{or} \quad b - d = 0 \ldots (\beta_2).$$

It follows from the second (β_2) of these equations, that $b = d$; and combining this conclusion with the first equation (α_2), it will likewise follow that $a = c$: or in other words, "that the opposite sides as well as the opposite angles of parallelograms are equal to each other."

570. The values of the adjacent sides a and b of the parallelogram as well as of the angle A are perfectly indeterminate, and are, therefore, assumable at pleasure: if we assume A to be a right angle, then all the other angles are right angles, and the parallelogram becomes a *rectangle:* if we suppose a and b to be equal to each other, leaving A indeterminate, or that a, b, c, d are equal to each other, and form, therefore, a parallelogram, then the parallelogram becomes a *rhomb:* if we suppose a, b, c, and d to be equal to each other, and A to be a right angle,

In case of a rectangle, rhomb, or square.

then the figure is a *square*, and all the other angles are likewise right angles.

Equations of figure and equation of angles.

571. In every quadrilateral figure there are four sides and four angles, which are involved in its complete determination: there are two equations (α) and (β) involving four sides and three angles, which may be called the *equations of figure*, inasmuch as they must be satisfied, in order that the four sides which are algebraically involved in them, may admit of being formed into a figure completely bounded: there is a third equation (γ) which regards the four angles only, and which may, therefore, be called the *equation of angles*, which shews that any three of them determine the fourth; or in other words, three only of the four angles can be considered as arbitrary: it follows, therefore, that there are only five of those elements of quadrilateral figures which can be considered as arbitrary and assumable at pleasure, which are, two sides and three angles, or three sides and two angles, or four sides and one angle: if the values of five such elements be given, whether explicitly or by means of equations which involve them, all the elements of the figure, and, therefore, the figure itself may be considered as determined or determinable.

General equations of figure and angles.

572. It will be very easy to investigate the *equations of figure* as well as the *equation of angles*, for figures of n sides: thus if a, a_1, $a_2,\ldots a_{n-1}$, be the arithmetical values of a series of lines, and θ_1, $\theta_2\ldots\theta_{n-1}$, the angles which the $(n-1)$ last of them form with the primitive line or axis a, then the equations of figure will be found, in the same manner as in the case of quadrilateral figures, to be

$$a + a_1 \cos\theta_1 + a_2 \cos\theta_2 + \ldots\ldots + a_{n-1} \cos\theta_{n-1} = 0\ldots(a),$$

$$a_1 \sin\theta_1 + a_2 \sin\theta_2 + \ldots\ldots + a_{n-1} \sin\theta_{n-1} = 0\ldots(b).$$

If we further suppose the successive interior angles to be A, A_1, $A_2,\ldots A_{n-1}$, and the successive and corresponding exterior angles to be

$$A', \; A_1', \; A_2',\ldots A'_{n-1},$$

then since

$$A' + A_1' + A_2' + \ldots A'_{n-1} = 2\pi,$$

we shall have

$$A + A_1 + A_2 + \ldots A_{n-1} = n\pi - 2\pi = (n-2)\,\pi, \; (c),$$

which is the *equation* of *angles*.

If we replace $\theta_1, \theta_2, \ldots \theta_{n-1}$ severally by

$$A_1', \; (A_1' + A_2'), \ldots\ldots (A_1' + A_2' + \ldots A'_{n-1}),$$

then the equations (a) and (b) become

$$a + a_1 \cos A_1' + a_2 \cos (A_1' + A_2') + \ldots\ldots$$
$$+ a_{n-1} \cos (A_1' + A_2' + \ldots A'_{n-1}) = 0 \ldots\ldots (a'),$$
$$a_1 \sin A_1' + a_2 \sin (A_1' + A_2') + \ldots\ldots$$
$$+ a_{n-1} \sin (A_1' + A_2' + \ldots A'_{n-1}) = 0 \ldots\ldots (b').$$

If we further replace A_1', $(A_1' + A_1'), \ldots\ldots$

$$(A_1' + A_2' + \ldots A'_{n-1}) \text{ by } \pi - A_1, \; 2\pi - (A_1 + A_2), \ldots$$
$$(n-1)\,\pi - (A_1 + A_2 + \ldots A_{n-1}),$$

then we shall get

$$a - a_1 \cos A_1 + a_2 \cos (A_1 + A_2) - \ldots\ldots$$
$$+ (-1)^{n-1} a_{n-1} \cos (A_1 + A_2 + \ldots A_{n-1}) = 0 \ldots\ldots (a''),$$
$$a_1 \sin A_1 - a_2 \sin (A_1 + A_2) + \ldots\ldots$$
$$+ (-1)^{n} a_{n-1} \sin (A_1 + A_2 + \ldots A_{n-1}) = 0 \ldots\ldots (b'').$$

573. In a figure of n sides, there are $2n$ elements, and only three equations involving them, one of which is the equation of angles: there are, therefore, $2n-3$ elements which are perfectly arbitrary and assumable at pleasure, which are n

sides, and $(n-3)$ angles, or $(n-1)$ sides and $(n-2)$ angles, or $(n-2)$ sides and $(n-1)$ angles: in order to determine the figure, there are, therefore, three classes of data, the first admitting of $\frac{n(n-1)(n-3)}{1.2.3}$ variations, the second of $n.\frac{n.(n-1)}{1.2}$ variations, and the third likewise of $n.\frac{n.(n-1)}{1.2}$ variations: when the independent data are fewer than $n-3$, the figure is not determined: if they exceed $n-3$ in number, they may be inconsistent with the results which $(n-3)$ data would give, or if not, they are superfluous, being necessary consequences of them.

Definition of similar figures.

574. "Similar figures are defined to be such as have their angles severally equal to each other, and the sides, about the equal angles in each, proportionals."

Number of conditions implied by this definition.

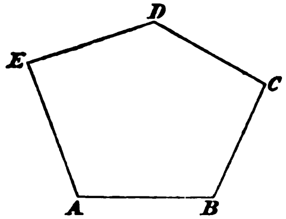
575. If this geometrical definition of similar figures be interpreted algebraically, it would impose as a condition of such figures, that all their angles should be given, and also the ratios of their sides: in other words, if a, a_1, $a_2, \dots a_{n-1}$ represented the sides, the ratios

$$\frac{a}{a_1}, \quad \frac{a_1}{a_2}, \quad \frac{a_2}{a_3}, \dots \frac{a_{n-2}}{a_{n-1}}$$

would be given, and also the angles A, A_1, A_2, A_{n-1}; but if the value of any one of these sides be given, such as a, all the others would be given, inasmuch as there would be n independent equations involving them successively; it follows, therefore, that the preceding definition of similar figures implies the fulfilment of $2n-1$ conditions, which exceeds by 2 the number of conditions requisite for the complete determination of the figure: it will be found, however, that only $2n-4$ conditions are required to determine this similarity, and that the other three conditions are, therefore, superfluous, or are not required to be included in the test by which this similarity is ascertained.

576. If we supposed any number of lines AB, BC,

Investigation of the number of conditions requisite to determine the similarity of figures.



CD, DE, to be algebraically represented by

$$a,\ a_1 (\cos \theta_1 + \sqrt{-1} \sin \theta_1),$$
$$a_2 (\cos \theta_2 + \sqrt{-1} \sin \theta_2),$$
$$\ldots a_{n-2} (\cos \theta_{n-2} + \sqrt{-1} \sin \theta_{n-2})$$

then if a be arithmetically equal to AE, their sum or

$$a + a_1 (\cos \theta_1 + \sqrt{-1} \sin \theta_1) + a_2 (\cos \theta_2 + \sqrt{-1} \sin \theta_2) + \ldots + a_{n-2} (\cos \theta_{n-2} + \sqrt{-1} \sin \theta_{n-2})$$
$$= a (\cos A + \sqrt{-1} \sin A) \ldots\ldots\ldots (\delta),$$

inasmuch as it is equal to AE (estimated in the direction from A to E), making the angle A with the primitive line, and completing the figure, and denominated, therefore, its *complement*, which is completely determined both in position and magnitude by the resolution of the single equation (δ), into two: if we likewise suppose $(n-1)$ other lines, bearing respectively to the former the same ratio e, and making the same angles θ_1, θ_2, ... θ_{n-2} respectively with the primitive line; then the *complement* of the corresponding figure would be equal to

$$ea + ea_1 (\cos \theta_1 + \sqrt{-1} \sin \theta_1) + ea_2 (\cos \theta_2 + \sqrt{-1} \sin \theta_2) + \ldots ea_{n-2} (\cos \theta_{n-2} + \sqrt{-1} \sin \theta_{n-2})$$
$$= ea (\cos A + \sqrt{-1} \sin A) \ldots\ldots\ldots\ldots (\delta');$$

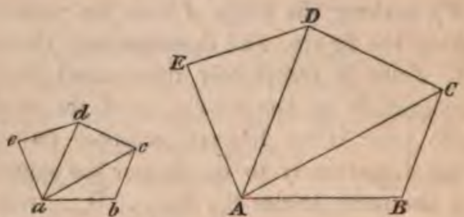
it follows, therefore, that the complements of these figures would bear to each other the same ratios with the other sides respectively, and that they would make likewise the same angle with the primitive lines: the sides of such figures satisfy, therefore, the required conditions of similarity; and inasmuch as $\theta_1, \theta_2, \ldots \theta_{n-2}$ determine $n-2$ exterior, and, therefore, $n-2$ interior angles, and since the angle A is determined from the determination of the *complement*, and the remaining angle from the equation of angles of the figure, it will follow that all the angles are determined and are, therefore, respectively the same in both figures; and, consequently, all the conditions of similarity are fully satisfied, which the definition requires.

There are $2n-4$ conditions of similarity in a figure of n sides.

577. In the preceding investigation, we have $n-2$ equations between $n-1$ successive sides, and $n-2$ angles are assigned, making $2n-4$ conditions: and it appears that all figures are similar, which have these conditions in common: without enquiring into other conditions of similarity, which are suggested by this conclusion, we shall content ourselves with pointing out a geometrical consequence of them, which furnishes the most convenient geometrical test of the similarity of figures.

Geometrical conditions of similarity.

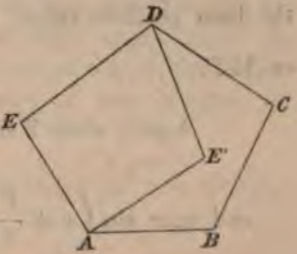
578. If we suppose the ratios AB and BC, of BC to CD, and of CD to DE to be the same as those of ab to bc, of bc to cd, and of cd to de respectively, and the angles B, C, D, to be equal respectively to the angles b, c, d; then the ratio of the complement of AB and BC or of AC to BC, and, therefore, of AC to CD is the same as that of the complement of ab and bc or ac to ab, and, therefore,



of *ac* to *cd*: in the same manner, the ratio of the complement of *AC* and *CD* or *AD* to *CD*, and, therefore, of *AD* to *DE* is the same as that of the ratio of the complement of *ac* and *cd* or *ad* to *cd*, and, therefore, of *ad* to *de*: lastly, the ratio of the complement of *AD* and *DE* or *AE* to *DE* is the same as that of the complement of *ad* and *de* or *ae* to *de*: it follows, therefore, that the three triangles *ABC*, *ACD*, *ADE*, and *abc*, *acd*, *ade* similarly formed in each, are similar to each other, and the same conclusion would equally follow, whatever was the number of the sides of the figure: it follows, therefore, that those rectilinear figures are similar to each other, in which all the triangles similarly formed in each are similar to each other.

Equations of figures in the case of re-entrant figures.

579. In the preceding investigations of the *equations of figure*, and in their applications to particular cases, we have tacitly supposed that the exterior angles of the figures were always positive, or that the figures possessed no re-entrant angles: in other words, we have not only supposed that the sum of each pair of exterior and interior angles was equal to π, but likewise that the interior angle was always less than π: it may be proper to examine a case where this condition is not fulfilled.



Let us suppose a pentagon *ABCDE* whose sides are severally equal to each other, and two of its angles *B* and *C* equal to each other and to $\frac{3\pi}{5}$ or 108°: and let it be required to determine the remaining angles of the pentagon: the equations of figure become

$$1+\cos 72^\circ+\cos 36^\circ+\cos(36^\circ+D)-\cos(36^\circ+E+D)=0,$$

$$\sin 72^\circ+\sin 36^\circ-\sin(36^\circ+D)+\sin(36^\circ+E+D)=0;$$

if we make $36+D=x$, and replace the cosines and sines of 72° and 36° by their numerical values, we shall find

$$\cos x - \cos (x + E) = \quad .5 \quad \ldots\ldots \ (\alpha),$$

$$\sin x - \sin (x + E) = 1.5383418 \ \ldots\ldots \ (\beta).$$

Dividing (α) by (β), and replacing them severally by equivalent forms (Art. 487), we shall get

$$\frac{\cos x - \cos (x + E)}{\sin x - \sin (x + E)} = \frac{2 \sin \left(x + \frac{E}{2}\right) \sin \frac{E}{2}}{2 \cos \left(x + \frac{E}{2}\right) \sin \frac{E}{2}}$$

$$= \tan \left(x + \frac{E}{2}\right) = .325012,$$

which is equally the tangent of 18°, $\pi + 18°$, $2\pi + 18°$, &c.

Consequently,

$$x + \frac{E}{2} = \quad 18°, \text{ or } \pi + 18°, \text{ or } 2\pi + 18°;$$

and, therefore,

$$D + \frac{E}{2} = -18°, \text{ or } \pi - 18°, \text{ or } 2\pi - 18°:$$

the least positive value, therefore, of $D + \frac{E}{2}$ is $\pi - 18°$, or 162°.

$$\text{Again, since } 2 \sin \left(x + \frac{E}{2}\right) \sin \frac{E}{2} = .5,$$

$$\text{and since } \sin \left(x + \frac{E}{2}\right) = .309017, \text{ or} -.309017,$$

it will follow that

$$\sin \frac{E}{2} = .809017, \text{ or } - 809017,$$

which are the sines of 54° or $\pi - 54°$ and of $\pi + 54°$ or $2\pi - 54°$:

consequently the values of E are $108°$, or $2\pi - 108°$, excluding all such as are greater than 2π.

If we suppose $E = 108°$, then $D + \frac{E}{2} = 162°$ and therefore $D = 108°$: the remaining angle A (determined by the equation of angles,) is also $108°$, and the figure formed is a regular pentagon without re-entrant angles.

If we suppose $E = 2\pi - 108 = 252°$, then $D + \frac{E}{2}$ or $D + 126° = 162°$, and, therefore, $D = 36°$: the remaining angle at A (determined by the equation of angles,) is also $36°$: the pentagon, therefore, is not regular, the angle E being equal to $252°$, and therefore re-entrant.

Expressions for the co-ordinates of the ultimate point of any number of lines which are placed consecutively.

580. There is another view which may be taken of these formulæ for the representation of lines in the same plane and their sums, which is frequently extremely important: we have before seen, (Art. 507.) that the expression

$$a\,(\cos\theta + \sqrt{-1}\,\sin\theta)$$

may be considered as involving the expressions for the co-ordinates of the point at the extremity of the equivalent line, which are $a\cos\theta$ upon the primitive line or axis, and $a\sin\theta$ upon the axis at right angles to it: it is an obvious and a necessary consequence of this interpretation, that the algebraical sum of any number of expressions such as

$$a,\ a_1(\cos\theta_1 + \sqrt{-1}\sin\theta_1),\ a_2(\cos\theta_2 + \sqrt{-1}\,.\,\sin\theta_2),$$
$$+ \ldots\ldots a_{n-1}(\cos\theta_{n-1} + \sqrt{-1}\sin\theta_{n-1}),$$

would involve the co-ordinates of the point which was at the extremity of the last of the series of equivalent lines, when those lines were placed consecutively: these co-ordinates are

$$a + a_1\cos\theta_1 + a_2\cos\theta_2 + \ldots\ldots a_{n-1}\cos\theta_{n-1},$$
$$\text{and } a_1\sin\theta_1 + a_2\sin\theta_2 + \ldots\ldots a_{n-1}\sin\theta_{n-1},$$

which are respectively equal to zero, when the ultimate point coincides with the origin of the co-ordinates.

The co-ordinates of a point moved successively through given spaces in given directions.

581. If we should conceive a point moved *successively* through spaces represented in magnitude by

$$a,\ a_1,\ a_2,\ \ldots,\ a_{n-1},$$

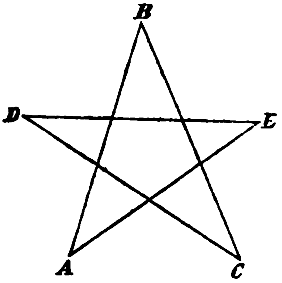
and in directions making angles θ_1, θ_2, θ_{n-1}, with the direction of the primitive line, then the expressions in the last Article would determine the position of the point in the plane at the end of the motion, and they would clearly be equal respectively to zero, when the moving point returned to the point which was the origin of the motion.

Nature of the figure described by a point which returns to the origin of the motion when the sum of the angles of transfer is a multiple of 2π.

582. If the sum of the angles of transfer of the lines which determine the motion, be equal to 2π, and the sum of the lines themselves be equal to zero, the lines will form, as we have already seen, a completely bounded figure: if however we should suppose the sum of the angles of transfer to be 4π, and the sum of the lines themselves to be equal to zero, then the point will still return to the origin of its motion, after describing an *intersecting* or *stellated* figure: the same would be the case likewise if the point in motion should return to the origin of its motion, when the lines of its transport were transferred through 6π, 8π or any other multiple of 2π, which did not exceed the number of lines of transport.

In all these cases, however, the sum of the interior and corresponding exterior angle is equal to π, and the sum of all those angles together are, therefore, $n\pi$, where n is the number of lines of transport: the sum of the interior angles, therefore, in the first case of these stellated figures is $(n-4)\pi$, in the second, $(n-6)\pi$, in the third, $(n-8)\pi$, and in the r^{th} $\{n-2(r+1)\}\pi$; thus forming *equations of angles* which are distinctive of the several classes of figures to which they correspond.

Description of a stellated regular pentagon.



583. Thus, if we suppose *AB* to be the primitive line, the lines of transport to be all equal to each other and to *AB*, and the angles of transfer to be equal to each other and to $\frac{4\pi}{5}$ (the supplement of *ABC*), then we shall find that the point in motion will return again to its origin *A*, after five transports: for the co-ordinates of the point, under such circumstances, are,

$$a\left\{1+\cos\frac{4\pi}{5}+\cos\frac{8\pi}{5}+\cos\frac{12\pi}{5}+\cos\frac{16\pi}{5}\right\}\cdot$$

$$a\left\{1+\frac{\cos\frac{5}{2}\left(\frac{4\pi}{5}\right)\sin\frac{4}{2}\left(\frac{4\pi}{5}\right)}{\sin\frac{2\pi}{5}}\right\}=a(1-1)=0\,^{*};$$

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* The series

$$\cos\theta+\cos 2\theta+\cos 3\theta+\ldots\ldots\ldots\cos n\theta,$$

is equivalent to

$$\frac{\epsilon^{\theta}+\epsilon^{-\theta}}{2}+\frac{\epsilon^{2\theta}+\epsilon^{-2\theta}}{2}+\frac{\epsilon^{3\theta}+\epsilon^{-3\theta}}{2}+\ldots\frac{\epsilon^{n\theta}+\epsilon^{-n\theta}}{2}$$

$$=\tfrac{1}{2}\left\{\epsilon^{\theta}+\epsilon^{2\theta}+\ldots\epsilon^{n\theta}\right\}+\tfrac{1}{2}\left\{\epsilon^{-\theta}+\epsilon^{-2\theta}+\ldots\epsilon^{-n\theta}\right\}$$

$$=\tfrac{1}{2}\cdot\frac{\epsilon^{(n+1)\theta}-\epsilon^{\theta}}{\epsilon^{\theta}-1}+\tfrac{1}{2}\cdot\frac{\epsilon^{-(n+1)\theta}-\epsilon^{-\theta}}{\epsilon^{-\theta}-1}$$

$$=\frac{1}{2\left(\epsilon^{\frac{\theta}{2}}-\epsilon^{-\frac{\theta}{2}}\right)}\left\{\epsilon^{\left(\frac{2n+1}{2}\right)\theta}-\epsilon^{-\left(\frac{2n+1}{2}\right)\theta}-\left(\epsilon^{\frac{\theta}{2}}-\epsilon^{-\frac{\theta}{2}}\right)\right\}$$

$$=\frac{1}{2\sin\frac{\theta}{2}}\left(\sin\frac{2n+1}{2}\theta-\sin\frac{\theta}{2}\right)$$

$$=\frac{\cos\frac{n+1}{2}\theta\,.\,\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$

and

$$a\left(\sin\frac{4\pi}{5}+\sin\frac{8\pi}{5}+\sin\frac{12\pi}{5}+\sin\frac{16\pi}{5}\right)$$

$$=a\left\{\frac{\sin\frac{5}{2}\left(\frac{4\pi}{5}\right)\sin\frac{4}{2}\left(\frac{4\pi}{5}\right)}{\sin\frac{2\pi}{5}}\right\}=0:$$

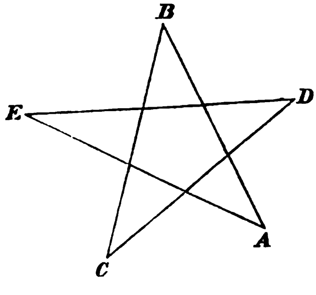
and the sum of the angles of transfer (when the motion is a second time transferred to AB in the primitive direction) is $5.\frac{4\pi}{5}=4\pi$: but as the sum of all the interior and exterior angles at A, B, C, D and E is 5π, and the sum of the exterior angles is 4π, it follows that the sum of the interior angles of the stellated figure or

$$A+B+C+D+E=5\pi-4\pi=\pi,$$

which is the equation of angles.

A similar stellated regular pentagon differently described.

584. If we make the same suppositions, merely assuming the common angle of transfer to be $\frac{6\pi}{5}$ instead of $\frac{4\pi}{5}$, the same figure will be formed, though in a different position with respect to the primitive line, the line BC being in this case transferred from the right to the left of the line AB: the equations of figure or the co-ordinates of the ultimate position of the point after five transports, are



The sum of the series

$$\sin\theta+\sin 2\theta+\sin 3\theta+\ldots\sin n\theta,$$

may be shewn in a similar manner to be equal to

$$\frac{\sin\frac{(n+1)}{2}\theta\,.\,\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$

$$a\left(1+\cos\frac{6\pi}{5}+\cos\frac{12\pi}{5}+\cos\frac{18\pi}{5}+\cos\frac{24\pi}{5}\right)$$

$$=a\left\{1+\frac{\cos 3\pi\,.\,\sin\dfrac{12\pi}{5}}{\sin\dfrac{3\pi}{5}}\right\}=a(1-1)=0,$$

and $a\left(\sin\frac{6\pi}{5}+\sin\frac{12\pi}{5}+\sin\frac{18\pi}{5}+\sin\frac{24\pi}{5}\right)$

$$=a\,.\left\{\frac{\sin 3\pi\,.\,\sin\dfrac{12\pi}{5}}{\sin\dfrac{3\pi}{5}}\right\}=0:$$

and inasmuch as $5\,.\,\frac{6\pi}{5}=6\pi$, the equation of angles, or

$$A+B+C+D+E=5\pi-6\pi=-\pi:$$

in this case, therefore, each of the interior angles is negative and equal to $-\frac{\pi}{5}$.

The Definition of a regular pentagon includes these stellated figures.

585. If we should define a regular pentagon to be a figure of five equal sides and five equal angles, it is obvious that these stellated figures would equally answer the conditions of such a definition with the regular and completely bounded pentagon of Geometry: the essential distinction between them exists in the equation of angles, which in Geometry is assumed to be always the same for figures which have the same number of sides; and it is for this reason that our attention in that science is exclusively confined to one figure in each class: in the absence however of any such restriction upon our hypotheses respecting the geometrical existence of such figures, it is obvious that the distinction between the two last stellated figures, which we have considered, is entirely algebraical.

There is no stellated hexagon.

586. The angle of transfer in a regular hexagon is $\frac{\pi}{3}$ or 60^0: if we should suppose the sum of the exterior angles doubled, then the angle of transfer would be doubled or become 120^0: the corresponding hexagon would therefore *degenerate* into an equilateral or *regular* triangle, and the describing point would pass twice over each of the sides: if we supposed the angle of transfer tripled, the figure corresponding would become a straight line; if quadrupled, it would become the regular hexagon of Geometry, where each of the interior angles was negative and equal to $-\frac{\pi}{3}$: and if quintupled, it would again become an equilateral triangle.

Different species of heptagons and octagons.

587. The series of regular heptagons would form a succession of figures, five out of seven being stellated, the heptagon of Geometry, forming the 1^{st}, 8^{th}, 15^{th}, &c. terms of the series, and a straight line the 7^{th}, 14^{th}, &c. The second of a similar series of octagons is a square, each of whose sides is twice described; the third is a stellated figure; the fourth is a line, eight times described; the fifth is a stellated figure; the sixth a square; the seventh a stellated figure, and the eighth a line: the ninth term is the primitive octagon and so on, the same figures recurring in the same order. It is not necessary for us to extend these remarks to the series of regular figures of a greater number of sides.

Algebraical representation of forces and their effects.

588. If we suppose forces in Mechanics (invested by definitions or assumptions with mathematical properties) to be represented in quantity and direction by straight lines, then they will be equally capable of algebraical expression with the lines themselves: it will remain to consider whether two or a greater number of forces can be replaced by a single equivalent force, and also whether such equivalent forces can be represented by a line which is the algebraical sum of the lines by which the component forces are represented.

It will form a necessary preparation to this investigation, to examine the mathematical first principles of the science of forces and their effects, and the nature of their connection with the corresponding physical principles, which are deducible from the observed operations and laws of the material world.

The immediate first principles of the application of mathematics to natural philosophy must be mathematical and not physical.

589. In the application of mathematics to natural philosophy, the first principles of our reasoning, whether assumed or not, must occupy the place of arbitrary assumptions or definitions with respect to the conclusions which are founded upon them: in other words, the validity of such conclusions must be referred to those principles alone, and not to their accordance with the laws or phenomena of nature: but in order that those sciences which are strictly mathematical, as far as the connection of their component propositions is concerned, may not be merely speculative, but may be considered as expressing, by means of proper interpretations, results which are transferable to the corresponding physical sciences, it is requisite that their fundamental principles should express or involve the most simple conclusions, independent of each other, respecting those sciences, which are deducible by observation, or experiment, or by reasonings upon them: and in order that such principles may possess a strictly mathematical character, they must be divested of all those causes of variation which are not consequences of the principles themselves, and which are therefore not capable of expression or estimation by means of them.

The points of the application of forces are mathematical.

590. "The points which are the objects of the actions of forces, are supposed to possess the properties of mathematical points."

If extension is to be considered in the point to which forces are applied, there will necessarily be different geometrical points in it, and the effect of the force or forces upon the physical point will vary with the geometrical point of their application: there would arise therefore from this source, variations in the effect produced which would be incapable of estimation, the cause of which must be excluded therefore from the first principles of the science.

A physical or material point admits of no physical properties, which are independent of extension, except *mobility*, which it possesses in common with a geometrical point: if *weight* be considered (hypothetically) to belong to it under such circumstances, it may be taken as an expression which designates a force (of which *weight* is the measure) acting in the direction of gravity.

Principle of the transmission of forces.

591. "Forces may be transmitted or may be supposed to be transmitted, without any alteration of their effect, to any point or points in the direction of their action."

Effect of two equal and opposite forces when applied at the same point.

592. Forces are represented in magnitude and direction by lines, and are therefore algebraically equivalent (that is, as far as their symbolical representation is concerned), when the lines which represent them are equal to each other and algebraically parallel: the condition imposed by the principle just stated, would shew that they would also be equivalent in *effect*, if the forces were algebraically equal and applied to the same or different points in a line which is coincident with the direction of their action.

If two forces which are equal and in opposite directions act upon a point, their algebraical sum is equal to zero, as well as the algebraical sum of their effects: for the magnitudes of the forces are equal, and their affections only differ in one of them being preceded with the sign +, and the other with the sign −: and the effects are identical with the forces, since they are applied to the same point: in such a case, one force may be said to be *contravalent* to the other, when their effects are equal and in *opposite* directions and when the natural state therefore, of the point, whether of rest or motion, is not affected by the action of the forces.

When applied at different points in the same straight line.

593. The same would be the case, if the forces were equal and in opposite directions, but applied to different

points in the line which is common to the direction of their action: for in this case, the sum of the forces is zero, since their algebraical representation would be precisely the same as in the case last considered: and inasmuch as the effect of the force upon one point may be transmitted, in conformity with the general principle abovementioned, without alteration to the other, the joint effects will be precisely the same as in the last case, and one will be therefore *contravalent* to the other.

The natural effect of force is to produce motion.

594. Physically speaking, the effect of a force upon a point is to produce motion in the physical point of its action, and the same may be *assumed* to be the natural effect of a mathematical force upon a mathematical point: in one case the intensity of the force, as well as of the effect of different forces, would be measured by the velocity communicated, when the mass of the body moved was not considered, or by the product of the velocity and mass of the body moved, when both of them were considered, or by other means, which it is not our present business to investigate: in the other, the magnitude and affection of the effect would be measured by the magnitude and affection of the *contravalent* force: for if the contravalent force be zero, the natural state of repose or motion of the point is not affected by the action of the forces: and if the *contravalent* force be not zero, then its magnitude and affection would be the proper measure of the magnitude and affection of the effect which would ensue from the action of such forces upon it, by whatever means it may be thought proper to estimate it.

Contravalent force.

In all cases, however, the equilibrium of the forces, or the *nullity* of their effect upon the point will exist, if the *contravalent* force be assumed to exist in common with the other forces.

Whatever be the number of forces acting upon a point there is always one equivalent force.

595. If the effect of forces upon a point, whether immediate or transmitted, be to produce motion, there can be only *one* effect of any number of forces, and consequently only one *contravalent* force: for if motion is the consequence of

the simultaneous action of any number of forces upon a point; it can only take place in one direction, and it therefore may be always produced by a single force in that direction, and, consequently, always counteracted by a *contravalent* force: the first force would be called the *equivalent* of the primitive forces, whatever was their number: and the general object of research in the science of mechanics is to determine, under all circumstances this *equivalent* force, and from thence, when required, the corresponding *contravalent* force.

The principle of transmission of force is physically true.

596. We are so ignorant of the essential constitution of bodies, that we could not conclude, by any reasoning independent of experiment and observation, that the effect of physical forces could be transmitted without alteration from one point to any other point in that body, which was in the line of its action: if we suppose bodies rigid, or the succession of physical points of which they are composed to be immoveably connected with each other, then such a conclusion would be probable; and, as far as experiments are capable of application, it would be found by means of them to be approximately true: it is by comparing the results of such observations or experiments with each other, and by reasoning from *analogy* upon the effect which would be produced by the removal of the different causes of variation, whether in the application of the forces, or in the estimation of their magnitude or in the magnitude of their effects, or in the sensible constitution of the bodies to which the forces are applied or through which they are transmitted, that we may be said to attain to a physical principle coincident, in the terms in which it is expressed, with the mathematical principle which is assumed above, and may, consequently, be enabled to apply the mathematical results which it gives, to the explanation or interpretation of the effects of the action of physical forces upon physical bodies.

597. In speaking of the physical communication of forces, it would be improper not to notice the transmission of them

Principle of pullies.

by means of flexible strings acting over points, or pullies, or other surfaces: thus two equal weights or forces acting at the extremities of such a string, will be found to sustain each other, under all circumstances, when the influence of the string itself is so small as to admit of being neglected, or equally contributes to the effect of both the forces: if we consider the string as composed of a succession of physical points invariably connected with each other, then the force may be considered as transmitted from point to point, and every point as acted upon by equal and opposite forces, and, consequently, at rest. The physical principle involved in this explanation, has sometimes been called the *principle of pullies*, in consequence of its admitting of very ready application to the estimation of the effects of forces transmitted by means of them: and the force transmitted has been denominated the *tension* of the string or other transmitting substance, whatever it might be, being the measure of the intensity of the force by which its successive parts or successive points are *tended* or drawn from each other. Mathematically speaking, however, this physical principle is included in the general principle given above, which is equally applicable, with very slight adaptations, to the explanation of the same classes of mechanical effects.

Action and re-action are equal and in opposite directions.

598. We have assumed it as a physical as well as a mathematical principle, that equal forces acting upon a point in opposite directions, would not affect its natural state of existence, whatever that state might be: in one case, it may be considered as the result of experiment and observation: in the other, it is a necessary consequence of the representation of forces, and of the assumed general principle of their transmission: when expressed in a different but equivalent form, that "*action* and *reaction are equal and in opposite directions*," it comprehends every species of mechanical action from which equilibrium ensues, however different in the circumstances of the application of the forces: whether of two equal and opposite forces acting upon the same point, or transmitted to it through the medium of a body; whether they act to or from each

other; or whether a single force acts upon a mass of matter, which continues at rest, whether from the action of a single force or a system of forces, which are brought into action by means of it, when an equal and *contravalent* force may be supposed to be instantaneously generated in the point of its application: for in whatever manner such a point is connected with any system of points, if it be the object of mechanical action, and be kept at rest, it must be so in consequence of the action of equal and opposite forces.

Principle of the superposition of equilibrium.

599. "The effect of a force or of a system of forces acting upon a point, whether directly or by transmission, is not affected by the action of any other forces whose equivalent is zero, or which would not by themselves affect its previous state of rest or motion."

This is assumed as a general mathematical principle, and is *technically* denominated the *principle of the superposition of equilibrium.*

A mathematical, but not a physical consequence of other principles.

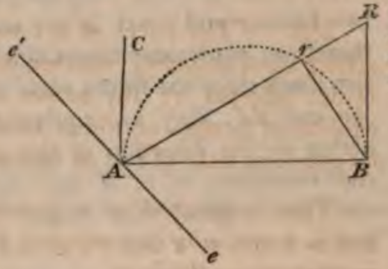
600. If considered mathematically in connection with the other principles which determine the mathematical effects of forces, it may be considered as a necessary consequence of them, inasmuch as we have assumed the nullity of the effect of forces upon a point, whose equivalent is zero, and consequently that such a point would be in the same circumstances with respect to the effect of any other forces upon it, as if the first forces had never been applied.

If this principle be viewed, however, as a physical principle, we could not conclude by any reasoning *a priori*, independent of experiment or observation, that a body or physical point acted upon by forces which sustained each other, or which did not affect its previous state of rest or motion, would be in the same physical state with respect to the effect of the action of other forces, as if the first system of forces had never been applied: in assuming this principle, therefore, to be physically as well as mathematically true, we must rest upon the same species of evidence as we have found it necessary to resort to in the establishment of all other physical principles of the effects of the action of forces upon bodies*.

Parallelogram of forces.

601. Having now examined the mathematical principles upon which forces are represented, and their effects estimated, it remains to shew that the algebraical *sum* of two or more forces acting upon the same point, whether directly or by transmission, will, in all cases, represent their *equivalent* or *resultant*.

Two forces at right angles to each other.



Let us suppose two *given* forces AB (a) and AC (b) at right angles to each other, acting upon the point A, whether directly or by transmission, the effects in both cases being the same: and let it be proposed to determine the magnitude and direction of their equivalent or *resultant* AR (R).

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* The establishment of the different physical principles in Mechanics which serve as guides to the assumption of the corresponding mathematical principles, and enable us to connect their results, by means of interpretation, with real mechanical phenomena, will be found to be dependent, in every case, in their ultimate analysis, upon experiment or observation: the remarks, which we have already had occasion to make, would shew the necessity of such a foundation of them, in all cases which regard the transmission of forces; but there are other cases where the conclusions might seem to be necessary consequences of the principle of sufficient reason, and which a superficial examination might induce us to mistake for the necessary results of reasoning: of this kind, are the mechanical propositions, "that two equal forces acting upon the same point in opposite directions, would keep it at rest:" "that three or a greater number of forces which were equal to each other and acted upon a point in directions which made equal angles with each other, would keep it at rest:" "that a perfectly uniform rod symmetrical in all its parts, placed upon its middle point, would balance itself:" and other propositions of a similar kind: in all those cases it is said, that there is no reason why motion should ensue in one direction in preference to any other; and, therefore, it is concluded, that it can take place in no direction whatever: but, in the first place it ought to be observed, that the hypotheses are mathematical and not physical, and that we cannot conceive the application of different physical forces to the same point, independently of transmission: and secondly, that the principle of sufficient reason itself, or that like causes will produce like effects, in its physical applications, must derive its authority from the *observed* universality of its operation.

We may consider the forces a and b as themselves the respective equivalents of two pairs of forces r and e, r' and e', where r and r' act in the same direction AR and e and e' in opposite directions Ae and Ae', at right angles to AR; and under such circumstances it will be indifferent whether we consider AR or R as the equivalent of the two forces a and b, or of the four forces r, r', e, e'. But the two forces r and r' act in the same direction AR, and are, therefore, equivalent to a single force $r + r'$ in that direction: and since the forces e and e' act in opposite directions Ae and Ae', they are equivalent to a single force $e - e'$, acting in the direction of the greater of them: but since the resultants a, b, R are similarly situated (or make the same angles) with respect to the pairs of component forces r and e, e' and r', a and b, it will follow that the respective resultants are proportional to their corresponding component forces*: consequently

$$e : a :: b : R, \quad \text{and, therefore,} \quad e = \frac{ab}{R};$$

$$e' : b :: a : R, \quad \text{and, therefore,} \quad e' = \frac{ab}{R};$$

$$a : r :: R : a, \quad \text{and, therefore,} \quad r = \frac{a^2}{R};$$

$$b : r' :: R : b, \quad \text{and, therefore,} \quad r' = \frac{b^2}{R}.$$

---

* This conclusion may be considered as a necessary consequence of the principle of sufficient reason, whether viewed in its mathemathical or physical sense: for in the first place, when the component forces are given in magnitude and position, the resultant must be determinable likewise in magnitude and position, inasmuch as one effect *only* can arise from the same causes: in the second place, if a resultant be given in magnitude and position, with respect to its component forces, the component forces must be determinable likewise, for the same reason: and, lastly, if we double or increase in any ratio the resultant in a given position, we must double or increase the component forces in the same ratio: for under the same circumstances the effect will vary as the cause or causes and conversely (Art. 376.)

It follows, therefore, that $e=e'$, and, consequently, the equivalent of the four forces r, r', e, e' is identical with that of r and r', which is equal to and coincident with, AR or R: therefore,

The magnitude of the resultant determined.

$$R=r+r'=\frac{a^2}{R}+\frac{b^2}{R},$$

or

$$R^2=a^2+b^2.$$

Having thus determined the magnitude of the resultant R, it will remain to determine its position with respect to its component forces.

The position of the resultant determined.

Since r and e the component forces of a, are at right angles to each other, it will follow from the preceding investigation that

$$r^2+e^2=a^2,$$

and that consequently r and e, or lines equal to them respectively, may form a right-angled triangle described upon a, that is, upon a line equal to a: let such a triangle ArB be formed upon AB, and since the subordinate component force r coincides with the resultant AR or R, it will follow that the side $Ar=r$ of this triangle will coincide with, and therefore determine the position of, this resultant.

Also, since $R^2=a^2+b^2$, and since $AR=R$, and $AB=a$, it will follow that BR is at right angles to AB and equal to b: for since $r=\frac{a^2}{R}$, it follows that

$$AR : AB :: AB : Ar$$

or the sides about the common angle A of the two triangles ABR, and ABr are proportionals: therefore, the angle ABR is equal to ArB, and, therefore, to a right angle: consequently $AB^2+BR^2=AR^2$, and, therefore, $BR^2=R^2-a^2=b^2$, and $BR=b$, which was required to be determined.

It thus appears, that if two forces at right angles to each other, be represented in magnitude and direction, by two sides of a right-angled triangle or by two adjacent sides of a rectangle, the magnitude and direction of their resultant will be represented by the hypothenuse of the right-angled triangle or by the diagonal of the rectangle; a most important proposition, which will enable us to bring the estimation of the equivalents of any number of forces acting in any manner upon a point or otherwise, under the dominion of Geometry or geometrical Algebra.

The resultant of two equal forces making any angle with each other, is their sum.

602. Thus, if a and $b\sqrt{-1}$ represent the magnitude and *co-ordination* of two forces, their *sum* $a + b\sqrt{-1}$ will represent the magnitude and direction also of their equivalent: in a similar manner, two forces represented (with respect to the former) by a and $-b\sqrt{-1}$, will have for their equivalent $a - b\sqrt{-1}$, which represents a line equal to the line denoted by $a + b\sqrt{-1}$, but making an equal angle on the opposite side of the primitive line or a: their equivalent or *sum* is $2a$, which is the diagonal of the parallelogram constructed upon them which they include: *their difference* or $2b\sqrt{-1}$ is the other diagonal of this parallelogram, or it is the *equivalent* or *sum* of two forces which are represented by

$$a + b\sqrt{-1}, \text{ and } -(a - b\sqrt{-1}),$$

the last of which is equal and opposite $a - b\sqrt{-1}$.

The resultant of two or any number of forces acting upon a point, is their sum.

603. More *generally*, the equivalent of any two forces whatever acting in *any* manner upon a point, is their algebraical *sum*, and is, therefore, represented in magnitude and direction by the diagonal of the parallelogram constructed upon the lines which represent them in quantity and direction: this conclusion is a necessary consequence of the resolution of any single force into the component forces a and $b\sqrt{-1}$, or into the sides a and b of a right-angled triangle constructed upon it, and is deduced precisely in the same manner with the corresponding proposition in

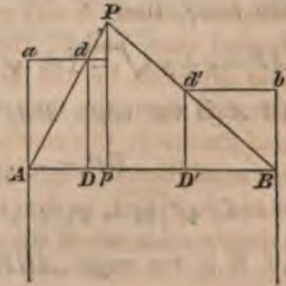
Equations of equilibrium.

Art. 512: in a similar manner we may determine and represent the equivalent of any number of forces acting upon the same point, when the equation or equations of equilibrium will be found to coincide with the equation or equations of figure which have been considered in Art. 571: in other words, forces acting upon a point which are in equilibrio, may be denoted by the sides of a *complete* figure, taken in order, which represent their magnitudes and are parallel to their directions: and in all other cases, when such forces are not in equilibrio, their *contravalent* force will be the side taken in the same order with the others, which is necessary to complete the figure.

Many cases which differ in their physical, coincide in their mathematical conditions.

604. In the view of the theory of forces and their effects, which we have given above, we have made no distinction between the case of forces acting upon a point immediately or by transmission: in the application, however, of this theory to the explanation, by means of interpretation, of the effects of the action of forces upon machines, and otherwise, we shall meet with innumerable cases, which are distinguished from each other by the physical agents for the transmission of the forces, though they may be reducible by means of hypotheses, to the same general mathematical conditions: in other words, the distinctions which those cases present, will not be capable of mathematical estimation: there is one case, however, where the action of parallel forces is considered, which is essentially different from the others in its mathematical conditions, which we shall now proceed to consider.

Parallel forces.



605. Let *Aa* (*a*) and *Bb* (*b*) represent the magnitudes of two forces acting upon a body, in directions which are parallel to each other: and as their effects will be the same at whatever points of their direction they may be applied, we shall assume *A* and *B*, in a line which

is perpendicular to them both, for the points of their application: their effect likewise upon the body (considered as a system of mathematical points invariably connected with each other, and capable, therefore, of transmitting the forces unaltered in their proper directions) will not be affected by supposing two equal forces β and $-\beta$ applied at the points A and B in the directions AB and BA: or in other words, the resultant of the four forces a, b, β and $-\beta$ will be the same as that of a and b: but the two forces a and $\beta\sqrt{-1}$ may be replaced by the single force $a+\beta\sqrt{-1}$, or by Ad, the diagonal of the rectangle $AadD$ constructed upon $Aa=a$, and $AD=\beta$: and in a similar manner, the two forces b and $-\beta\sqrt{-1}$ may be replaced by the single force $b-\beta\sqrt{-1}$ or by Bd', the diagonal of the parallelogram $Bbd'D'$, if $Bb=b$, and $BD'=\beta$: the two forces Ad and Bd' by which the four forces a, β, b, $-\beta$, or the two forces a and b, have been replaced, are no longer parallel to each other, but have a common point P: they have, therefore, a *single* resultant which is their *sum* or

$$(a+\beta\sqrt{-1})+(b-\beta\sqrt{-1}) \text{ or } a+b,$$

which is the arithmetical sum of the simple forces, if they act in the same direction, and their arithmetical difference, if they act in opposite directions.

606. Again, if we denote the distance AB of the two points A and B by d, and call x and y the co-ordinates of P (considering A as the origin of the co-ordinates, and AB as the axis), then

$$AP=x+y\sqrt{-1}=\sqrt{(x^2+y^2)}\,(\cos A+\sqrt{-1}\sin A),$$

(if A is the angle BAP), and

$$BP=-(d-x)+y\sqrt{-1}$$
$$=\sqrt{\{(d-x)^2+y^2\}}\,(-\cos B+\sqrt{-1}\sin B)$$

(if B is the angle ABP):

The locus of the contravalent force of two parallel forces.

consequently

$$x = \sqrt{(x^2 + y^2)} \cos A = \sqrt{(x^2 + y^2)} \frac{\beta}{\sqrt{(a^2 + \beta^2)}},$$

replacing $\cos A$ by $\frac{\beta}{\sqrt{(a^2 + \beta^2)}}$:

and, therefore,

$$x^2 - \frac{\beta^2 x^2}{a^2 + \beta^2} = \frac{\beta^2 y^2}{a^2 + \beta^2},$$

$$\text{or } a^2 x^2 = \beta^2 y^2, \quad \text{or } ax = \beta y:$$

in a similar manner, we shall find

$$d - x = \sqrt{\{(d - x)^2 + y^2\}} \frac{\beta}{\sqrt{(b^2 + \beta^2)}},$$

and, therefore, also

$$b(d - x) = \beta y:$$

consequently

$$ax = b(d - x),$$

$$\text{or } (a + b) x = bd \text{ and } x = \frac{bd}{a + b}.$$

It follows, therefore, that x is always of the same magnitude, whatever be the value of β, and consequently the point P will be always in the same straight line, parallel to Aa and Bb: and likewise, since

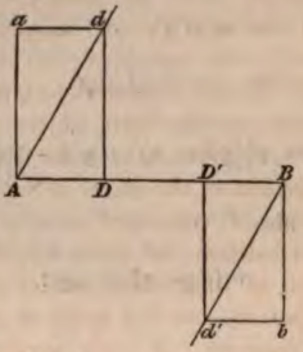
$$\frac{x}{d - x} = \frac{b}{a},$$

it follows that the line which is the locus of P, at any point of which the *contravalent* force may be equally applied, will divide the distance AB in the inverse ratio of the forces themselves.

Parallel forces which are equal and in opposite directions have no single equivalent force.

607. The expression $x = \dfrac{bd}{a+b}$ for the distance of the line which is the locus of the points of application of the contravalent forces, would denote *infinity*, if a and b were equal to each other and with opposite signs: in such a case there is no single equivalent force, inasmuch as there is no equivalent system of forces which have a common point: for under such circumstances

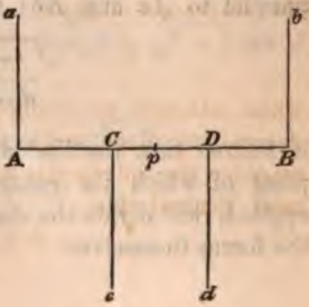
$$\cos A = \frac{\beta}{\sqrt{(a^2+\beta^2)}}, \text{ and } \cos B = \frac{-\beta}{\sqrt{(a^2+\beta^2)}} = \cos(\pi - A),$$



or the lines Ad and Bd' continue parallel to each other, whatever be the value of β.

Theory of couples.

608. In the absence, however, of any single contravalent force, there is an infinite number of similar pairs or *couples* of forces which will produce equilibrium in the system of points, and which may be considered, therefore, as the *contravalent* systems which are proper to such cases: thus if Aa and Cc represent such a couple of forces, applied at the points A and C and Bb and Dd, a couple of forces equal to the former, and at the same distance BD from each other in a reversed position: then there is a single equivalent force which is equal to the sum of Aa and Bb,



which is parallel to them, and the locus of whose points of application bisects AB in p: and there is also a single equivalent force, parallel to Cc and Dd, and equal to their sum, the locus of whose points of application bisects CD in p, which is likewise the middle point of AB; these two equivalent forces are, therefore, equal to each other, and act in the same line in opposite directions, and are, therefore, in equilibrio.

609. The preceding examination of the first principles and fundamental propositions of statical equilibrium, is so intimately connected with the theory of the algebraical representation of straight lines in magnitude and in position, and furnishes so instructive an illustration of its application, that we have ventured to introduce it, though apparently foreign to the main object of this Treatise: it would not be very difficult to make similar applications of it to the estimation of the effects of any number of forces upon any system of connected points, with a view to determine the conditions of their equilibrium, and also to the estimation of their effects under all circumstances, when they produce motion in a point or a body: the very first elements, however, of such applications, with the necessary introductory examination of the mathematical and physical first principles of Dynamics, with a view to establish their connection with each other, would lead to details which would be inconsistent with the limits of this work, and we feel compelled, therefore, to omit them.

Meaning of the sign $\cos\theta + \sqrt{-1}\sin\theta$ applied to a symbol designating a plane surface.

610. We have before considered the interpretation of the succession of signs

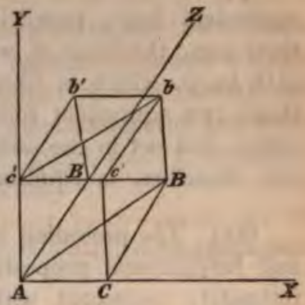
$$+1,\ +\sqrt{-1},\ -1,\ -\sqrt{-1},$$

when attached to symbols or combinations of symbols, designating plane surfaces (Art. 449, 450.): it remains to consider the interpretation of similar applications of the more general sign

$$\cos\theta + \sqrt{-1}\sin\theta.$$

When the plane surface is rectangular.

Let us assume AX as a primitive axis, and let AY be an axis at right angles to AX, and let AZ be a third axis at right angles to AX and AY, and, therefore, to the plane passing through them: we shall thus form three co-ordinate axes AX, AY, AZ which are at right angles to each other, and three co-ordinate planes passing severally through the different pairs of those axes AX and AY, AX and AZ, AY and AZ, which are also at right angles to each other: let us suppose a line AB in the plane of AX and AZ moving into different positions round the point A, and, therefore, round the axis AY: if θ be the angle which AB in any assigned position makes with AX, then the line AB (if arithmetically equal to a), will be algebraically represented by



$$a\,(\cos\theta + \sqrt{-1}\,\sin\theta):$$

if we farther suppose the line AB in its movement round AY, to carry with it the rectangular plane $ABbc'$, contained by AB and Ac' (b), or by AB and Bb, then the plane which is arithmetically represented by ab, will be algebraically represented by

$$ab\,(\cos\theta + \sqrt{-1}\,\sin\theta):$$

for the succession of its positions corresponding to the successive affectations of ab with

$$\cos\theta + \sqrt{-1}\,\sin\theta,$$

will coincide with the successive corresponding positions of the line AB: and the observations made in Art. 459 and 460, and elsewhere, respecting the representation of the different positions of a line may be transferred, without any change, to the case which we are now considering.

611. If the plane surface was not rectangular, but bounded by any other figure, whether rectilinear or not, whose area was arithmetically equal to ab, or a^2, and which was coincident with the rectangle represented in the last Article, then it would equally be represented in position and in magnitude by

When the plane surface is not rectangular.

$$a^2 (\cos \theta + \sqrt{-1} \sin \theta):$$

for the succession of positions corresponding to different affectations of a^2, by

$$\cos \theta + \sqrt{-1} \sin \theta,$$

would be altogether independent of the form of the area, which a^2 may represent arithmetically.

612. The projections of the line AB upon the axes AX and AZ, or AC and AB', are equal respectively to $a \cos \theta$, and $a \sin \theta$: the corresponding projections $ACcc'$ and $AB'b'c'$ of the rectangle $ABbc'$ upon the co-ordinate planes XAY and ZAY are equal to $ab \cos \theta$, and $ab \sin \theta$ respectively: if the projections of the line AB be denoted by α and β, the line itself will be represented by

Primary and supplimental projections.

$$\alpha + \beta \sqrt{-1},$$

and the corresponding rectangle by

$$\alpha b + \beta b \sqrt{-1}:$$

if we call, therefore, α and β the *primary* and *supplemental* projections of the line, then αb and βb, or $ab \cos \theta$, and $ab \sin \theta$ may be called likewise the *primary* and *supplemental* projections of the plane surface, which is arithmetically equal to ab, and which makes an angle θ with the *plane* of *reference* XAY.

Plane of reference.

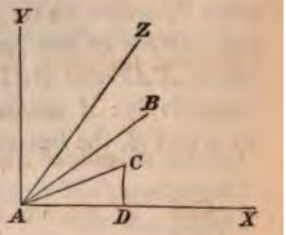
613. It follows, therefore, that the representation of a rectangular plane area, under the circumstances which we have considered, is dependent upon the representation of a line which may be called its *modulus*, which is in the

Modulus of a plane surface.

same plane with it, and perpendicular to its intersection with the plane of reference: if the position and magnitude of the *modulus* be inferred from its algebraical representation, then it is obvious, that the same algebraical representation will equally correspond to all *moduli* which are equal and parallel to each other: it will equally follow, therefore, that all equal plane surfaces which correspond to such *moduli*, or, in other words, all equal plane surfaces which are parallel to each other, will have the same algebraical representation: it is for this reason that we shall confine our attention to the same modulus which passes through the origin of the co-ordinates for all planes whose moduli are parallel to it: and since there is no necessary connection, independently of other data, between the length of the modulus and the area of the corresponding plane surface, we shall regard the position of the modulus only, and not its magnitude, in the enquiries which follow.

Relative plane.

614. If we call the plane passing through the modulus at right angles to the *plane* of *reference*, which is ZAX in the figure in Art. 610., the *relative plane*, then we may conceive this plane, with the modulus and its corresponding plane surface, to move round AZ as an axis, into any other position, such as ZAC in the figure before us, making an angle ϕ or XAC with its primitive position: under such circumstances, the angle $BAC\ (\theta)$ which measures the inclination of the given plane surface to the plane of reference XAY, remains unaltered: and if a^2 represent the area of this surface in its proper plane, then the surface continues to be represented in magnitude and also in position with respect to the plane of reference by



$$a^2(\cos\theta + \sqrt{-1}\sin\theta).$$

615. The *supplemental* projection

$$a^2\sin\theta \quad \text{or} \quad a^2\sin\theta\sqrt{-1}$$

(if its position with respect to the plane of reference be considered) makes an angle equal to $\frac{\pi}{2} + \phi$ with the co-ordinate plane ZAX: for this is the angle made by the *trace* of the supplemental projection, or by its section with the plane of reference, with AX, which is the measure likewise of the angle made by one plane with the other: the *primary* and *supplemental* projections of $a^2 \sin\theta$ upon the co-ordinate planes ZAX and ZAY are, therefore,

Projections of the plane surface upon the co-ordinate planes.

$$-a^2 \sin\theta \sin\phi \text{ and } a^2 \sin\theta \cos\phi,$$

which are likewise equal to the projections of the given plane upon the same co-ordinate planes.

If we should assume θ' and θ'' to represent the angles of inclination of the given plane to the planes ZAX and ZAY, then the projections upon those planes are

$$a^2 \cos\theta' \text{ and } a^2 \cos\theta'';$$

it follows, therefore, that

$$a^2 \cos\theta' = -a^2 \sin\theta \sin\phi,$$

$$\text{and } a^2 \cos\theta'' = a^2 \sin\theta \cos\phi.$$

616. If we add together the squares of the several projections upon the co-ordinate planes, we shall find

Equation of projections.

$$\begin{aligned} & a^4 (\cos^2\theta + \cos^2\theta' + \cos^2\theta'') \\ &= a^4 \cos^2\theta + a^4 \sin^2\theta \sin^2\phi + a^4 \sin^2\theta \cos^2\phi \\ &= a^4 \{\cos^2\theta + \sin^2\theta (\sin^2\phi + \cos^2\phi)\} \\ &= a^4 (\cos^2\theta + \sin^2\theta) = a^4; \end{aligned}$$

or the sum of the squares of the several projections is equal to the square of the area of the plane surface.

617. It follows, therefore, that the position of a plane surface will be given, if its inclination to the plane of reference and the position of the relative plane be given: or if its inclination to the plane of reference, and the

Conditions which determine the position of a plane.

position of its *trace* upon the same plane be gi
its inclinations to any two co-ordinate planes be given: for,

$$\cos^2\theta + \cos^2\theta' + \cos^2\theta'' = 1,$$

and, therefore, any two of the quantities θ, θ', θ'', will determine the third: and if any two of them be given, then θ and ϕ may be determined from the equations in Art. 615.

Different algebraical modes of representing a plane surface in position and magnitude.

618. A plane will be completely represented in magnitude and position, by the two independent expressions

$$a^2\cos\theta + a^2\sqrt{-1}\sin\theta \;\dots\dots\dots\dots\dots\dots\dots\dots\;(\alpha),$$

and

$$-a^2\sin\theta\sin\phi + \sqrt{-1}\,a^2\sin\theta\cos\phi \;\dots\dots\dots\;(\beta),$$

where $a^2\cos\theta$ expresses the primary projection of the plane, $a^2\sqrt{-1}\sin\theta$ its supplemental projection, and where the second expression (β) represents in a similar manner the supplemental projection referred to the co-ordinate plane which makes an angle ϕ with the relative plane.

In a similar manner, if $-t\sin\phi + \sqrt{-1}\,t\cos\phi$, expressed the position and magnitude of the *trace* of the plane surface upon the plane of reference, θ its inclination to that plane, and if $at = a^2$, then the position and magnitude of the plane surface would be expressed by

$$a^2\cos\theta + \sqrt{-1}\,a^2\sin\theta \;\dots\dots\dots\dots\dots\dots\dots\dots\;(\alpha),$$

and

$$-at\sin\theta\sin\phi + \sqrt{-1}\,at\sin\theta\cos\phi \dots\dots\dots(\beta').$$

619. If α, β and γ represent the projections of a plane surface (a^2) upon the three co-ordinate planes, then we should have

$$\alpha^2 + \beta^2 + \gamma^2 = a^4,$$

Planes represented algebraically in magnitude and position in terms of their projections.

$$\cos\theta = \frac{\alpha}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} = \frac{\alpha}{\alpha^2},$$

$$\cos\theta' = \frac{\beta}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} = \frac{\beta}{\alpha^2},$$

$$\cos\theta'' = \frac{\gamma}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} = \frac{\gamma}{\alpha^2},$$

$$\cos\phi = \frac{\cos\theta''}{\sin\theta} = \frac{\gamma}{\sqrt{(\beta^2+\gamma^2)}},$$

$$\sin\phi = -\frac{\cos\theta'}{\sin\theta} = \frac{-\beta}{\sqrt{(\beta^2+\gamma^2)}}.$$

Under such circumstances, the corresponding plane will be represented in position and magnitude by

$$\alpha + \sqrt{-1}\,\sqrt{(\beta^2+\gamma^2)},$$

and

$$-\beta + \sqrt{-1}\,\gamma;$$

or by

$$\sqrt{(\alpha^2+\beta^2+\gamma^2)}\left\{\frac{\alpha}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} + \sqrt{-1}\,.\,\frac{\sqrt{(\beta^2+\gamma^2)}}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}}\right\},$$

and

$$\sqrt{(\alpha^2+\beta^2+\gamma^2)}\left\{\frac{-\beta}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} + \sqrt{-1}\,.\,\frac{\gamma}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}}\right\}.$$

Algebraical representation of the faces of a regular tetrahedron.

620. In order to exemplify the preceding formulæ, let it be required to represent the faces of a regular tetrahedron by means of them.

For this purpose, we shall commence by assuming the base of the tetrahedron for the plane of reference, and one of its edges t to coincide with the principal axis: under such circumstances, the *traces* of the three inclined faces will be represented by t,

$$t\left(\cos\frac{2\pi}{3} + \sqrt{-1}\sin\frac{2\pi}{3}\right),$$

and

$$t\left(\cos\frac{2\pi}{3} - \sqrt{-1}\sin\frac{2\pi}{3}\right):$$

if the area of each face be $a^2 = at$, then the planes corresponding to those traces will be represented by

$$\left(\text{since } \cos\frac{2\pi}{3} = -\frac{1}{2}, \text{ and } \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\right),$$

$$at\cos\theta + \sqrt{-1}\,at\sin\theta, \text{ and} - at\sin\theta \ldots\ldots\ldots\ldots(1),$$

$$\left.\begin{array}{r} at\cos\theta + \sqrt{-1}\,at\sin\theta, \\ \text{and } at\sin\theta\left(\frac{1}{2} + \sqrt{-1}\,\frac{\sqrt{3}}{2}\right) \end{array}\right\}\ldots\ldots\ldots\ldots(2),$$

$$\left.\begin{array}{r} at\cos\theta + \sqrt{-1}\,at\sin\theta, \\ \text{and } at\sin\theta\left(\frac{1}{2} - \sqrt{-1}\,\frac{\sqrt{3}}{2}\right) \end{array}\right\}\ldots\ldots\ldots\ldots(3).$$

Arithmetical sum of the primary projections.

The sum of the three primary projections is

$$3at\cos\theta = at,$$

and, therefore,

$$\cos\theta = \frac{1}{3}, \text{ and } \theta = 70^{\circ}.32',$$

which is the angle of inclination of the faces to each other.

The sum of the projections upon each of the two coordinate planes at right angles to the plane of reference will be equal to zero.

Algebraical sums of the projections upon the co-ordinate planes are severally zero.

621. We have considered the primary projections of the inclined faces as possessing the same algebraical sign, a circumstance which must take place when they are derived from the expressions for those planes considered with regard to the plane of reference only: but if we consider the position of those primary projections with reference

to the traces of the inclined faces, then they will be respectively represented by $at \cos\theta$,

$$at \cos\theta \left(\cos\frac{2\pi}{3} + \sqrt{-1}\sin\frac{2\pi}{3}\right),$$

and

$$at \cos\theta \left(\cos\frac{2\pi}{3} - \sqrt{-1}\sin\frac{2\pi}{3}\right),$$

when their sum will be equal to zero, equally with the sums of the projections upon the other co-ordinate planes.

Algebraical representation of the faces of a regular parallelopipedon.

622. Again, let it be required to represent, in a similar manner, the several faces of a parallelopipedon.

Let t, t', t'' represent three adjacent edges of the solid, and let t and t' coincide with the plane of reference, and t with the principal axis: let A, A', A'' be the angles made by t and t', t and t'', t' and t'': and let θ and $\theta_{,}$* be the angles of inclination of two adjacent inclined faces of the solid to the plane of reference: it follows, therefore, that the *traces* of the inclined faces are represented respectively by t,

$$t'(-\cos A + \sqrt{-1}\sin A),$$

$-t$, and

$$t'(\cos A - \sqrt{-1}\sin A):$$

and the corresponding planes are represented by

$$\left.\begin{aligned} &tt'' \sin A' (\cos\theta + \sqrt{-1}\sin\theta), \\ \text{and } &tt' \sin A' \sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}\ldots\ldots(1),$$

$$t't'' \sin A'' (\cos\theta_{,} + \sqrt{-1}\sin\theta_{,}),$$

---

* Since the opposite planes are parallel, it will follow that the perpendiculars from any points in one of them upon the other, are equal to each other: consequently $t'' \sin A' \sin\theta = t'' \sin A'' \sin\theta_{,}$, and, therefore, any three of the quantities A', A'', θ and $\theta_{,}$, will determine the fourth.

and $t't'' \sin A'' \sin \theta_{,} \, (-\cos A + \sqrt{-1} \sin A) \ldots (2),$

$$\left.\begin{array}{r} -\,tt'' \sin A' (\cos\theta + \sqrt{-1} \sin\theta), \\ \text{and } -tt'' \sin A' \sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots(3),$$

$$\left.\begin{array}{r} t't'' \sin A'' (\cos\theta_{,} + \sqrt{-1} \sin\theta_{,}), \\ \text{and } t't'' \sin A'' \sin\theta_{,} \, (\cos A - \sqrt{-1} \sin A) \end{array}\right\} \ldots\ldots(4).$$

The remaining face of the solid is parallel and equal to the base, and, therefore, represented by $tt' \sin A$: but if it is considered with reference to its proper algebraical sign, as determined by the sign of t in one case, and by that of $-t$ in the other, then it will be found to be denoted by $-tt' \sin A$, if the base be $tt' \sin A$, or conversely.

If the primary projections are required to be represented in position upon their proper plane, as well as in magnitude, then we shall find their algebraical sum, or

$$tt'' \sin A' \cos\theta + t't'' \sin A'' \cos\theta_{,} \, (-\cos A + \sqrt{-1} \sin A)$$
$$-\,tt'' \sin A \cos\theta + t't'' \sin A'' \cos\theta_{,} \, (\cos A - \sqrt{-1} \sin A) = 0,$$

which is equal to zero equally with the sum of the projections upon the other two co-ordinate planes.

Enunciation of a general theorem respecting solid figures.

623. The general conclusion, of which the one just given may be considered as a particular case, is as follows:

"The algebraical sum of the projections of any bounded solid contained by plane surfaces, upon a co-ordinate plane, whose signs are determined by the signs of the traces of its edges upon it, is equal to zero."

The equations which arise from this general property of solids correspond to the equations of figure of bounded plane figures; and it would be possible to trace a similar analogy between other algebraical properties of solids and plane figures, particularly with reference to the conditions which determine them, or which determine the similarity of solids to each other: the great extent, however, to which such enquiries would necessarily lead us, compels us to omit them.

Determination of the n values of $(a+b\sqrt{-1})^{\frac{1}{n}}$ and $(a-b\sqrt{-1})^{\frac{1}{n}}$.

624. In Article 495, and in those which immediately follow it, we have determined the n values of $(1)^{\frac{1}{n}}$, which lead likewise to the determination of the n values of $(a)^{\frac{1}{n}}$, if the corresponding arithmetical root of a could be assigned: in a similar manner, we shall be able to determine the n values of $(a+b\sqrt{-1})^{\frac{1}{n}}$, or of $(a-b\sqrt{-1})^{\frac{1}{n}}$: for

$$(a+b\sqrt{-1})^{\frac{1}{n}} = (a^2+b^2)^{\frac{1}{2n}}\left\{\frac{a}{\sqrt{(a^2+b^2)}}+\frac{b\sqrt{-1}}{\sqrt{(a^2+b^2)}}\right\}^{\frac{1}{n}}$$

$$= \rho\,(1)^{\frac{1}{n}}(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{n}},$$

if ρ be the arithmetical value of $(a^2+b^2)^{\frac{1}{2n}}$, and if

$$\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}}:$$

but

$$(1)^{\frac{1}{n}} = \cos\frac{2\gamma\pi}{n}+\sqrt{-1}\sin\frac{2\gamma\pi}{n},$$

$$\text{and } (\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{n}}$$

$$= \cos\left(\frac{2\gamma'\pi+\theta}{n}\right)+\sqrt{-1}\sin\left(\frac{2\gamma'\pi+\theta}{n}\right),$$

where γ and γ' are any terms of the series 0, 1, 2, 3, ...: consequently

$$(1)^{\frac{1}{n}}(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{n}}$$

$$= \left(\cos\frac{2\gamma\pi}{n}+\sqrt{-1}\sin\frac{2\gamma\pi}{n}\right)\times$$

$$\left\{\cos\left(\frac{2\gamma'\pi+\theta}{n}\right)+\sqrt{-1}\sin\left(\frac{2\gamma'\pi+\theta}{n}\right)\right\}$$

$$=\cos\left(\frac{2(\gamma+\gamma')\pi+\theta}{n}\right)+\sqrt{-1}\,.\sin\left(\frac{2(\gamma+\gamma')\pi+\theta}{n}\right)$$

$$=\cos\left(\frac{2\gamma\pi+\theta}{n}\right)+\sqrt{-1}\,.\sin\left(\frac{2\gamma\pi+\theta}{n}\right),$$

replacing $\gamma+\gamma'$ by γ, since they equally designate terms in the same series.

It follows, therefore, that

$$(a+b\sqrt{-1})^{\frac{1}{n}}$$

$$=\rho\left\{\cos\left(\frac{2\gamma\pi+\theta}{n}\right)+\sqrt{-1}\,.\sin\left(\frac{2\gamma\pi+\theta}{n}\right)\right\};$$

and, in a similar manner, we shall find

$$(a-b\sqrt{-1})^{\frac{1}{n}}$$

$$=\rho\left\{\cos\left(\frac{2\gamma\pi+\theta}{n}\right)-\sqrt{-1}\sin\left(\frac{2\gamma\pi+\theta}{n}\right)\right\}:$$

from these expressions, all the values (which are n in number) of $(a+b\sqrt{-1})^{\frac{1}{n}}$ and $(a-b\sqrt{-1})^{\frac{1}{n}}$ may be determined.

Examples.

625. As an example, let it be required to calculate the values of

$$\{25+\sqrt{(-104)}\}^{\frac{1}{3}}.$$

In this case

$$\rho=(a^2+b^2)^{\frac{1}{6}}=(729)^{\frac{1}{6}}=3,$$

and $\cos\theta=\frac{25}{27}$, and, therefore, $\theta=22^0\,.\,12'$.

Consequently, the three values required are

$$3\,(\cos 7^0\,.\,24'+\sqrt{-1}\,.\sin 7^0\,.\,24')$$

$$=2.9750136+.3863868\sqrt{-1};$$

$$3\,(\cos 127^0 . 24' + \sqrt{-1} . \sin 127^0 . 24')$$

$$= -1.8221274 + 2.3832438\sqrt{-1};$$

$$3\,(\cos 112^0 . 36' - \sqrt{-1} . \sin 112^0 . 36')$$

$$= -1.1528859 - 2.7696306\sqrt{-1}.$$

The corresponding geometrical values are three lines equal to each other and to 3, making angles of

$$7^0 . 24', \quad 127^0 . 24', \text{ and } 247^0 . 24', \quad \text{or } -112^0 . 36',$$

with the primitive line or axis.

Again, let it be required to calculate the different values of the expression

$$\sqrt[3]{\left\{-6 + \sqrt{\left(-\frac{1225}{27}\right)}\right\}}$$

$$+ \sqrt[3]{\left\{-6 - \sqrt{\left(-\frac{1225}{27}\right)}\right\}}.$$

The three values of

$$\sqrt[3]{\left\{-6 + \sqrt{\left(-\frac{1225}{27}\right)}\right\}},$$

calculated as in the last example, are

(α) $\quad 1.5 + 1.4434\sqrt{-1},$

(β) $\quad -2 + .5773\sqrt{-1},$

(γ) $\quad .5 - 2.0206\sqrt{-1}.$

The three values of $\sqrt[3]{\left\{-6 - \sqrt{\left(-\frac{1225}{27}\right)}\right\}}$, are

(α') $\quad 1.5 - 1.4434\sqrt{-1},$

(β') $\quad -2 - .5773\sqrt{-1},$

(γ') $\quad .5 + 2.0206\sqrt{-1}.$

The nine different values of the proposed expression, which result from combining the values of the first and second parts of it, are

$$\begin{array}{ll}(1) & (\alpha) + (\alpha') = 3,\\ (2) & (\beta) + (\beta') = -4,\\ (3) & (\gamma) + (\gamma') = 1,\end{array}$$

$$\left.\begin{array}{ll}(4) & (\alpha) + (\beta') = -.5 + .8661\sqrt{-1}\\ (5) & (\beta) + (\alpha') = -.5 - .8661\sqrt{-1}\end{array}\right\},$$

$$\left.\begin{array}{ll}(6) & (\alpha) + (\gamma') = 2 + 3.4640\sqrt{-1}\\ (7) & (\gamma) + (\alpha') = 2 - 3.4640\sqrt{-1}\end{array}\right\},$$

$$\left.\begin{array}{ll}(8) & (\beta) + (\gamma') = -1.5 + 2.5979\sqrt{-1}\\ (9) & (\gamma) + (\beta') = -1.5 - 2.5979\sqrt{-1}\end{array}\right\}.$$

The pairs of values α and α', β and β', γ and γ', may be considered as representing lines which are severally equal to each other and to

$$\sqrt{\frac{13}{3}} \text{ or } 2.0818,$$

and which are placed at angles of $43^0.54'$, $163^0.54'$, and $76^0.6'$ on each side of the primitive line: the sums of those pairs of lines which make equal angles with the primitive line, or the diagonals of the parallelograms constructed upon them, coincide with the primitive line, and are denoted by 3, -4 and 1 respectively: the remaining sums form pairs of lines making equal angles with the primitive line: the first pair of these sums are severally equal to 1, and make angles equal to 120^0, the second equal to 4 making angles equal to 60^0, and the third equal to 3 and making angles equal to 120^0, with the primitive line.

626. The values of $(a+b\sqrt{-1})^{\frac{1}{n}}$, or of combinations of such quantities may be estimated, therefore, in the

The number of values of $(a+b\sqrt{-1})^{\frac{1}{n}}$ or of $(a-b\sqrt{-1})^{\frac{1}{n}}$ cannot exceed n.

same manner as the values of $(a)^{\frac{1}{n}}$, or of combinations of them: in one case, we multiply the arithmetical root into the n values of $(1)^{\frac{1}{n}}$ or $(-1)^{\frac{1}{n}}$; in the other case, we multiply the arithmetical root into the n values of

$$\left\{\frac{a}{\sqrt{(a^2+b^2)}}+\frac{b\sqrt{-1}}{\sqrt{(a^2+b^2)}}\right\}^{\frac{1}{n}}:$$

it appears, likewise, that all the values of

$$\left\{\frac{a}{\sqrt{(a^2+b^2)}}+\frac{b\sqrt{-1}}{\sqrt{(a^2+b^2)}}\right\}^{\frac{1}{n}}$$

may be found by multiplying one of them into the different values of $(1)^{\frac{1}{n}}$: but under no circumstances can the number of such values, which are symbolically different from each other, in whatever manner they are determined, be made to exceed n: it is necessary to attend to these observations, in order to ascertain and exhibit the complete equivalence of different expressions which involve them, which are deducible from each other.

The complete form of the series for $(1+x)^n$ and $(a+x)^n$.

627. Thus, the equation

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\&c.$$

which is complete, when n is a whole number, exhibits the *arithmetical root* only, when n is a fraction: in order to be complete, generally, it must be put under the form

$$(1+x)^n=(1)^n\left\{1+nx+\frac{n(n-1)}{1.2}x^2+\&c.\right\}$$

$$=(\cos 2\gamma n\pi+\sqrt{-1}\sin 2\gamma n\pi)\times$$

$$\left\{1+nx+\frac{n(n-1)}{1.2}x^2+\&c.\right\}$$

The equation

$$(a+x)^n=a^n\left\{1+\frac{nx}{a}+\frac{n(n-1)}{1.2}\frac{x^2}{a^2}+\&c.\right\}$$

is complete under all circumstances: for a^n is equivalent to $a^n \{\cos 2\gamma n\pi + \sqrt{-1} \sin 2\gamma n\pi\}$, where a^n is the arithmetical value of a^n, corresponding to $\gamma = 0$: but if the same equation was exhibited under the form

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2} x^2 + \&c.$$

then it can only be considered as complete upon the supposition that the same value of γ was assumed in every case, when a^n, a^{n-1}, a^{n-2}, &c. were replaced by their equivalent forms

$$a^n (2\gamma n\pi + \sqrt{-1} \sin 2\gamma n\pi),$$

$$a^{n-1} \{\cos 2\gamma (n-1)\pi + \sqrt{-1} \sin 2\gamma (n-1)\pi\},$$

$$a^{n-2} \{\cos 2\gamma (n-2)\pi + \sqrt{-1} \sin 2\gamma (n-2)\pi\}, \&c.$$

The number of absolute values of a series may be different from the number of its values considered as the result of a developement.

628. This last remark is extremely important, inasmuch as it will impose upon us a very different course of proceeding, when we are required to estimate the values of a series involving fractional indices, considered as the result of a developement, and when considered *absolutely* without reference to its origin: thus, the series

$$\sqrt{ax} - \frac{1}{2} \cdot \frac{x^{\frac{3}{2}}}{\sqrt{a}} - \frac{1 \cdot 1}{2^2 \cdot 1 \cdot 2} \cdot \frac{x^{\frac{5}{2}}}{a^{\frac{3}{2}}} - \frac{1 \cdot 1 \cdot 3}{2^3 \cdot 1 \cdot 2 \cdot 3} \cdot \frac{x^{\frac{7}{2}}}{a^{\frac{5}{2}}} - \&c.$$

continued indefinitely, would have an infinite number of values, arising from the different combinations of positive and negative values of its several terms: but if the same series be considered as the developement of $\sqrt{ax - x^2}$, then it is at once reducible to the equivalent form

$$\sqrt{ax} \left\{1 - \frac{1}{2} \cdot \frac{x}{a} - \frac{1 \cdot 1}{2^2 \cdot 1 \cdot 2} \cdot \frac{x^2}{a^2} - \frac{1 \cdot 1 \cdot 3}{2^3 \cdot 1 \cdot 2 \cdot 3} \cdot \frac{x^3}{a^3} - \&c.\right\}$$

which necessarily possesses the same number of values with the function from which it is derived and no more.

629. In considering the nature of the series which may be supposed to correspond to the developements of assigned expressions, or which may be supposed to represent quantities whether algebraically expressible in a finite form or not, it may be at once inferred that the series can admit of no more values than the expression or quantity from which it is derived: if the value of the quantity be single, therefore, there can be no fractional indices in its developement: if double, there must be a fractional index whose denominator is 2, involved in one term at least: if triple, there must be a fractional index, whose denominator is 3, and so on in other cases.

The nature of a series may be in some degree inferred from the single or multiple values of the expression from which it arises.

630. In many cases formulæ have been deduced upon a particular hypothesis, and afterwards generalized, without a sufficient attention to the influence of the hypothesis upon the result which is obtained: or, in other words, the principle of the permanence of equivalent forms has been tacitly assumed to apply, before the forms themselves were in a state adapted to its application: as an example, where such errors both have been, and may be, committed, let us consider the following equivalent form of $(\cos x)^m$.

Investigation of the series for $(2\cos x)^m$.

Since $2\cos x = \epsilon^x + \epsilon^{-x}$, Art. 470, it follows, that

$$2^m (\cos x)^m = e^{mx} + m\epsilon^{(m-2)x} + \frac{m(m-1)}{1.2}\epsilon^{(m-4)x} + \&c.$$

$$= \cos mx + m\cos(m-2)x + \frac{m(m-1)}{1.2}\cos(m-4)x + \&c.$$

$$+\sqrt{-1}\left\{\sin mx + m\sin(m-2)x + \frac{m.(m-1)}{1.2}\sin(m-4)x + \&c.\right\};$$

replacing ϵ^{mx}, $\epsilon^{(m-2)x}$, $\epsilon^{(m-4)x}$, &c.

by $\cos mx + \sqrt{-1}\sin mx$,

$\cos(m-2)x + \sqrt{-1}\sin(m-2)x$,

$\cos(m-4)x + \sqrt{-1}\sin(m-4)x$, &c.

Its arithmetical value.

If we assume ρ to represent the *arithmetical* value of $2^m \cos x^m$, which is determined without reference to its sign, then its complete value is

$$\rho\,(\cos 2m\gamma\pi + \sqrt{-1}\,\sin 2m\gamma\pi),$$

when $\cos x$ is positive, and

$$\rho\,\{\cos m\,(2\gamma+1)\,\pi + \sqrt{-1}\,.\sin m\,(2\gamma+1)\,\pi\},$$

when $\cos x$ is negative: and since $\cos x$ is the same, if x be replaced by

$$x+2\pi,\quad x+4\pi,\quad \ldots\, x+2\gamma\pi,\ \&c.,$$

it will follow that the equation will be complete when put under the form

Its complete value.

$$\rho\,(1)^m \text{ or } \rho\,(-1)^m = \cos m\,(x+2\gamma\pi)$$

$$+ m\cos(m-2)\,(x+2\gamma\pi) + \frac{m\,(m-1)}{1\,.\,2}\cos(m-4)\,(x+2\gamma\pi)$$

$$+\sqrt{-1}\,\{\sin m\,(x+2\gamma\pi) + m\sin(m-2)\,(x+2\gamma\pi)$$

$$+\frac{m\,(m-1)}{1\,.\,2}\sin(m-4)\,(x+2\gamma\pi)+\&c.\}\ \ldots\ldots\ (1).$$

If m be a whole number, there is only one value of each member of this equation: but if m be a fraction such as $\frac{1}{n}$ or $\frac{p}{n}$, then the number of values of each member of the equation is n, and no more.

Its complete value expressed under an abbreviated form.

631. If we agree to denote

$$\cos m\,(x+2\gamma\pi) + m\cos(m-2)\,(x+2\gamma\pi)$$

$$+\frac{m\,(m-1)}{1\,.\,2}\cos(m-4)\,(x+2\gamma\pi)+\&c.$$

by X_γ, and

$$\sin m\,(x+2\gamma\pi) + m\sin(m-2)\,(x+2\gamma\pi)$$

$$+\frac{m\,(m-1)}{1\,.\,2}\sin(m-4)\,(x+2\gamma\pi)+\&c.$$

by X_γ', where the subscribed γ denotes some term in the series 0, 1, 2, 3, &c., then we shall have, when $\cos x$ is positive,

When $\cos x$ is positive.

$$\rho\,(\cos 2m\gamma\pi + \sqrt{-1}\sin 2m\gamma\pi) = X_\gamma + X_\gamma'\sqrt{-1}\,;$$

and, therefore,

$$\rho\cos 2m\gamma\pi = X_\gamma, \quad \text{and} \quad \rho\sin 2m\gamma\pi = X'\,;$$

and when $\cos x$ is negative,

When $\cos x$ is negative.

$$\rho\,\{\cos m\,(2\gamma+1)\,\pi + \sqrt{-1}\sin\,(2\gamma+1)\,\pi\}$$
$$= X_\gamma + X_\gamma'\sqrt{-1}\,;$$

and, therefore,

$$\rho\cos m\,(2\gamma+1)\,\pi = X_\gamma, \quad \text{and}\ \rho\sin m\,(2\gamma+1)\,\pi = X'_\gamma.$$

632. If m be a whole number, and $\cos x$ *positive*, then

m a whole number and $\cos x$ positive.

$$\cos 2m\gamma\pi = 1, \quad \text{and} \quad \sin 2m\gamma\pi = 0\,;$$

therefore the only value of $2^m\cos x^m$ or ρ, is X, since X_γ is the same, whatever γ may be, and $X_\gamma' = 0$.

If m be an *even* whole number, and $\cos x$ *negative*, then

m an even whole number and $\cos x$ negative.

$$\cos m\,(2\gamma+1)\,\pi = 1, \quad \text{and}\ \sin m\,(2\gamma+1)\,\pi = 0\,;$$

therefore, the only value of $2^m\cos x^m$ or ρ, is X, since X_γ has only one value, and $X_\gamma' = 0$.

If m be an *odd* whole number, and $\cos x$ *negative*, then

m an odd whole number and $\cos x$ negative.

$$\cos m\,(2\gamma+1)\,\pi = -1, \quad \text{and}\ \sin m\,(2\gamma+1)\,\pi = 0\,;$$

therefore the only value of $2^m\cos x^m$ or $-\rho$ is X, since X_γ has only one value, and $X'_\gamma = 0$.

633. If m be a fraction such as $\frac{p}{n}$, and $\cos x$ *positive*, then ρ corresponds to $\gamma = 0$, or

m a fraction and $\cos x$ positive.

$$\rho = 2^m\cos x^m = X_0.$$

For all other values of γ, under such circumstances, we have

$$\rho = \frac{X_\gamma}{\cos 2m\gamma\pi} = \frac{X_\gamma'}{\sin 2m\gamma\pi};$$

If $\gamma = 0$, we find

$$\rho = \frac{X_0'}{0} = \frac{0}{0} = X_0,$$

a result which will admit of interpretation by the aid of principles which will be established in the following Chapter.

m a fraction and $\cos x$ negative.

634. If m be a fraction such as $\frac{p}{n}$, and $\cos x$ *negative*, then

$$\rho = \frac{X_\gamma}{\cos m(2\gamma + 1)\pi} = \frac{X_\gamma'}{\sin m(2\gamma+1)\pi},$$

which will equally express the arithmetical value of $2^m \cos x^m$.

Cases in which ρ may be expressed by simple values of X_γ or X_γ'.

It remains to examine in what cases, under such circumstances the value of ρ, or the arithmetical value of $2^m \cos x^m$ may be expressed by simple values of X_γ or X_γ'.

Let $m = \frac{p}{n}$ (in its lowest terms,) where p is an even and n an odd number: in this case, if we make $2\gamma + 1 = n$, then $\cos m(2\gamma + 1)\pi = \cos p\pi = 1$, and $\sin p\pi = 0$: consequently $\rho = X_{\frac{n-1}{2}}$, which is the only value of X_γ or of X_γ' which expresses it: for since $\frac{p(2\gamma+1)}{n}$ is a whole number, and n is prime to p, it follows that $2\gamma + 1 = n$ or $2n$ or $3n$, or some term of that series (Art. 192): but the second of these values makes $\gamma = \frac{2n-1}{2}$, which is not a whole number: and the third makes $\gamma = \frac{3n-1}{2}$ which is greater than n, and, therefore, produces a value of X_γ which is identical with $X_{\frac{n-1}{2}}$; similar observations apply to all the other terms of the series.

If p and n be both of them odd numbers, then if we make $2\gamma + 1 = n$, we shall have

$$\cos m\,(2\gamma+1)\,\pi = \cos p\pi = -1, \quad \text{and } \sin p\pi = 0:$$

consequently $\rho = -X_{\frac{n-1}{2}}$, which is the only value of $-X_\gamma$ or of X_γ', which expresses it.

If p be odd and n even, then $\frac{p.(2\gamma+1)}{n}$ cannot become a whole number, and there is no value of X_γ which is equal to ρ: but if n be an even number of the form $2r$ (*impariter par*) where r is an odd number, that is, if n be any term of the series 2, 6, 10, 14, &c., then

$$m\,(2\gamma+1)\,\pi = \frac{p\,(2\gamma+1)\,\pi}{n} = \frac{p\,(2\gamma+1)\,\pi}{2r} = \frac{p\pi}{2},$$

if $2\gamma + 1 = r$: consequently under such circumstances,

$$\cos m\,(2\gamma+1)\,\pi = 0, \quad \text{and } \sin m\,(2\gamma+1)\,\pi = 1, \text{ or } -1,$$

according as p is a term of the series 1, 5, 9, &c. or of the series 3, 7, 11, &c.: in the first case we shall find, therefore, $\rho = X'_{\frac{\frac{n}{2}-1}{2}}$, and in the second $\rho = -X'_{\frac{\frac{n}{2}-1}{2}}$.

If p be an odd number, and n an even number of the series 4, 8, 12, 16, &c. then there is no simple value of X_γ or of X_γ' which is equal to ρ.

◆

CHAP. XIII.

ON INDETERMINATE COEFFICIENTS.

Direct and indirect methods of deducing equivalent forms.

635. THE methods employed for the determination of equivalent forms, may be considered as constituting two classes, according as they are *direct* or *indirect:* direct methods are those in which the transition from the primitive to the equivalent form, is effected by means of defined or definable operations, such as multiplication, division, the raising of powers, and the extraction of roots: whilst indirect methods are generally resorted to, when we are unable to express in words or otherwise the nature of the operation or operations which connect the primitive forms with others which may be equivalent to them: under such circumstances, however, the primitive form will commonly furnish us with the conditions which the secondary form must satisfy, in order that its equivalence may be determined.

To this second class may be referred the method of indeterminate coefficients, which admits of the most extensive applications, and which may be frequently advantageously employed, when direct methods are likewise available: it will be best understood, from its application to a particular example.

Examples.

636. The fraction $\frac{a}{a+bx}$ is readily shewn to be equal to the indefinite series

$$1 - \frac{bx}{a} + \frac{b^2x^2}{a^2} - \frac{b^3x^3}{a^3} + \frac{b^4x^4}{a^4} - \&c.,$$

by the actual division of the numerator by the denominator: if we had assumed, therefore,

$$\frac{a}{a+bx} = 1 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \&c.$$

where A_1, A_2, A_3, A_4, &c. are indeterminate coefficients, it is obvious that the required conditions would be satisfied, if we made

$$A_1 = -\frac{b}{a},\ A_2 = \frac{b^2}{a^2},\ A_3 = -\frac{b^3}{a^3},\ A_4 = \frac{b^4}{a^4},\ \&c.$$

In this case we have commenced with the developement of the series, and we consequently gain nothing by subsequently assuming a series with indeterminate coefficients: but if we had begun by assuming in the first instance, from our knowledge of the form of the series, in whatever manner derived,

$$\frac{a}{a+bx} = 1 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \&c.$$

and had considered that $\frac{a}{a+bx}$, and the equivalent series, must produce the same result, when multiplied by the same quantity $a + bx$, then we should get

$$\frac{a}{a+bx} \times (a+bx)$$

$$= (a+bx)\,(1 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \&c.),$$

$$\text{or } a = a + (A_1a + b)\,x + (A_2a + A_1b)\,x^2 + (A_3a + A_2b)\,x^3$$

$$+ (A_4a + A_3b)\,x^4 + \&c.;$$

two results, which can only become *identical*, by supposing the coefficients of x and of its powers in the second of them to be severally equal to zero: consequently we have

$$A_1 a + b = 0, \text{ and } \therefore A_1 = -\frac{b}{a};$$

$$A_2 a + A_1 b = 0, \text{ and } \therefore A_2 = -\frac{A_1 b}{a} = \frac{b^2}{a^2};$$

$$A_3 a + A_2 b = 0, \text{ and } \therefore A_3 = -\frac{A_2 b}{a} = -\frac{b^3}{a^3};$$

$$A_4 a + A_3 b = 0, \text{ and } \therefore A_4 = -\frac{A_3 b}{a} = \frac{b^4}{a^4};$$

we should thus obtain the law of the indefinite series which is algebraically equal to $\frac{a}{a+bx}$, without the necessity of resorting to the operation of the division of a by $a + bx$.

Again, let us assume, as a second example,

$$\frac{a+bx}{a+\beta x+\gamma x^2} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.,$$

where the first term, which involves no power of x, is denoted by A_0, and the other coefficients by the indeterminate quantities A_1, A_2, A_3, A_4, &c., the subscript number being in each case equal to the index of x in the corresponding term: if, therefore, the series assumed be equivalent to the primitive form, we shall have

$$\frac{a+bx}{a+\beta x+\gamma x^2} \times (a+\beta x+\gamma x^2)$$
$$= (a+\beta x+\gamma x^2) \{A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.\};$$

and, therefore,

$$\begin{aligned} a + bx = A_0 a &+ A_1 a x + A_2 a x^2 + A_3 a x^3 + A_4 a x^4 + \&c. \\ &+ A_0 \beta x + A_1 \beta x^2 + A_2 \beta x^3 + A_3 \beta x^4 + \&c. \\ &\qquad + A_0 \gamma x^2 + A_1 \gamma x^3 + A_2 \gamma x^4 + \&c. \end{aligned}$$

In order that these two results may be *identical* with each other, we must have

$$A_0\alpha = a, \qquad \text{and } \therefore A_0 = \frac{a}{\alpha},$$

$$A_1\alpha + A_0\beta = b, \qquad \text{and } \therefore A_1 = \frac{b}{\alpha} - \frac{A_0\beta}{\alpha} = \frac{b}{\alpha} - \frac{a\beta}{\alpha^2},$$

$$A_2\alpha + A_1\beta + A_0\gamma = 0, \text{ and } \therefore A_2 = -\frac{A_1\beta}{\alpha} - \frac{A_0\gamma}{\alpha}$$

$$= \frac{b\beta}{\alpha^2} - \frac{a\beta^2}{\alpha^3} - \frac{a\gamma}{\alpha^2},$$

$$A_3\alpha + A_2\beta + A_1\gamma = 0, \text{ and } \therefore A_3 = -\frac{A_2\beta}{\alpha} - \frac{A_1\gamma}{\alpha}.$$

The examination of the formation of the successive terms of this series would shew that any one coefficient after the second was formed by multiplying the two preceding coefficients by $\frac{-\beta}{\alpha}$ and $\frac{-\gamma}{\alpha}$ respectively, and connecting the results together with their proper signs: similar observations may be made with respect to the formation of the successive terms of other series derived in a similar manner, as well as to many others*: thus, if we assume

$$\frac{a + bx + cx^2}{1 + \alpha x + \beta x^2 + \gamma x^3} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.$$

we shall find

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* Such series are sometimes called *recurring* series, and the multipliers connected with their proper signs, constitute the *scale of relation*: thus the scale of relation of the series we have just considered, is $\frac{-\beta}{\alpha} - \frac{\gamma}{\alpha}$: and the scale of relation of the series which follows is $-\alpha - \beta - \gamma$.

Scale of relation.

$$A_0 = a,$$

$$A_1 + A_0\alpha = b,$$

$$A_2 + A_1\alpha + A_1\beta = 0,$$

$$A_3 + A_2\alpha + A_1\beta + A_0\gamma = 0,$$

$$A_4 + A_3\alpha + A_2\beta + A_1\gamma = 0,$$

and it appears, therefore, that the coefficient of any one term of this series after the third, may be formed by multiplying the coefficients of the three preceding terms by $-\alpha$, $-\beta$, and $-\gamma$, respectively, and connecting the results with their proper signs.

Identity of the results obtained.

637. The determination of the assumed indeterminate coefficients in the preceding examples, (and the same will be the case also in those which follow) supposes the algebraical *identity* of the results which are produced: thus the series which is equivalent to $\dfrac{a}{a+bx}$ must be of such a kind, that if it be multiplied into $a+bx$, it will produce the same result as the product of $\dfrac{a}{a+bx}$ and $a+bx$: and inasmuch as the result in the latter case is a, the result likewise in the first case must be *identical* with it, and consequently the first term of the product must be a, and the coefficients of all the other terms must be equal to zero: for under no other circumstances can the assumed series be considered as equivalent to $\dfrac{a}{a+bx}$: the same observations may be very readily adapted to the other series which we have considered.

638. If two series or expressions such as

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.$$

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \&c.$$

be identical with each other, they must not only present themselves under the same form, but the coefficients of the corresponding terms arranged, if possible, according to the powers of the same letter, or of the same combination of letters, must be equal to, and identical with, each other: thus, in the case before us, we must have

Meaning of the identity of two expressions or series.

$$a_0 = A_0,\ a_1 = A_1,\ a_2 = A_2,\ a_3 = A_3,$$

and so on: it is the fulfilment of these conditions which constitutes the *identity* of such expressions, and which may in all cases be considered as equivalent to the definition of the meaning of the term.

Distinction between the terms *identical* and *equivalent*.

639. We have before (Art. 121.) had occasion to consider the meaning of the term *equivalent*, as the interpretation of the meaning of the sign =, placed between two expressions, which are deducible from each other, or from a third expression, by the ordinary laws and operations of Algebra: and it would from thence appear, that expressions which were *identical*, were also *equivalent*, but not conversely: thus $\frac{a^3 - x^3}{a - x}$ is equivalent to $a^2 + ax + x^2$, but not identical with it, inasmuch as it does not present itself under the same form: and the same remark applies to all other pairs of expressions connected by the sign =, where the transition from one to the other is effected by means of an operation, whether definable or not: but if two expressions or series under the *same* form present themselves as equivalent to a third expression, then they must be identical with each other, in order that they may satisfy the definition of equivalence; and the corresponding terms, whether involving coefficients which were assumed to be indeterminate or not, must be identical with each other.

An erroneous assumption of the form of a series with indeterminate coefficients does not lead to an erroneous result.

640. The process for the conversion of expressions into others which are equivalent to them under an assigned form, appears to assume arbitrarily the existence of such equivalent forms, whether they really exist or not: a little consideration, however, will shew that the process employed for the

determination of the indeterminate members of the assumed series, must involve the conditions upon which the equivalence of the two expressions depends: the failure, therefore, in the determination of the coefficients which were improperly or unnecessarily assumed, or their entire disappearance from the final result, would furnish the proper correction of the first assumption, or, in other words, might be considered as the proper indication of the non-existence of the equivalent expression or series under the form which was assigned to it.

Examples.

641. Thus, if we should assume

$$\frac{1}{1+x} = \frac{A_{-1}}{x} + A_0 + A_1 x + A_2 x^2 + \&c.,$$

and proceed to the determination of the indeterminate coefficients, by the comparison of the terms of the two resulting identical expressions, we should find $A_{-1} = 0$, and therefore the assumption of a term which can have no existence in the equivalent expression or series, will lead to no error in the final result: again, if we should assume

$$\frac{1-x^2}{1+x^2-x^4} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 + \&c.$$

and proceed in a similar manner to the determination of the several coefficients of the series, we should find A_1, A_3, A_5, and all the coefficients of the terms which involve odd powers of x, severally equal to zero, and the correct series

$$1 - 2x^2 + 3x^4 - 5x^6 + 8x^8 - \&c.$$

would be determined in the same manner as if we had commenced with the assumption of the series

$$A_0 + A_2 x^2 + A_4 x^4 + A_6 x^6 + \&c.$$

in which all superfluous terms were omitted.

642. Though the correct assumption of the form of the equivalent series in the first instance, is not essential to the correct determination of the series itself, yet it will

more or less simplify the process for that purpose, by lessening the number of quantities to be determined, and by not encumbering the equations whose solutions are required with unnecessary symbols: it is for this reason, that it becomes extremely important to avail ourselves of any considerations which may serve to guide us in the primitive assumption of the series.

Observations which guide us to the correct assumption of series.

The general principle of the method of indeterminate coefficients, as we have already seen, is to deduce two expressions or series, which, from the nature of the process by which they are obtained, are identical with each other; and it is obvious that this identity will exist for all values of the symbols from infinite and negative, to infinite and positive, and, therefore, including zero: it likewise follows, from our investigations in a former chapter (X.), that the primitive expression, and the assumed equivalent series, must be algebraically equivalent for all values of the symbols included between the same limits: but the first can only be considered as arithmetically equal to the second, for those values of the symbols which make the series convergent from some determinable term: if the series proceed, therefore, according to positive powers of a symbol or combination of symbols, then this arithmetical equality will exist likewise for all those values of this symbol or combination of symbols, which make the series convergent or finite, and therefore when that value is zero: but if the series proceeds according to negative powers of this symbol or combination of symbols, then zero would no longer be included in the succession of values upon which the arithmetical equality of the two members of the equation depends: if, however, the series, though involving terms with negative powers of this symbol or combination of symbols, involves likewise others with positive powers, according to which it still continues to be arranged, then zero may be likewise considered as included in the succession of values, and infinity considered as the true corresponding arithmetical value or sum of the series.

643. The proper indication of the first case which we have noticed above, would be a value of the primitive ex-

Determination of the first term of the equivalent series.

pression equal to a finite quantity, or zero, corresponding to a zero value of the symbol or combination of symbols, according to the positive powers of which the terms of the series were required to be arranged: thus the series for $(1+x)^n$ (if its existence be taken for granted) arranged according to powers of x, would have its first term (or term independent of x) equal to 1, inasmuch as that is the value of $(1+x)^n$, when $x=0$: in a similar manner, the first term of the series for $(a+x)^n$ proceeding according to powers of x, would be a^n, for the same reason: the same would be the first term of the series for $(a+x)^n$, proceeding according to powers of $\frac{x}{a+x}$ or $\frac{x}{a-x}$: but the first term of the series for $(a+x)^n$, proceeding according to powers of $\frac{a-x}{a+x}$, would be $2^n a^n$, inasmuch as that would be the value of $(a+x)^n$, when $a-x=0$, or $a=x$: the first term of a series proceeding according to powers of x, which should be equivalent to a^x, would be found to be 1, inasmuch as $a^x=1$, when $x=0$: the first term of a series for $\cos x$, proceeding likewise according to powers of x, would be 1, since $\cos 0=1$: but the first term of a series, likewise for $\cos x$, which should proceed according to powers of $\frac{\pi}{2}-x$, would be 0, inasmuch as $\cos x=\cos\frac{\pi}{2}=0$, if $\frac{\pi}{2}-x=0$: in a similar manner the first term of a series for $\sin x$, proceeding according to powers of x, would be 0, inasmuch as $\sin 0=0$: but the first term of a series for the same quantity proceeding according to powers of $\frac{\pi}{2}-x$, would be 1, inasmuch as $\sin x=\sin\frac{\pi}{2}=1$, if $\frac{\pi}{2}-x=0$.

644. If the series proceed entirely according to inverse powers of a symbol, or combination of symbols, then the primitive expression will be zero, when the symbol, or combination of symbols, is infinite, which, under such circumstances, is the arithmetical sum of the series: thus

the series for $\frac{1}{x+1}$, proceeding according to inverse powers of x, involves one of those powers in every term, and when x is infinite, its arithmetical sum is zero: the same is the case with the series for $\frac{1}{\sqrt{ax+x^2}}$, considered as the developement of $\frac{1}{x\sqrt{\left(1+\frac{a}{x}\right)}}$: if the series, though still arranged according to inverse powers of a symbol or combination of symbols, contain a term which is independent of it, or a term or terms which involve its positive powers, then under the same circumstances, the primitive expression will become equal to that term in one case and to infinity in the other, and those values are likewise the corresponding arithmetical sums of the series: thus the series for $\sqrt{\left(1+\frac{a}{x}\right)}$ has its first term equal to 1, which is its arithmetical sum when x is infinite: whilst the series for $\sqrt{(x^3+ax^2)}$, becomes infinite under the same circumstances, which is likewise its arithmetical sum; lastly, if the series in question proceed according to ascending powers of a symbol or combination of symbols, but involves a term or terms with its negative or inverse powers, then infinity is also the arithmetical sum of the series, when the symbol or combination of symbols is zero: thus the series for $\frac{1}{\sqrt{ax+x^2}}$, arranged according to ascending powers of x, is equal to infinity when x is zero, which is likewise the corresponding value of $\frac{1}{\sqrt{ax+x^2}}$, and the arithmetical sum of the series: in a similar manner, the series for cot x arranged according to ascending powers of x, is infinite under the same circumstances, and there is one term at least in it which involves a negative power of x.

Indications of series proceeding entirely according to odd or even powers of a symbol.

645. There are many other considerations which enable us to predict other properties of the series which are equivalent to particular expressions, before the developements themselves are effected: thus, if the expression, whose developement is required, remains unaltered upon a change of the sign {from + to − and from − to +}, of the symbol or combination of symbols, according to which the series is arranged, then it can involve its *even* powers only: if under the same circumstances, the expression changes its sign entirely, then it must involve its *odd* powers only: an example of the first kind would be furnished by the series for cos x, and of the second by the series for sin x: we should, therefore, be justified in assuming

$$\cos x = 1 + A_2 x^2 + A_4 x^4 + A_6 x^6 + \&c.$$

and

$$\sin x = A_1 x + A_3 x^3 + A_5 x^5 + \&c.$$

Indications of the form of a series from the number of its values.

646. It would appear, from the observations made at the conclusion of the last Chapter, that a series which is *completely* equivalent to an expression, must possess the same number of values with it, and no more: if the primitive expression, therefore, possesses only one value, the corresponding equivalent series must involve integral powers of its symbols only, whether ascending or descending; if the series possesses, or may be shewn from indirect considerations to possess, one term which has the same number of values with the primitive expression, then that term is either singular and does not arise from the same operation or operations which lead to the other terms of the series, or the symbolical portion of it upon which its multiple values depends, may be considered as a common factor of some or of all the other terms of the series, and may be conceived, therefore, to be multiplied into a series of rational terms; but if different terms of the series involve symbolical portions which admit of different multiple values, then under all circumstances, the series may be resolved into different products whose respective

pairs of factors are the *recipients*, Art. 435. of the multiple values and rational quantities or quantities with arithmetical values only, it being of course necessary that the number of different combinations of these multiple values should be equal to and not exceed the number of different values of the primitive expression whose developement is required.

Examples.

647. Thus, the series for

$$\frac{a+x}{a-x}, \quad \frac{a+bx+cx^2}{a+\beta x+\gamma x^2}, \quad a^x, \ \sin x, \ \cos x,$$

and other expressions admitting but of one value for the same values of the symbols, must proceed according to integral powers of x, or of any other symbol or combination of symbols according to which the series are arranged, whether ascending or descending: the series for $(a+x)^n$ has for its first term a^n, and since a^n is the *recipient* of the multiple values of $(a+x)^n$, when those values are more than one, and since no term in the series can be independent of a^n, it will follow, that the series required may be considered as the product of a^n and of a series of rational terms: in a similar manner the series for

$$\sqrt{(ax-x^2)}, \quad \frac{1}{\sqrt[3]{(ax+x^2)}},$$

$$(a-bx+cx^2)^{\frac{1}{n}} \text{ and } \sqrt[5]{\left(\frac{a-x}{a+x}\right)}$$

are the products of

$$\sqrt{(ax)}, \quad \frac{1}{\sqrt[3]{(ax)}}, \quad a^{\frac{1}{n}} \text{ and } \sqrt[5]{1}$$

respectively, into series proceeding according to integral powers of the symbols which are involved.

648. If we take any expression whatever, involving its symbols algebraically or transcendentally, which we desig-

Form of the series for an expression involving a symbol x, when x is replaced by $x+h$,

nate by u, and if we replace any one of its symbols, such as x, by $x + h$, and designate it under its new form by u', then we shall be justified in assuming the following form of the developement of u' proceeding according to powers of h: namely,

$$u' = u + c_1 h + c_2 h^2 + c_3 h^3 + \&c.$$

For in the first place, if we suppose $h = 0$, u' becomes identical with u, and the series which represents the developement becomes identical with its first term: consequently u, which is the primitive expression, must necessarily form the first term of the equivalent series, or the term which is independent of h.

In the second place, the series for u' can involve integral powers of h only; for otherwise u' would have more values than u, which is clearly impossible, inasmuch as no additional values are introduced into u by replacing x by $x + h$.

In the third place, this series can involve no negative powers of h, inasmuch as the terms which should involve them would become infinite when $h = 0$, and the series for u' would not, therefore, under such circumstances, become identical with u.

The course of reasoning to which this enquiry will conduct us, will shew that the powers of h will ascend regularly, coinciding with the series of natural numbers 1, 2, 3, &c. and also that when the coefficient of any one term of the series becomes zero, all those which follow it, will become zero likewise; this is a question, however, for subsequent consideration, and we are at liberty at present to allow the possibility of any one or more of those terms, becoming zero in any order whatever.

649. It will immediately follow from the assumption which we have made and justified, that

$$\frac{u' - u}{h} = c_1 + c_2 h + c_3 h^3 + c_4 h^4 + \&c,:$$

and inasmuch as the hypothesis that $h=0$, makes u' identical with u, $\frac{u'-u}{h}=\frac{0}{0}$, and reduces the corresponding series to its first term c_1, it will follow that

Expression for the coefficient of the second term of the preceding series.

$$\frac{u'-u}{h}=\frac{0}{0}=c_1.$$

650. We have explained in our Note on Art. 323. the meaning of expressions which become, under certain circumstances, $\frac{0}{0}$, and the principles upon which their real algebraical values, when different from zero and infinity, may be determined and exhibited: in addition to the examples which we have given there, we shall consider the value of this expression, when $u=x^n$.

Example.

In this case $\frac{u'-u}{h}=\frac{(x+h)^n-x^n}{h}$

$$=\frac{(x^n+nx^{n-1}h+n\frac{(n-1)}{1.2}x^{n-2}h^2+\text{\&c.})-x^n}{h}$$

$$=nx^{n-1}+\frac{n(n-1)}{1.2}x^{n-2}h+n\frac{(n-1)(n-2)}{1.2.3}x^{n-3}h^2+\text{\&c.}:$$

consequently, when $h=0$, we get

$$\frac{u'-u}{h}=\frac{0}{0}=nx^{n-1}=c_1.$$

If $x=1$, and $u=1^n$, and, therefore, $u'=(1+h)^n$, then we shall find

$$\frac{u'-u}{h}=\frac{0}{0}=n=c_1.$$

651. It follows from the preceding theory, that the determination of the value of $\frac{u'-u}{h}$, when it becomes $\frac{0}{0}$,

is equivalent to the determination of the coefficient of h in the series for u', and conversely: or that the same law which enables us to pass from u to c_1 in the developement of u', will enable us to pass likewise from u to the value of $\frac{u'-u}{h}$, when it becomes $\frac{0}{0}$: it is clearly a matter of indifference, therefore, what name we give to this law of derivation, or to the quantities c_1 or $\frac{u'-u}{h} = \frac{0}{0}$, which are thus derived, or whether we determine the value of c_1 from that of $\frac{u'-u}{h} = \frac{0}{0}$, or conversely.

Law of dependence of the coefficients of the several terms of the series upon the coefficient of the second term.

652. Without stopping to examine or discover this law of derivation, for all the forms of algebraical or transcendental expressions which may present themselves, an enquiry of great importance, but, in some respects, foreign to our present object, we shall proceed to investigate the essential connection between the determination of the coefficients of the second and of the other terms of the series for u'.

We have already shewn, that

$$u' = u + c_1 h + c_2 h^2 + c_3 h^3 + \&c. \ldots\ldots\ldots (1),$$

and if we replace h by h_1 and designate the corresponding value of u by u_1, we shall also get

$$u_1 = u + c_1 h_1 + c_2 h_1^2 + c_3 h^3 + \&c. \ldots\ldots\ldots (2):$$

if we subtract the second of these series from the first, we shall find

$$u' - u_1 = c_1 (h - h_1) + c_2 (h^2 - h_1^2) + c_3 (h^3 - h_1^3) + \&c.:$$

and if we divide both members of this equation by $h - h'$, we shall get

$$\frac{u' - u_1}{h - h_1} = c_1 + c_2 \cdot \frac{h^2 - h_1^2}{h - h_1} + c_3 \cdot \frac{h^3 - h_1^3}{h - h_1} + \&c.$$

$$= c_1 + c_2 (h + h_1) + c_3 (h^2 + h h_1 + h_1^2) + \&c.:$$

if we now suppose $h = h_1$, and therefore $u' = u_1$, we shall get

$$\frac{u' - u'}{h - h} = \frac{0}{0} = c_1 + 2c_2h + 3c_3h^2 + \&c. \ldots (3).$$

But we have already seen, that $\frac{u - u}{0} = \frac{u - u}{h - h} = c_1$; and it will follow, therefore, that $\frac{u' - u'}{h - h} = c_1{}^1$, where the symbol x, which is involved in c_1, is replaced by $x + h$; for it is obvious that $\frac{u - u}{h - h} = c_1$, will be converted into $\frac{u' - u'}{h - h} = c_1{}^1$, by merely replacing x by $x + h$; and inasmuch as $c_1{}^1$ is convertible into an equivalent series

$$c_1 + C_2h + C_3h^2 + C_4h^3 + \&c.,$$

in the same manner, and for the same reasons, that u' is convertible into

$$u + c_1h + c_2h^2 + c_3h^3 + \&c.,$$

we must have, therefore,

$$\begin{aligned} \frac{u' - u'}{h - h} &= c_1 + C_2h + C_3h^2 + C_4h^3 + \&c. \\ &= c_1 + 2c_2h + 3c_3h^2 + 4c_4h^3 + \&c.: \end{aligned}$$

and since these series are identical, it will follow that

$$C_2 = 2c_2, \quad C_3 = 3c_3, \quad C_4 = 4c_4, \quad \&c.$$

It is obvious, however, that C_2, C_3, C_4, &c. are derived from c_1, in the same manner, or by the same law, that c_1, c_2, c_3, &c. are derived from u: or in other words, if we can derive c_1 from u, we can derive, by the same law, C_2 from c_1, C_3 from c_2, C_4 from c_3, and so on, as far as we choose to proceed: it follows, therefore, that if we can determine c_1 from u_1, we can also determine C_2, C_3, C_4, &c.,

and, therefore, also c_2, c_3, c_4, and so on, the coefficients of the developement of u', as far as we think it requisite to determine them*.

Taylor's series.

* If we choose to adopt a conventional notation to express the successive derivation of these several coefficients from the primitive expression, the proposition in the text will resolve itself into the celebrated series of Taylor, which is the foundation of the Differential Calculus, and of the general theory of the developement of series.

Thus, if we *agree* to denote c_1 by Du, where Du does not denote the product of D into u, as in ordinary algebraical notation, but the expression *derived* from u, which is the algebraical value of $\frac{u-u}{h-h}=\frac{0}{0}$, then we shall find by the same law

$$C_2 = Dc_1 = D\,(Du),$$

replacing c_1 by Du: if we further agree to denote $D\,(Du)$ by D^2u (following the analogy of indices in algebra, but abandoning their usual interpretation), then we shall find

$$c_2 = \tfrac{1}{2}\,C_2 = \frac{1}{1.2}\,.\,D^2u:$$

in a similar manner, we find from the proposition in the text, that

$$C_3 = Dc_2 = D\left(\frac{D^2u}{1.2}\right),$$

$\left\{\text{replacing } c_2 \text{ by } \frac{D^2u}{1.2}\right\}$; and therefore $C_3 = \frac{D^3u}{1.2}$, if we replace $D\,(D^2u)$ by D^3u, in conformity with the analogy above-mentioned: consequently

$$c_3 = \frac{C_3}{3} = \frac{D^3u}{1.2.3}:$$

if we proceed in the same manner, we shall find

$$c_4 = \frac{D^4u}{1.2.3.4},\quad c_5 = \frac{D^5u}{1.2.3.4.5},$$

and so on, as far as we choose to proceed: we thus are enabled to represent symbolically the law which expresses the derivation of the several terms of the series for u', and to transform the series

$$u' = u + c^1h + c_2h^2 + c_3h^3 + c_4h^4 + \&c.$$

into $u' = u + Du\,.\,h + D^2u\,.\,\frac{h^2}{1.2} + D^3u\,.\,\frac{h^3}{1.2.3} + D^4u\,.\,\frac{h^4}{1.2.3.4} + \&c.$

which is the series of Taylor.

In order to exemplify it, let us apply it to the developement of u' when $u = x^n$. In this case $Du = nx^{n-1}$, and the law of derivation expressed in words would be as follows.

"Multiply the quantity (x^n) by the index of x, and diminish that index by 1."

653. It would thus appear, that if $c_1 = 0$, then c_2, c_3, c_4, and all the subsequent coefficients of the series would be zero likewise: in other words, unless $u' = u$, which can only take place when u, in its most simple form, does not involve x, the series for u' must involve, as far as it proceeds, a succession of powers of h, coinciding with the series of natural numbers 1, 2, 3, 4, 5, &c.

If any one coefficient of the series become zero, all those which follow it will be zero likewise.

654. In all expressions (u) involving x, there exists a value of $\frac{u' - u}{h} = \frac{0}{0}$, which is also the coefficient of h in the developement u'; and the law of derivation which connects this value with u, involves the law of derivation of the coefficients of all the other terms of the series for u'.

There is always a value of $\frac{u'-u}{h} = \frac{0}{0}$, when u involves x.

655. It will likewise follow, that there is no expression involving x, or dependent in any way upon it, to which there will not be an equivalent corresponding series when x is replaced by $x + h$: in other words, the existence of such developements is necessary, and not hypothetical.

There is always a series equivalent to u'.

656. The preceding conclusions, which are of great importance and generality, were obtained by the comparison of two series independently deduced, which were identical with each other: the same artifice may be employed in many other cases, to effect particular developements, without any direct reference to the preceding conclusions, or to

The determination of the coefficients of the series for $(1+x)^n$ from the coefficient of the second term.

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therefore, also,

$$D^2 u = n(n-1)x^{n-2};$$

for the index of x in Du is $n-1$:

$$D^3 u = n(n-1)(n-2)x^{n-3};$$

for the index of x in $D^2 u$ is $n-2$:

$$D^4 u = n(n-1)(n-2)(n-3)x^{n-4};$$

$$D^5 u = n(n-1)(n-2)(n-3)(n-4)x^{n-5}:$$

……………………………………………………………………

consequently,

$$u' = (x+h)^n = x^n + nx^{n-1}.h + n(n-1)x^{n-2}.\frac{h^2}{1.2}$$

$$+ n(n-1)(n-2)x^{n-3}.\frac{h^3}{1.2.3} + \&c.$$

the general reasonings connected with them: of this kind is the following method, which is the one most commonly used, of deducing the coefficients of the series for $(1+x)^n$, from our knowledge of the coefficient of its second term.

Let $(1+x)^n = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \&c. \ldots\ldots (1)$,

and, therefore, replacing x by x_1,

$$(1+x_1)^n = 1 + c_1 x_1 + c_2 x_1^2 + c_3 x_1^3 + \&c.:$$

consequently,

$$(1+x)^n - (1+x_1)^n = c_1(x-x_1) + c_2(x^2-x_1^2) + c_3(x^3-x_1^3) + \&c.$$

and, therefore,

$$\frac{(1+x)^n-(1+x_1)^n}{(1+x)-(1+x_1)} = c_1 + c_1(x+x_1) + c_2(x^2+xx_1+x_1^2) + \&c.$$

If we now suppose $x = x_1$, then assuming $\dfrac{v^n - v^n}{v - v} = nv^{n-1}$, we shall also have

$$n\,(1+x)^{n-1} = c_1 + 2c_1 x + 3c_2 x^2 + \&c.$$

Multiplying both members of this equation by $1+x$, we get

$$n\,(1+x)^n = c_1 + (2c_2 + c_1)\,x + (3c_2 + 2c_2)\,x^2 + \&c.$$
$$= n + nc_1 x + nc_2 x^2 + \&c.,$$

being equal to n multiplied into the original series which was assumed for $(1+x)^n$.

Having thus obtained two identical series for $n\,(1+x)^n$, we equate their corresponding terms, which gives us

$$c_1 = n:$$

$$2c_2 + c_1 = nc_1, \quad \text{and } \therefore \quad 2c_2 = nc_1 - c_1 = (n-1)c_1$$

$$= n\,(n-1), \quad \text{and } \therefore \quad c_2 = \frac{n\,(n-1)}{1\,.\,2}:$$

$$3c_3 + 2c_2 = nc_2, \quad \text{and } \therefore \quad 3c_3 = nc_2 - 2c_2$$

$$= (n-2)\,c_2 = \frac{n\,(n-1)\,(n-2)}{1\,.\,2},$$

and, consequently,

$$c_3 = \frac{n(n-1)(n-2)}{1.2.3}.$$

In a similar manner we may find

$$c_4 = \frac{n(n-1)(n-2)(n-3)}{1.2.3.4},$$

$$c_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5},$$

and so on as far as we please to proceed.

657. If it can be generally shewn, that $\frac{v^n - v^n}{v - v} = nv^{n-1}$ for all values of n, then it is obvious, that the preceding method of deducing the remaining coefficients of the series is equally general likewise: it remains to consider, therefore, the grounds upon which that proposition is founded.

Cases in which it can be directly proved that $\frac{v^n - v^n}{v - v} = nv^{n-}$

In the first place, if n be a whole number, then the actual division of $v^n - v_1{}^n$ by $v - v_1$ would terminate after n terms, which would be all of them equal to each other, and to v^{n-1}, if $v = v_1$: therefore under such circumstances

$$\frac{v^n - v^n}{v - v} = nv^{n-1}.$$

If we should replace n by the fraction $\frac{m}{p}$ in its lowest terms, and if we make $v^{\frac{1}{p}} = z$, and, therefore, $v^n = v^{\frac{m}{p}} = z^m$, we shall have

$$\frac{v^n - v_1{}^n}{v - v_1} = \frac{v^{\frac{m}{p}} - v_1{}^{\frac{m}{p}}}{v - v_1} = \frac{z^m - z_1{}^m}{z^p - z_1{}^p} = \frac{\left(\dfrac{z^m - z_1{}^m}{z - z_1}\right)}{\left(\dfrac{z^p - z_1{}^p}{z - z_1}\right)} = \frac{mz^{m-1}}{pz^{p-1}}$$

$$= \frac{m}{p} . z^{m-p} = \frac{m}{p} v^{\frac{m}{p}-1} = nv^{n-1},$$

if $v = v_1$.

If we should replace n by $-m$, where m is a whole or a fractional number, then we should have

$$\frac{v^{-m}-v_1^{-m}}{v-v_1}=-\frac{1}{v^m v_1^m}\cdot\frac{v^m-v_1^m}{v-v_1}=-\frac{1}{v^{2m}}\cdot mv^{m-1}$$

$$=-mv^{-m-1}=nv^{n-1},$$

if $v=v_1$.

The preceding conclusion generalized by the principle of the permanency of equivalent forms.

658. If, however, we should seek to pass from the value of $\frac{v^n-v^n}{v-v}$ when n is a whole number to its value when n is a general symbol, we should conclude by the principle of the permanency of equivalent forms (Art. 132.) that the values would be symbolically the same in both cases: and this conclusion can be deduced in no other manner: for though we are enabled to ascertain its truth for all values of n, which are finite numerical fractions, whether positive or negative, where the operation which the index designates admits of being defined or interpreted, by means of direct processes which depend upon such interpretations, yet all similar processes must necessarily fail from the very nature of the case when those interpretations have no existence, or if they do exist, are either not discoverable or not assigned.

No general proof of the binomial theorem which is independent of that principle.

659. The same observations must, more or less, apply to all demonstrations of the binomial theorem which are independent of the principle of the permanency of equivalent forms: for it is obvious, from the very nature of the case, that the developement of $(1+x)^n$ must depend upon our knowledge of the meaning of the operation which the index n designates: for under no other circumstances can we either perform the operation, if the developement required is to be effected by a direct process, nor determine the equivalence of any expression derived from it by known operations, to two identical series, which involve independently the indeterminate coefficients of the series required: it will follow, therefore, that the transition to the series

for $(1+x)^n$ under such circumstances must be effected by the principle referred to, inasmuch as the developements in all other cases must depend upon operations, which, however they may be varied, and in whatever manner disguised, are essentially connected with, and dependent upon, interpretations of specific values of the index*, (Art. 135).

Developement of a^x.

660. The indirect artifices which are generally employed in effecting developements by means of indeterminate coefficients, are more frequently resorted to for the purpose of shortening the process which would be otherwise requisite, than from any necessity of overcoming a difficulty which could not be conquered by other means: thus suppose it was required to investigate a series which should be equivalent to a^x, proceeding according to powers of x.

First process.

Let $a^x = 1 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.$

and, therefore,

$$a^h = 1 + A_1 h + A_2 h^2 + A_3 h^3 + \&c.:$$

also

$$a^{x+h} = 1 + A_1 (x+h) + A_2 (x+h)^2 + A_3 (x+h)^3 + \&c.:$$

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[*] The opinion expressed in the text is expressly opposed to the great authority of Euler, who, in a species of preface to a demonstration of the Binomial Theorem which is given in the Petersburg Acts for 1774, has denied the universality of the principle of the permanency of equivalent forms: it is not a little remarkable, however, that the validity of the demonstration which he has given, even if limited to those values of the index which are usually considered, is perhaps more than any other dependent upon the truth of the principle in question, and altogether incapable of being extended to general values of the index, without the aid of its authority.

The exception to the truth and universality of this principle which he quotes, is found in the very remarkable series

$$\frac{1-a^m}{1-a} + \frac{(1-a^m)(1-a^{m-1})}{1-a^2} + \frac{(1-a^m)(1-a^{m-1})(1-a^{m-2})}{1-a^3} + \&c.$$

the law of the series being sufficiently manifest from that of its three first terms: if m be a whole number, the sum of this series is m, which he says is not the case for other values: but he appears to have confounded together, as is very commonly the case, the algebraical and arithmetical sum of the series, the first of which only is involved in the principle in question.

but $a^x \times a^h = a^{x+h}$, and, therefore, also the product of the two series for a^x and a^h must be equal to the series for a^{x+h}: if this multiplication be effected, and the powers of $x + h$ in the series for a^{x+h} be developed, we shall get two identical series which are deduced independently of each other, by the comparison of whose corresponding terms, a series of equations may be obtained, from which the values of A_2, A_3, A_4, &c. may be determined in terms of A_1.

Second process.

The process which we have indicated above, however, is extremely tedious, and may be greatly shortened by the following artifice.

Assuming as before

$$a^x = 1 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.,$$

and multiplying both members of the equation by a, we shall get

$$\begin{aligned} a^{1+x} &= 1 + A_1 (1+x) + A_2 (1+x)^2 + A_3 (1+x)^3 + \&c. \\ &= 1 + A_1 + A_2 + A_3 + \&c. \\ &\quad + x (A_1 + 2A_2 + 3A_3 + \&c.) \\ &\quad + x^2 (A_2 + 3A_3 + \&c.) \\ &\quad + \&c. \\ &= a \times a^x = a + A_1 a x + A_2 a x^2 + A_3 a x^3 + \&c. \end{aligned}$$

Consequently, by equating corresponding terms and coefficients in these two identical series, we shall obtain

$$a = 1 + A_1 + A_2 + A_3 + \&c.,$$

and, therefore,

$$\begin{aligned} A_1 a &= A_1 + A_1^2 + A_1 A_2 + A_1 A_3 + \&c. \\ &= A_1 + 2A_2 + 3A_3 + 4A_4 + \&c. \end{aligned}$$

Consequently,

$$2A_2 = A_1^2, \qquad \text{and, therefore, } A_2 = \frac{A_1^2}{1.2},$$

$$3A_3 = A_1A_2 = \frac{A_1^3}{1.2}, \qquad \text{and, therefore, } A_3 = \frac{A_1^3}{1.2.3},$$

$$4A_4 = A_1A_3 = \frac{A_1^4}{1.2.3}, \text{ and, therefore, } A_4 = \frac{A_1^4}{1.2.3.4};$$

we thus get

$$a^x = 1 + A_1x + \frac{A_1^2x^2}{1.2} + \frac{A_1^3x^3}{1.2.3} + \frac{A_1^4x^4}{1.2.3.4} + \&c.$$

Determination of the coefficient of the second term in the series for a^x.

661. The preceding process of developement leaves the coefficient A_1 (which is the value of $\frac{a^x - a^x}{x - x}$ when $x = 0$) indeterminate: but inasmuch as

$$a = 1 + A_1 + A_2 + A_3 + \&c.$$

$$\text{or } a - 1 = A_1 + \frac{A_1^2}{1.2} + \frac{A_1^3}{1.2.3} + \&c.$$

it is obvious that there exists a series proceeding according to powers of $(a-1)$, which is equivalent to it: and inasmuch as

$$a = 1 + (a-1),$$

we have, therefore,

$$a^x = \{1 + (a-1)\}^x,$$

which, developed according to powers of $(a-1)$ by the binomial theorem, gives us

$$a^x = 1 + x(a-1) + \frac{x(x-1)}{1.2}(a-1)^2$$
$$+ \frac{x(x-1)(x-2)}{1.2.3}(a-1)^3 + \&c.:$$

if this series could be transformed, so as to proceed according to powers of x, it would necessarily be identical with the series for a^x, which we have given in the last Article: it will be sufficient however for our purpose, to determine the coefficient of the second term, which may be effected as follows.

The last terms of the numerators of the successive coefficients x, $x(x-1)$, $x(x-1)(x-2)$, &c., when the factors are multiplied together, are

$$x, \quad -x, \quad 1.2.x, \quad -1.2.3.x, \text{ \&c.},$$

and the corresponding parts of the entire coefficients which involve x, are

$$1, \ \frac{-1}{2}, \ \frac{1}{3}, \ \frac{-1}{4}, \ \frac{1}{5}, \ -\text{\&c.};$$

consequently the coefficient of the term of the transformed series required is

$$(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{\&c.}$$

which is therefore equal to A_1.

Determination of the value of a which makes $A_1 = 1$.

662. It is not our present intention to enquire into the conversion of this series for A_1 into another form, from which its arithmetical value may be calculated, whenever it may be necessary to determine it, nor to investigate the other important consequences connected with it, which will be more properly considered in the following Chapter: we shall merely notice the particular value of a, which makes $A_1 = 1$, and which produces a form of the series more commonly referred to than any other.

If we call e the value of a, which makes $A_1 = 1$, we shall have

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \text{\&c.}:$$

and if we make $x=1$, this series will give us

$$e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\&c.$$

a converging series, from which we may calculate the arithmetical value of e.

If we collect together (by aggregation) 14 terms of this series, we shall find

$$e=2.7182818,$$

a value which is accurate as far as the last figure.

Equation which connects e, a and A_1.

663. The numerical quantity just determined, is extensively employed in analytical enquiries, being assumed as the base of Napierian logarithms, which will be more particularly considered in the following Chapter: for the present we shall merely give the equation which connects e, a and A_1 with each other.

$$\text{Since } e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\&c.,$$

it will follow, replacing x by A_1, that

$$e^{A_1}=1+A_1+\frac{A_1^{\,2}}{1.2}+\frac{A_1^{\,3}}{1.2.3}+\&c.$$
$$=a.$$

Series for $\cos x$ and $\sin x$ in terms of x.

664. The exponential expressions for $\cos x$ and $\sin x$ will furnish the most obvious and immediate method for their developement, though it would be very easy to deduce it from other consequences of their definition, whether algebraical or geometrical*: thus

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* The following method of deducing the series for $\cos x$ and $\sin x$, is analogous to the one employed in Art. 660. for deducing the series for a^x.

Assume

$$\cos x=1+B_2x^2+B_4x^4+\&c.$$

$$\cos x = \frac{\epsilon^{x} + \epsilon^{-x}}{2}$$

$$= \tfrac{1}{2}\left\{1 + B_1 x + \frac{B_1^2 x^2}{1.2} + \frac{B_1^3 x^3}{1.2.3} + \frac{B_1^4 x^4}{1.2.3.4} + \&c.\right\}$$

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and, therefore, making $x = 1$,

$$\cos 1 = 1 + B_2 + B_4 + \&c.$$

$$\text{Also}\quad \sin x = B_1 x + B_3 x^3 + B_5 x^5 + \&c.$$

and, therefore, making $x = 1$,

$$\sin 1 = B_1 + B_3 + B_5 + \&c.$$

Consequently

$$\begin{aligned}
\cos(1+x) &= 1 + B_2 (1+x)^2 + B_4 (1+x)^4 + \&c.\\
&= 1 + B_2 + B_4 + B_6 + \&c.\\
&\quad + (2B_2 + 4B_4 + 6B_6 + \&c.)\,x\\
&\quad + \left(B_2 + \frac{3.4}{1.2}.B_4 + \frac{5.6}{1.2}.B_6 + \&c.\right) x^2 + \&c.\\
&= \cos 1 \cos x - \sin 1 \sin x, \quad \text{(Art. 480.)}\\
&= (1 + B_2 + B_4 + \&c.)\,(1 + B_2 x^2 + B_4 x^4 + \&c.)\\
&\quad - (B_1 + B_3 + B_5 + \&c.)\,(B_1 x + B_3 x^3 + B_5 x^5 + \&c.)\\
&= 1 + B_2 + B_4 + B_6 + \&c.\\
&\quad - (B_1^2 + B_1 B_3 + B_1 B_5 + \&c.)\,x\\
&\quad + (B_2 + B_2^2 + B_2 B_4 + \&c.)\,x^2 + \&c.
\end{aligned}$$

Equating corresponding terms of these two identical series, we shall get

$$2B_2 + 4B_4 + 6B_6 + \&c. \qquad = - B_1^2 - B_1 B_3 - B_1 B_5 - \&c.$$

$$B_2 + \frac{3.4}{1.2}.B_4 + \frac{5.6}{1.2}.B_6 + \&c. = B_2 + B_2^2 + B_2 B_4 + \&c.$$

Consequently,

$$2B_2 = - B_1^2, \qquad \text{and, therefore,}\ B_2 = \frac{-B_1^2}{1.2},$$

$$\frac{3.4}{1.2}.B_4 = B_2^2 = \frac{B_1^2}{1.2.1.2}, \qquad \text{and, therefore,}\ B_4 = \frac{B_1^4}{1.2.3.4},$$

$$\frac{5.6}{1.2}.B_6 = B_2 B_4 = -\frac{B_1^6}{1.2.1.2.3.4}, \ \text{and, therefore,}\ B_6 = \frac{-B_1^6}{1.2.3.4.5.6}$$

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4 B_4

$$+\tfrac{1}{2}\left\{1-B_1x+\frac{B_1^{\,2}x^2}{1.2}-\frac{B_1^{\,3}x^3}{1.2.3}+\frac{B_1^{\,4}x^4}{1.2.3.4}+\&c.\right\}$$

$$=1+\frac{B_1^{\,2}x^2}{1.2}+\frac{B_1^{\,4}x^4}{1.2.3.4}+\&c.$$

In a similar manner, we find $\sin x=\dfrac{\epsilon^{x}-\epsilon^{-x}}{2\sqrt{-1}}$

$$=\frac{1}{2\sqrt{-1}}\left\{1+B_1x+\frac{B_1^{\,2}x^2}{1.2}+\frac{B_1^{\,3}x^3}{1.2.3}+\frac{B_1^{\,4}x^4}{1.2.3.4}+\&c.\right\}$$

$$-\frac{1}{2\sqrt{-1}}\left\{1-B_1x+\frac{B_1^{\,2}x^2}{1.2}-\frac{B_1^{\,3}x^3}{1.2.3}+\frac{B_1^{\,4}x^4}{1.2.3.4}-\&c.\right\}$$

$$=\frac{1}{\sqrt{-1}}\left\{B_1x+\frac{B_1^{\,3}x^3}{1.2.3}+\frac{B_1^{\,5}x^5}{1.2.3.4.5}+\&c.\right\}.$$

In this second case, $\sqrt{-1}$ must divide B_1, $B_1^{\,3}$, $B_1^{\,5}$, &c. so as to leave the results unaffected with $\sqrt{-1}$, inasmuch as the series is necessarily rational in all its terms, possessing only one value for one value of x: if we make therefore $B_1=C\sqrt{-1}$, we shall get

$$\cos x=1-\frac{C^2x^2}{1.2}+\frac{C^4x^4}{1.2.3.4}-\&c.\ldots (1).$$

$$\sin x=Cx-\frac{C^3x^3}{1.2.3}+\frac{C^5x^5}{1.2.3.4.5}-\&c.\ldots (2).$$

$4B_4=-B_1B_3=\dfrac{B_1^{\,4}}{1.2.3}$, and, therefore, $B_3=\dfrac{-B_1^{\,3}}{1.2.3}$,

$6B_6=-B_1B_5=\dfrac{-B_1^{\,6}}{1.2.3.4.5}$, and, therefore, $B_5=\dfrac{B_1^{\,5}}{1.2.3.4.5}$,

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We thus get, if we replace B_1 by the symbol C used in the text, and with which it is identical,

$$\cos x=1-\frac{C^2x^2}{1.2}+\frac{C^4x^4}{1.2.3.4}-\frac{C^6x^6}{1.2.3.4.5.6}+\&c.$$

$$\sin x=Cx-\frac{C^3x^3}{1.2.3}+\frac{C^5x^5}{1.2.3.4.5}-\&c.$$

The quantity ϵ in ϵ^x and C in the series for cos x and sin x are indeterminate as far as the definitions of those quantities are concerned.

665. The particular value of ϵ in the exponential expressions for cos x and sin x, which constituted the algebraical definition of those quantities, and therefore its relation to e the base of Napierian logarithms, has not hitherto been assigned, and in no part of the preceding investigations has its determination been required: for it is obvious that the conditions which the definitions of sine and cosine, (whether algebraical or geometrical) impose, would be equally satisfied, whatever was the value of ϵ, which is so far therefore essentially indeterminate: in a similar manner, the series in the last Article for cos x and sin x will equally answer the conditions of those definitions, whatever be the value which we assign to C: for the first will remain unaltered, and the second will change its sign, upon a change of the sign of x from + to − or conversely: whilst the third condition

$$\cos^2 x + \sin^2 x = 1$$

is equally satisfied by all values of x, and therefore it is indifferent whether we replace x by Cx, or conversely, in the several terms of those series: it is obvious, therefore, that the determination of the value of C, if its determination be required, must be sought for upon principles which are independent of the definition of the sine and cosine of an angle, and of the developement which is founded upon them.

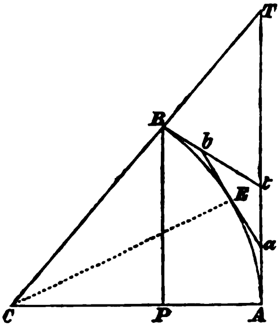
Transition from the geometrical value of an angle to its measure.

666. The fact is, that a given value of the sine or cosine corresponds to a determinate value of the angle, and we are thus enabled to determine the ratio which this angle bears to any other angle, or to a right angle: the transition, however, from such a value of the angle (considered as a real geometrical quantity) to its measure is perfectly arbitrary, inasmuch as any quantity may be assumed, as its measure, which increases or decreases in the same ratio with it: thus the arc of any *given* circle whatever, described round the angular point, which is intercepted by the lines which form the angle, may be assumed as its measure, if we adhere to the *same* circle for different values of the

angle: if we should assume however for this measure the ratio which the arc of this circle bears to its radius, which would give us the same quantity as the measure of the same angle, whatever was the radius of this circle, then we should possess a uniform measure, the magnitude of which would also remain the same, if the radius of this circle was equal to 1.

Essential connection between the assumed measure of an angle and the value of C.

667. But if we assume the radius of this circle to be equal to 1, then if ACB be any angle represented by x, the sides CP and BP of the right-angled triangle BCP, will represent the values of its cosine and sine, or will be arithmetically represented by $\cos x$ and $\sin x$, and the arc AEB will similarly represent the angle x: for under such circumstances, the ratios $\frac{CP}{AC}$, $\frac{BP}{AC}$, $\frac{AEB}{AC}$ will become *arithmetically* equal to CP, BP and AEB: it remains to shew in what manner this assumed measure of an angle will enable us to ascertain the value of C in the series for $\cos x$ and $\sin x$. (Art. 664).



In the first place, the ratio of the sine BP to the arc AEB or $\frac{BP}{AEB}$ is identical with $\frac{\sin x}{x}$, and therefore expressed by the series

$$C - \frac{C^3 x^2}{1.2.3} + \frac{C^5 x^4}{1.2.3.4.5} - \&c.,$$

the value of which, when x is zero, is C: in the second place, the ratio of the sine BP to the chord AB of the arc AEB, or

$\frac{BP}{AB}$, which is identical with $\frac{\sin x}{2\sin\frac{x}{2}}$, or $\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}}$ or

$\cos\frac{x}{2}$ is greater than $\frac{\sin x}{x}$, inasmuch as the chord AB is less than the arc AEB or x: in the third place, the ratio of the sine BP to the tangent AT or $\frac{BP}{AT}$ or $\frac{\sin x}{\tan x}$ or $\cos x$, is less than $\frac{\sin x}{x}$, inasmuch as AT is greater than the arc AEB or x*: it follows, therefore, that the value of $\frac{\sin x}{x}$ is included between those of $\cos\frac{x}{2}$ and $\cos x$ for all values of x; and inasmuch as $\cos\frac{x}{2}$ and $\cos x$ are both of them equal to 1, when $x=0$, it must necessarily follow, that $\frac{\sin x}{x}$ is equal to 1 under the same circumstances, which is likewise the value of C: or, in other words, the value of C which corresponds to the hypothesis which we have made respecting the measure of angles, and which is determined by it, is necessarily *unity*†.

The tangent of a circular arc is greater than the arc itself.

[*] This may be easily shewn from geometrical considerations: if from the extremity B of the arc AEB, we draw Bt a tangent meeting AT in t, then we have $At = Bt$: but Tt is greater than Bt, being opposite to a greater angle of the triangle BTt: therefore AT is greater than $At + Bt$: again, if from the middle point E of AEB, we draw a tangent to the arc, meeting At in a, and Bt in b, then we have $at + bt$ greater than ab, and therefore $At + Bt$, and *a fortiori* AT greater than $Aa + ab + Bb$: in a similar manner, if we bisect the arcs AE and BE, and from their middle points draw tangents meeting Aa, ab and Bb, then $Aa + ab + Bb$, and therefore *a fortiori* AT will be greater than the sum of the last formed tangents included between A and B: by continuing this process, we should increase the number, and diminish the magnitude, of these small tangents, until their sum should differ in length from the arc AEB by a quantity or line less than any that could be assigned, at the same time that it was necessarily less than AT: or, in other words, the arc AEB is necessarily less than the tangent AT.

[†] The conclusion in the text may likewise be considered as a consequence of the following more general proposition, and may be deduced in the same manner:

Proposition.

"If there be three quantities expressed by converging series proceeding according to powers of the same symbol, and if the first be necessarily greater than the second, and the second greater than the third, for the same value of that

668. If we refer to Art. 660., we find

$$\epsilon^x = 1 + B_1 x + \frac{B_1^2 x^2}{1.2} + \&c.$$

$$= 1 + C\sqrt{-1}\,x - \frac{C^2 x^2}{1.2} - \frac{C^3\sqrt{-1}.x^3}{1.2.3} + \frac{C^4 x^4}{1.2.3.4} + \&c.$$

$$= 1 + \sqrt{-1}.x - \frac{x^2}{1.2} - \frac{\sqrt{-1}.x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c.,$$

since $C = 1$: it will follow, therefore, from the equation given in Art. 663., that

$$\epsilon = e^{\sqrt{-1}},$$

---

that symbol, then if the first and third series have the same first term, the first term of the second series will be necessarily equal to it."

Thus if the converging series representing such quantities be

$$A_0 + A_1 x + A_2 x^2 + \&c.;$$

$$B_0 + B_1 x + B_2 x^2 + \&c.;$$

$$A_0 + a_1 x + a_2 x^2 + \&c.;$$

then values of x *exist* which will make their respective arithmetical values differ from their first terms by quantities less than any which can be assigned, Art. 326. and 327.: let such values be $A_0 + D$, $B_0 + \Delta$ and $A_0 + d$. Then, since $A_0 + D > B_0 + \Delta$, and $B_0 + \Delta >$ than $A_0 + d$, it will follow, that

$$A_0 - B_0 + (D - \Delta), \text{ and } (B_0 - A_0) + (\Delta - d)$$

are arithmetical values, having the same positive signs: if possible, let $B_0 = A_0 + b$ or $A_0 - b$: in the first case, the preceding expressions become

$$-b + (D - \Delta), \text{ and } \quad b + (\Delta - d),$$

and in the second

$$b + (D - \Delta), \text{ and } -b + (\Delta - d):$$

and inasmuch as these expressions are necessarily positive, it will follow that in the first case $D - \Delta$ must be greater than b, or D greater than $b + \Delta$, which is contrary to the hypothesis, inasmuch as D has been supposed less than any quantity which can be assigned: and, in the second case, $\Delta - d$ must be greater than b, or Δ greater than $b + d$, which is contrary to the hypothesis, since Δ has been supposed less than any quantity that can be assigned: it follows, therefore, that B_0 is necessarily equal to A_0, which was the proposition to be proved.

We

an equation which expresses the relation between ϵ and e which is the base of Napierian logarithms, the value of which has already been determined. Art. 662.

Ultimate forms of the exponential values of $\sin x$ and $\cos x$.

669. It will follow from hence, that

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2},$$

$$\text{and } \sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}};$$

the form under which the exponential expressions for the cosine and sine are commonly exhibited.

670. In a similar manner it will appear, that

$$\epsilon^x = e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\,\sin x;$$

We should have arrived immediately at the same conclusion, if we had supposed, as in the text, that x was equal to *zero*, or, in other words, if we had assumed *zero* as one of the arithmetical values of x, which might be equally considered with any others: such a conclusion, in fact, may be considered as a necessary consequence of the proposition which we have just demonstrated, and inasmuch as this consequence is usually the ultimate object of our investigations when this proposition is made use of, it is more convenient to refer at once to this consequence itself, than to the proposition upon which it depends.

It is very easy, however, if it should be required, to deduce the conclusion in the text, immediately from this proposition: for since $\cos\frac{x}{2}$ is greater than $\frac{\sin x}{x}$, and $\frac{\sin x}{x}$ greater than $\cos x$, and since

$$\cos\frac{x}{2} = 1 - \frac{C^2}{4}\cdot\frac{x^2}{1.2} + \frac{C^4}{16}\cdot\frac{x^4}{1.2.3.4} - \&c.$$

$$\frac{\sin x}{x} = C - \frac{C^3x^2}{1.2.3} + \frac{C^5x^4}{1.2.3.4.5},$$

$$\cos x = 1 - \frac{C^2x^2}{1.2} + \frac{C^4x^4}{1.2.3.4} - \&c.$$

it follows that $C = 1$: we equally conform, however, to the spirit of this proposition, though not to its form, if we begin by shewing that $\cos\frac{x}{2}$ and $\cos x$ are superior and inferior limits of $\frac{\sin x}{x}$, which become equal to each other, and to 1, when $x = 0$, and thence infer that the first term of the series for $\frac{\sin x}{x}$, proceeding according to powers of x, is 1.

$\epsilon^x = e^{x\sqrt{-1}}$

or the sign of affection ϵ^x or $\cos x + \sqrt{-1}\,\sin x$ may be replaced by $e^{x\sqrt{-1}}$: for these expressions are symbolically equivalent to each other, and consequently the assumption of the identity of their interpretation will give the same meaning to the same symbolical result, in whatever manner we may arrive at it.

The preceding investigations involve the most important elements of the general theory of the developement of series, and lead immediately to the fundamental propositions of the Differential Calculus: it is not our intention, however, to trace this connection further, and much less to venture upon the applications of a branch of algebraical science, of all others the most comprehensive, inasmuch as it completes the connection of Algebra with Geometry according to the most extensive meaning of the term, and also with nearly every other department of Natural Philosophy, the objects of which can be brought under the dominion of symbolical language.

Reversion of series.

671. When a quantity or symbol is expressed in a series proceeding according to powers of any other quantity or symbol, then the series is said to be *reversed*, when the second quantity or symbol is expressed in a series proceeding according to powers of the first: thus, if

$$y = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.,$$

then the *reversed* series gives

$$x = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \&c.:$$

in a similar manner, $\sin x$ and $\cos x$ have been expressed in series proceeding according to powers of x, and the corresponding *reversed* series would express the value of x in terms of $\sin x$ in one case, and of $\cos x$ in the other.

Example of the reversion of a series.

672. The *reversion* of series may be, in many cases, effected by means of indeterminate coefficients, aided by the polynomial theorem: thus, since

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \text{\&c.}$$

we may assume

$$x = A_0 \sin x + A_3 \sin^3 x + A_5 \sin^5 x + \text{\&c.}:$$

for it is obvious, that this series must be confined to odd powers of sin x, for the same reason that the first series was confined to odd powers of x (Art. 645): therefore,

$$\begin{aligned}
x &= A_0 \sin x + A_3 \sin^3 x + A_5 \sin^5 x + \text{\&c.} \\
-\frac{x^3}{1.2.3} &= -\frac{A_0^3}{1.2.3} \sin^3 x - \frac{3A_0^2 A_3}{1.2.3} \sin^5 x - \text{\&c.} \\
+\frac{x^5}{1.2.3.4.5} &= \qquad\qquad\qquad\qquad \frac{A_0^5}{1.2.3.4.5} \sin^5 x + \text{\&c.} \\
\cdots\cdots\cdots\cdots & \qquad\qquad\qquad\qquad \cdots\cdots\cdots\cdots
\end{aligned}$$

Consequently,

$$\sin x = A_0 \sin x + \left(A_3 - \frac{A_0^3}{1.2.3}\right) \sin^3 x$$

$$+ \left(A_5 - \frac{3A_0^2 A_3}{1.2.3} + \frac{A_5}{1.2.3.4.5}\right) \sin^5 x + \text{\&c.};$$

and since the members of this equation are identical with each other, we have

$$A_0 = 1,$$

$$A_3 - \frac{A_0^3}{1.2.3} = 0, \quad \text{and } \therefore A_3 = \frac{1}{1.2.3},$$

$$A_5 - \frac{3A_0 A_3}{1.2.3} + \frac{A_0^5}{1.2.3.4.5} = 0, \quad \text{and } \therefore A_5 = \frac{9}{1.2.3.4.5},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad\qquad \cdots\cdots\cdots\cdots\cdots$$

therefore,

$$x = \sin x + \frac{\sin^3 x}{1.2.3} + \frac{9 \sin^5 x}{1.2.3.4.5} + \text{\&c.}$$

673. If the series, whose reversion was required, involved a term which was independent of the symbol or combination of symbols according to whose powers it was arranged, then it would be found that a direct application of the method of indeterminate coefficients, such as has been made use of in the preceding example, without the aid of any other artifice, would fail to effect it: of this kind is the series

Cases in which the ordinary process for the reversion of series fails.

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.$$

which may be reversed, by such means, in a series proceeding according to powers of $1 - \cos x$, or versin x, but not according to powers of $\cos x$: of the same kind is the series considered in Art. 661.

$$a = 1 + A_1 + \frac{A_1^2}{1.2} + \frac{A_1^3}{1.2.3} + \&c.$$

which has been already reversed by an artifice, according to powers of $(a-1)$, if required to be reversed according to powers of a.

It is in consequence of the difficulty of determining the law of formation of the *reversed* series, by this method, in those cases in which it admits of application, and of its failure in others, that it is very rarely resorted to, being superseded by methods founded upon the Differential Calculus, the consideration of which, therefore, we feel compelled to omit.

674. There are many other applications of indeterminate coefficients, not immediately connected with the developement of series, which are extremely important, and which will be required on many occasions to meet the very various wants of algebraical science: the one which follows, is the only one which the proper limits of this Chapter will allow us to notice: it is the problem in which it is required to resolve an algebraical fraction with a compound denominator into a series of partial fractions whose

Resolution of rational algebraical fractions with compound denominators into partial fractions.

denominators are factors of the denominator of the primitive fraction.

We will begin with the consideration of one or two of its most simple cases.

Examples. To resolve the fraction $\frac{1}{1-x^2}$ into two others whose denominators are $1+x$ and $1-x$ respectively.

Assume,

$$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}.$$

Consequently, if we add the two partial fractions together, we shall get

$$\frac{1}{1-x^2} = \frac{A - Ax + B + Bx}{1-x^2};$$

and since these fractions are identical, their numerators are identical, and, therefore,

$$\left.\begin{aligned} A + B &= 1 \\ -A + B &= 0 \end{aligned}\right\}:$$

if we add these equations together, we get $2B = 1$, and, therefore, $B = \frac{1}{2}$: and if we subtract the second from the first, we get $2A = 1$, and, therefore, $A = \frac{1}{2}$: consequently,

$$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}.$$

Again, let it be required to resolve the fraction

$$\frac{6x^3 - 2x - 6}{(x^2+1)(x+2)(x-3)}$$

into partial fractions, whose denominators are x^2+1, $x+2$, and $x-3$.

Assume

$$\frac{6x^3-2x-6}{(x^2+1)\,(x+2)\,(x-3)} = \frac{A+Bx}{x^2+1} + \frac{C}{x+2} + \frac{D}{x-3}:$$

if we add together the partial fractions, and compare the several terms of the numerator of the resulting fraction with the corresponding and identical terms of the primitive fraction, we shall get

$$\begin{aligned} B+\ C+\ D&=6,\\ A-\ B-3C+2D&=0,\\ A+6B-\ C-\ D&=2,\\ 6A+3C-2D&=6; \end{aligned}$$

and by solving these equations, we shall find $A=1$, $B=1$, $C=2$, and $D=3$; and, therefore, the partial fractions required are

$$\frac{1+x}{1+x^2},\quad \frac{2}{x+2},\quad \text{and}\quad \frac{3}{x-3}.$$

Algebraical fractions distinguished into two classes as *proper* and *spurious*.

675. Fractions which are rational, and the terms of whose numerators and denominators are arranged according to the powers of a symbol such as x, have sometimes been distinguished into two classes, according as the highest power of x in their numerators was *not less* or *less* than the highest power of x in their denominators: fractions belonging to the first class have been called *spurious*, those belonging to the second *proper*, designations which, though not very appropriate, are frequently convenient, from superseding the necessity of a more specific description: all *spurious* fractions may be converted, by actual division, into a rational quotient involving no inverse power of x and a remainder which is a *proper* fraction, inasmuch as the division may be always continued without the

introduction of negative powers of x, until the index of the highest power of x in the remainder is less than the index of the highest power of x in the divisor.

Thus, $\frac{1+x^4}{1+x^2}$ may be converted by this process into the equivalent expression $x^2-1+\frac{2}{1+x^2}$: $\frac{x^5+7x^3+10x^2}{x^3-13x-12}$ into $x^2+20+\frac{22x^2+260x+240}{x^3-13x-12}$: and, similarly, in other cases.

When, therefore, the primitive fraction is *proper*, the partial fractions, whose algebraical sum is equal to it, must be proper likewise: and when the primitive fraction is *spurious*, we commence by reducing it to a quotient involving positive powers of x, and a remainder which is a proper fraction; and this remainder is subsequently resolved into proper partial fractions in the same manner as in the first case.

If, therefore, a factor of the denominator of the primitive proper fraction be $a+bx$, the corresponding partial fraction is $\frac{A}{a+bx}$. If this factor be $a+bx+cx^2$, the corresponding partial fraction is, or may be $\frac{A+Bx}{a+bx+cx^2}$.

If the factor be $(a+bx)^n$, the corresponding partial fraction is

$$\frac{A_0+A_1x+A_2x^2+\ldots A_{n-1}x^{n-1}}{(a+bx)^n},$$

which may be transformed into the equivalent form

$$\frac{a_0+a_1(a+bx)+a_2(a+bx)^2+\ldots a_{n-1}(a+bx)^{n-1}}{(a+bx)^n},$$

whose numerator may be made to involve the same succession of powers of x with the former: the separation of its several terms, striking out the common factors in each, gives us the n partial fractions

$$\frac{a_0}{(a+bx)^n}+\frac{a_1}{(a+bx)^{n-1}}+\frac{a_2}{(a+bx)^{n-2}}+\dots\frac{a_{n-1}}{a+bx}.$$

Process for determining independently the partial fraction corresponding to an assigned factor like $a+bx$.

676. The examples which we have given, sufficiently shew how very embarrassing the determination of such indeterminate coefficients would become if their number was considerable, from the number and complexity of the equations whose solutions are required: the following considerations will enable us to determine them successively, without the necessity of having recourse to the solution of such equations.

Let $\frac{M}{N}$ be a proper fraction, of the kind we are considering, and let one of the factors of N be $a+bx$, or $b\,(x-\alpha)$, where $\alpha=-\frac{a}{b}$; and let $N=(a+bx)\,Q$: consequently we may assume,

$$\frac{M}{N}=\frac{A}{a+bx}+\frac{P}{Q}=\frac{AQ+(a+bx)\,P}{N}.$$

Therefore

$$(a+bx)\,P=M-AQ,$$

$$\text{and }P=\frac{M-AQ}{a+bx}:$$

it will follow, therefore, that $M-AQ$ is divisible by $a+bx$ without a remainder, inasmuch as P is a rational numerator which involves no inverse power of x: if we make, therefore, $a+bx=0$, or $x=\alpha$, we shall find

$$P=\frac{0}{0},$$

or in other words, $M-AQ=0$, when $x=\alpha$: if we suppose, therefore, the values of M and Q when $x = a$, to be represented by m and q respectively, we shall have

$$m - Aq = 0, \quad \text{or } A = \frac{m}{q},$$

which is the value of the indeterminate quantity A, which was required to be found: and in a similar manner, the values of the numerators of the partial fractions corresponding to the other simple factors of the denominator of the primitive fraction may be determined.

Example.

As an example, let it be proposed to resolve the fraction

$$\frac{x^2}{(x+1)(x+2)(x+3)}$$

into partial fractions.

Assume

$$\frac{x^2}{(x+1)(x+2)(x+3)} = \frac{A_1}{x+1} + \frac{P_1}{Q_1}.$$

In this case $M = x^2$, $Q_1 = (x+2)(x+3)$ and $a_1 = -1$: therefore,

$$A_1 = \frac{m}{q_1} = \frac{1}{(2-1)(3-1)} = \frac{1}{2}.$$

Again, assume

$$\frac{x^2}{(x+1)(x+2)(x+3)} = \frac{A_2}{x+2} + \frac{P_2}{Q_2}.$$

In this case $M = x^2$, $Q_2 = (x+1)(x+3)$ and $a_2 = -2$: therefore,

$$A_2 = \frac{m}{q_2} = \frac{4}{(1-2)(3-2)} = -4.$$

Lastly, assume

$$\frac{x^2}{(x+1)(x+2)(x+3)} = \frac{A_3}{x+3} + \frac{P_3}{Q_3}.$$

In this case, $M = x^2$, $Q_3 = (x+1)(x+2)$ and $a_3 = -3$: therefore,

$$A_3 = \frac{m}{q_3} = \frac{9}{(1-3)(2-3)} = \frac{9}{2}.$$

Consequently,

$$\frac{x^2}{x^3+6x^2+11x+6}=\frac{1}{2(x+1)}-\frac{4}{2(x+2)}+\frac{9}{2(x+3)}.$$

Similar process for determining the partial fractions corresponding to a factor such as $(a+bx)^n$.

677. If two or more factors of the denominator of the primitive fraction be identical with each other, forming a factor of the form $(a+bx)^n$, we shall have

$$\frac{M}{N}=\frac{A_0}{(a+bx)^n}+\frac{A_1}{(a+bx)^{n-1}}+\ldots\frac{A_{n-1}}{a+bx}+\frac{P}{Q},$$

$$=\frac{A_0Q+A_1Q(a+bx)+\ldots A_{n-1}Q(a+bx)^{n-1}+P(a+bx)^n}{Q(a+bx)^n},$$

and, therefore,

$$P=\frac{M-A_0Q-A_1Q(a+bx)-\ldots\ldots-A_{n-1}Q(a+bx)^{n-1}}{(a+bx)^n};$$

a fraction whose numerator is completely divisible by $(a+bx)^n$; consequently, in the first place, $M-A_0Q$ is divisible by $a+bx$, since every other term of the numerator involves $a+bx$: therefore, $m-A_0q=0$, and $A_0=\frac{m}{q}$.

Again, make $\frac{M-A_0Q}{a+bx}=M_1$, and, therefore,

$$P=\frac{M_1-A_1Q-A_2Q(a+bx)\ldots-A_{n-1}Q(a+bx)^{n-2}}{(a+bx)^{n-1}},$$

which gives us $A_1=\frac{m_1}{q}$ for the same reason as before: if we proceed in a similar manner to make

$$\frac{M_1-A_1Q}{a+bx}=M_2,\quad \frac{M_2-A_2Q}{a+bx}=M_3,\quad \frac{M_3-A_3Q}{a+bx}=M_4,$$

and so on, we shall find,

$$A_2=\frac{m_2}{q},\quad A_3=\frac{m_3}{q},\quad A_4=\frac{m_4}{q},$$

and so on, until the successive numerators of the partial fractions corresponding to $(a + bx)^n$ are all of them determined.

Example. As an example, let it be required to resolve the fraction

$$\frac{x^2+2x+1}{(x-2)^2\,(x+4)}$$

into partial fractions.

Assume

$$\frac{x^2+2x+1}{(x-2)^2\,(x+4)} = \frac{A_0}{(x-2)^2} + \frac{A_1}{(x-2)} + \frac{P}{Q}.$$

In this case $M = x^2 + 2x + 1$, $Q = x + 4$, and $a = 2$: therefore,

$$A_0 = \frac{m}{q} = \frac{3}{2}.$$

Again,

$$M_1 = \frac{M - \frac{3}{2}Q}{x-2} = x + \frac{5}{2};$$

and, therefore,

$$A_1 = \frac{m_1}{q} = \frac{3}{4}.$$

Lastly, assuming

$$\frac{x^2+2x+1}{(x-2)^2\,(x+4)} = \frac{B}{x+4} + \frac{P_1}{Q_1},$$

we find $M = x^2 + 2x + 1$, $Q_1 = (x-2)^2$, and $a = -4$: therefore,

$$B = \frac{m}{q} = \frac{1}{4},$$

consequently,

$$\frac{x^2+2x+1}{x^3-12x+16}=\frac{3}{2\,(x-2)^2}+\frac{3}{4\,(x-2)}+\frac{1}{4\,(x+4)},$$

The application of this method of resolving a fraction with a compound denominator into partial and more simple fractions, pre-supposes a knowledge of one or more of the component factors of its denominator; and the resolution will be complete when all those factors are given or discoverable: if, however, the denominator of the primitive fraction be given, without its factors, their discovery, as well as their necessary existence, is inseparably connected with the general theory of equations and of their solution.

CHAP. XIV.

ON LOGARITHMS AND LOGARITHMIC TABLES, AND THEIR APPLICATIONS.

Definition of the term *logarithm.*

678. In the equation

$$a^x = n,$$

it is usual to give the general name of *logarithm*, Art. 364, to the index of that power of the same symbol a, which is equal to another symbol or quantity n: or in other words, x is called the *logarithm* of n to the base a.

System of logarithms.

679. A system of logarithms is the series of indices of the *same base* which correspond to the succession of values of n.

Napierian logarithms.

680. Thus, if the base be $e = 2.7182818\ldots$, then the corresponding indices form the Napierian system of logarithms, inasmuch as this was the base which was adopted by the inventor of logarithms, Lord Napier: it is this system which is almost exclusively used in algebraical formulæ; and the abbreviated form log n, always means the Napierian logarithm of n, unless the contrary be expressed, or be inferred from its position and usage.

Tabular logarithms.

681. If the base be the number 10, which is the *radix* or *base* of our scale of numeration and arithmetical notation, then the corresponding indices are called *tabular* logarithms, being such as are recorded in our ordinary tables of logarithms, and which are *exclusively* used in arithmetical calculations.

Mode of designating tabular logarithms.

682. The term log n, when used in such calculations, would always mean the *tabular* logarithm of n, unless the contrary was expressed: for its position and usage would, under such circumstances, determine its meaning, without the aid of any specific designation: in other cases, it might be conveniently designated by *t log*, where the letter *t* is prefixed to *log*, to indicate that it is the tabular logarithm which is thus designated.

Logarithms of other systems.

683. The quantities e and 10 are the only bases of logarithms which are used in analytical enquiries, though the general symbol a is employed to designate any base whatever, whether different from e and 10, or the same, the use of which may seem more adapted to the generality of algebraical investigations.

There is only one arithmetical value of the logarithm of the same number or quantity.

684. If in the equation $a^x = n$, there are more symbolical values of x than one, which satisfy the required conditions, they are equally logarithms* of n: it is very easily shewn, however, that there is only one *arithmetical* value of x which makes a^x equal to an arithmetical value of n: for if y be such a value, different from x, then $a^x = n$ and $a^y = n$, and, therefore, $\frac{a^x}{a^y} = a^{x-y} = 1$, an equation which can only be arithmetically satisfied by making $x - y = 0$; or, in other words, x is equal to y and identical with it: we shall confine our attention, in the first instance, entirely to such arithmetical logarithms, and afterwards proceed to consider symbolical logarithms, as distinguished from the former.

Properties of logarithms.

685. The properties of logarithms are the properties of the indices of the same symbol: thus if $a^x = n$, and $a^{x'} = n'$, then $a^{x+x'} = nn'$, $a^{x-x'} = \frac{n}{n'}$, $a^{px} = n^p$, and $a^{\frac{x}{p}} = n^{\frac{1}{p}}$: therefore, if x be denoted by log′ n (where log′ is different from *log* or *t log*, unless $a = e$ or $a = 10$), and x' by log′ n, we shall have

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* The term logarithm is used strictly in the sense of this definition, without any reference to the very limited meaning of it which is noticed in Art. 364.

(1) Log′ $nn' = \log' n + \log' n'$, or the logarithm of the product of two numbers or quantities, is equal to the sum of the logarithms of the two factors, and conversely.

(2) Log′ $\frac{n}{n'} = \log' n - \log' n'$, or the logarithm of the quotient of two numbers or quantities, is the logarithm of the dividend diminished by the logarithm of the divisor, and conversely.

(3) Log′ $n^p = p \log' n$, or the logarithm of the p^{th}, or any power of a number is found by multiplying p, or the index of the power, into the logarithm of the number, and conversely.

(4) Log′ $n^{\frac{1}{p}} = \frac{1}{p} \log' n$: or the logarithm of the p^{th}, or any *root* of a number is found by dividing the logarithm of the number by the number which expresses the denomination of the root, and conversely.

Examples. 686. Thus if $n = 37$, and $n' = 185$, then we shall find from the tables,

$$(\alpha)\quad \text{t log } 37 \times 185 = \text{t log } 37 + \text{t log } 185$$

$$= 1.5682017 + 2.2671717$$

$$= 3.8353733 = \text{t log } 6845.$$

$$(\beta)\quad \text{t log } \frac{185}{37} = \text{t log } 185 - \text{t log } 37$$

$$= .6989700 = \text{t log } 5.$$

$$(\gamma)\quad \text{t log } 37^2 = 2 \text{ t log } 37$$

$$= 3.1364034 = \text{t log } 1369.$$

$$(\delta)\quad \text{t log } 37^3 = 3 \text{ t log } 37 = 4.7046051 = \text{t log } 50653.$$

(ε) $\text{t} \log \sqrt{185} = \frac{1}{2} \text{t} \log 185$

$$= 1.1335858 = \text{t} \log 13.611.$$

(η) $\text{t} \log \sqrt[3]{185} = \frac{1}{3} \text{t} \log 185$

$$= .7557239 = \text{t} \log 5.6980.$$

(ζ) $\text{t} \log \sqrt[5]{185} = \frac{1}{5} \text{t} \log 185$

$$= .4534343 = \text{t} \log 2.8408.$$

Arithmetical operations performed by logarithms adapted to the same system.

687. If the logarithms of all numbers, whether whole or decimal, which belong to the same system, were registered in tables, it is obvious that the operations of the multiplication and division of such numbers would be performed by means of the addition and subtraction of their logarithms, and that the operations of raising any powers, and extracting any roots of such numbers, would be performed by means of multiplying and dividing the logarithms of the numbers by the numbers which express the denominations of the powers and roots required: for by such means we obtain the logarithms of the results, and consequently by a reference to such tables we obtain the corresponding results themselves.

Superior convenience and brevity of tables of logarithms to base 10.

688. A very little consideration, however, would shew that the extent of tables which would be required to comprehend a succession of the logarithms of all numbers, both whole and decimal, even if confined within the narrowest limits which are required for the purposes of calculation, would be much too great to be either easily registered or referred to: and this is an objection which applies to the logarithms in all systems whose bases do not coincide with the radix of the scale of notation: if, however, the base be 10, the logarithms of all numbers or quantities expressed by the formulæ $10^n \times N$ and $\frac{N}{10^n}$, will be at once known from the logarithm of N: for $\text{t} \log 10^n \times N = \text{t} \log 10^n + \text{t} \log N = n + \text{t} \log N$, and also $\text{t} \log \frac{N}{10^n} =$

$\text{t} \log N - \text{t} \log 10^n = \text{t} \log N - n$, inasmuch as n is the logarithm of 10^n to the base 10: it follows from hence, that we are required in this system to register the succession of integral numbers only, inasmuch as the logarithms of all numbers, whether whole or decimal, which are expressed by the same significant digits, similarly placed with respect to each other, are known from each other.

Exemplified.

$$\begin{aligned}
\text{Thus} \qquad & \text{t} \log 96498 = 4.9845228,\\
\text{t} \log 96498 \times 10 = {} & \text{t} \log 964980 = 5.9845228,\\
\text{t} \log 96498 \times 10^2 = {} & \text{t} \log 9649800 = 6.9845228,\\
\text{t} \log \frac{96498}{10} = {} & \text{t} \log 9649.8 = 3.9845228,\\
\text{t} \log \frac{96498}{10^2} = {} & \text{t} \log 964.98 = 2.9845228,\\
\text{t} \log \frac{96498}{10^3} = {} & \text{t} \log 96.498 = 1.9845228,\\
\text{t} \log \frac{96498}{10^4} = {} & \text{t} \log 9.6498 = .9845228,\\
\text{t} \log \frac{96498}{10^5} = {} & \text{t} \log .96498 = \overline{1}.9845228,\\
\text{t} \log \frac{96498}{10^6} = {} & \text{t} \log .096498 = \overline{2}.9845228.
\end{aligned}$$

……

In the two last cases, the sign — is placed above the number 1 in one case, and above the number 2 in the other, to shew that it must be subtracted from the decimal part of the logarithm: in other words, the logarithms of decimal numbers less than 1 are *negative*, but are expressed in this manner, in order to preserve the decimal part of the logarithm the same, for all numbers expressed by the same significant digits in the same order.

Mantissæ and *characteristics:* their meaning.

689. The decimal part of a logarithm, which remains unaltered for all numbers or quantities which only differ from each other in being multiplied or divided by powers of 10, has been denominated by authors who have used the Latin language, the *mantissa*, a term which we shall retain, in the absence of any simple designation which could be supplied by our own language: the integral part is called the *characteristic:* if the number be between 1 and 10, its logarithm is between 0 and 1, and the *characteristic* is 0: if the number be between 10 and 10^2, its logarithm is between 1 and 2, and its *characteristic* is 1: if the number be between 10^2 and 10^3, its logarithm is between 2 and 3, and its *characteristic* is 2, and so on, the number which forms the characteristic being *less by unity* than the number of integral places in the number of whose logarithm it is a part: if the number is between 1 and $\frac{1}{10}$, the characteristic is -1: if the number is between $\frac{1}{10}$ and $\frac{1}{10^2}$, its characteristic is -2: and so on for other fractions or equivalent decimals in this descending series.

Tables of the logarithms of numbers contain *mantissæ* only.

690. It is for the reasons above-mentioned, that the tables give the *mantissæ* only of logarithms with the significant digits in their proper order of the corresponding numbers, inasmuch as the *characteristic* may be always supplied from the number of integral places in the number whose logarithm is sought for, or in case it has no integral places, from the number of zeros (if any) which succeed the decimal point, the negative characteristic exceeding by 1 the number of such zeros: thus the *mantissa* of the logarithm of 53399 is .7275331, which is all that is given by the tables: the complete logarithm is 4.7275331, where 4 is less by 1 than the number of integral places in the corresponding number: in a similar manner the complete logarithm of .00053399 is $\bar{4}.7275331$ where 4 exceeds by 1 the number of zeros which immediately follow the decimal point.

Tables of logarithmic sines, co-sines, tangents, &c.

691. Inasmuch as the sines, cosines, tangents, secants, &c. of angles enter into formulæ which are the subjects of calculation equally with other symbols possessing assigned numerical values, a register of the logarithms of their successive numerical values becomes equally necessary with that of the logarithms of the series of natural numbers: tables of *natural* sines, cosines, tangents, co-tangents, secants, co-secants, &c. will contain the successive numerical values of the sine, cosine, tangent, co-tangent, secant, co-secant of every minute (and in some cases of every second) of the quadrant from $1'$ to 45^0, and consequently as far as 90^0, if taken in an inverse order, when the sine is replaced by the cosine, the tangent by the co-tangent, and the secant by the co-secant: tables of logarithmic sines and cosines, tangents and co-tangents, secants and co-secants, will contain the logarithms of the natural values *increased by the number* 10, arranged in the same order, each page of the natural values being opposed to a corresponding page of the logarithmic values.

The reason why the logarithms of the numerical values of the sines, cosines, &c. are increased by 10.

692. A very little consideration will shew the great convenience, for the purposes of calculation, of increasing the logarithms of the goniometrical quantities, as recorded in the tables, by the number 10: for the natural values of the sines and cosines are included between 0 and 1, and the *characteristics* of their logarithms are therefore negative: thus:

$\sin 1' = .0002909$: its natural logarithm $\bar{4}.4637261$

$\sin 1^0 = .0174524$ .... $\bar{2}.2418553$

$\sin 50^0 = .7660444$ .... $\bar{1}.8842540$

$\cos 30' = .9999619$ .... $\bar{1}.9999836$

$\cos 30^0 = .8660254$ .... $\bar{1}.9375306$

$\cos 85^0 = .0871557$ .... $\bar{2}.9402960$

If such logarithms were registered in tables, their *characteristics* and *mantissæ* would have different signs, and great confusion would be thus occasioned in the calculation by means of them of the values of those formulæ in which such quantities occurred, without any reference to the apparent violation of propriety in the necessary introduction of negative quantities (with independent signs) into processes which are strictly arithmetical: it is for these reasons, that their natural logarithms are increased by 10, and the logarithms of sin 1′, sin 1^0, sin 50^0, cos 30′, cos 30^0, cos 85^0, present themselves, therefore, in the tables with the following values:

6.4637261, 8.2418553, 9.8842540,
9.9999836, 9.9375306, 8.9402960.

Transition from the tabular logarithms of sines, cosines, &c. to their natural logarithms, and conversely.

693. The number 10 is the logarithm of 10^{10}, and the *registered* logarithms of the sines, cosines, and other goniometrical quantities, are the natural logarithms of those quantities multiplied by 10^{10}: we might consider the tabular sines, cosines, tangents, &c. as the natural sines, cosines, tangents, &c. multiplied by 10^{10}, or 10000000000, as is most commonly done[*]; but in all cases we may replace the natural logarithms of such quantities, when they occur in formulæ reduced to logarithmic computation, by the tabular logarithm diminished by 10: thus the natural logarithm of cos θ will be replaced by (t log cos $\theta - 10$), that of $\cos^2 \theta$ by 2 (t log cos $\theta - 10$), and similarly in other cases: when this is done, the formulæ may be afterwards adjusted to arithmetical computation, in any manner which may be considered most convenient.

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[*] Such a view of the meaning of tabular sines, cosines, tangents, &c. would be adapted to the ordinary definitions of those terms: if we refer to the figure in Art. 465., AD is defined to be cosine, and CD the sine of the angle BAC, in a circle whose radius is AB: in a similar manner if we refer to the figure in Art. 488., BT is defined to be the tangent, and AT the secant of the angle BAC in a circle whose radius is AB: if the radius of this circle be 1, the sines, cosines, tangents, and secants thus defined will coincide with the sines, cosines, tangents and secants of the same angle, as determined by our definition (Art. 465. and 466.); if the radius be 10^{10}, the sines, cosines, tangents and secants corresponding, will coincide with the *tabular* sines, cosines, tangents and secants, whose logarithms to the base 10, are registered in the tables.

Examples.

694. The following are examples of the applications of logarithms to the computation of formulæ involving goniometrical quantities.

Given the hypothenuse and the angle at the base of a right-angled triangle, to find the sides.

(1) If $x = 393 \cos 72^{0}.9'$, and $y = 393 \sin 72^{0}.9'$, to find the values of x and y.

$$\log x = \log 393 + \log \cos 72^{0}.9'$$

$$= \log 393 + (\text{t} \log \cos 72^{0}.9' - 10)$$

$$\begin{array}{rl} \log 393 = & 2.5943926 \\ \text{t} \log \cos 72^{0}.9' = & 9.4864674 \\ \hline & 12.0808600 \\ & 10 \\ \hline \log 120.467 = & 2.0808600 \end{array}$$

and, therefore, $x = 120.467$

$$\log y = \log 393 + (\text{t} \log \sin 72^{0}.9' - 10)$$

$$\begin{array}{rl} \log 393 = & 2.5943926 \\ \text{t} \log \sin 72^{0}.9' = & 9.9786198 \\ \hline & 12.5730124 \\ & 10 \\ \hline \log 374.13 = & 2.5730124 \end{array}$$

and, therefore, $y = 374.13$

In this case x and y are the base and perpendicular of a right-angled triangle whose hypothenuse is 393, and the angle at whose base is $72^{0}.9'$ (Art. 466.)

Given two sides of a triangle and the angle opposite to the greater: to find the remaining angles and side.

(2) The sides of a triangle are 17.09 and 93.451, and the angle opposite to the greater of them is $93^{0}.16'$: to find the angle opposite to the less.

In this case

$$\sin A = \frac{a}{b} \cdot \sin B = \frac{17.09}{93.451} \times \sin 93^{0}.16' = \frac{17.09}{93.451} \times \sin 86^{0}.44'.$$

(Art. 523 and 478).

Consequently,

$$(\text{t} \log \sin A - 10) = \log 17.09$$

$$+ (\text{t} \log \sin 86^{0}.44' - 10) - \log 93.451;$$

or striking out 10 from both sides of the equation,

$$\text{t}\log\sin A = \log 17.09 + \text{t}\log\sin 86^\circ.44' - \log 93.451:$$

$$\begin{aligned}
\log 17.09 &= 1.2327421\\
\text{t}\log\sin 86^\circ.44' &= \underline{9.9992938}\\
&\ 11.2320359\\
\log 93.451 &= \underline{1.9705840}\\
\text{t}\log\sin 10^\circ.31' &= \underline{9.2614519}
\end{aligned}$$

and, therefore, $A = 10^\circ.31'$.

If the remaining angle and side of the triangle be required, we shall find $C = 180^\circ - 93^\circ.16' - 10^\circ.31' = 76^\circ.13'$.

Also $c = \dfrac{a\sin C}{\sin A}$, and, therefore,

$$\begin{aligned}
\log c &= \log a + (\text{t}\log\sin C - 10) - (\text{t}\log\sin A - 10)\\
&= \log a + \text{t}\log\sin C - \text{t}\log\sin A,
\end{aligned}$$

$$\begin{aligned}
\log 17.09 &= 1.2327421\\
\text{t}\log\sin 76^\circ.13' &= \underline{9.9873103}\\
&\ 11.2200524\\
\text{t}\log\sin 10^\circ.31' &= \underline{9.2614519}\\
\log 90.908 &= 1.9586005
\end{aligned}$$

and, therefore, $c = 90.908$.

(3) If two sides of a triangle be 27.04 and 74.67, and the angle included between them be $117^\circ.20'$, to find the remaining side and angles.

Given two sides and the included angle of a triangle: to find the remaining parts.

The formulæ required for the solution of this example are given Art. 525.

Since

$$\tan\frac{(A-B)}{2} = \frac{(a-b)}{a+b}\tan\frac{(A+B)}{2}\,[*],$$

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[*] If $\tan\theta = \frac{a}{b}$, then $\tan(\theta - 45^\circ) = \frac{a-b}{a+b}$, and, therefore, $\tan\left(\frac{A-B}{2}\right) = \tan(\theta - 45^\circ)\tan\left(\frac{A+B}{2}\right)$, a variation of form which has sometimes been given, but which in no respect shortens the calculation of the remaining angles.

we have

$$\text{t}\log\tan\frac{(A-B)}{2} - 10 = \log(a-b)$$

$$+\left\{\text{t}\log\tan\frac{(A+B)}{2} - 10\right\} - \log(a+b),$$

or

$$\text{t}\log\tan\frac{(A-B)}{2} = \log(a-b)$$

$$+\,\text{t}\log\tan\frac{(A+B)}{2} - \log(a+b):$$

but

$$a-b = 47.63,\quad a+b = 101.71,\quad \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2} = 31^0.20':$$

and

$$\begin{array}{rr}
\log 47.63 = & 1.6778806 \\
\text{t}\log\tan 31^0.20' = & \underline{9.7844784} \\
 & 11.4623590 \\
\log 101.71 = & \underline{2.0073637} \\
\text{t}\log\tan 15^0.55' = & 9.4549953
\end{array}$$

$$\therefore\ \frac{(A+B)}{2} + \frac{(A-B)}{2} = A = 47^0.15',$$

$$\frac{(A+B)}{2} - \frac{(A-B)}{2} = B = 15^0.25'.$$

Also, $c = \dfrac{a\sin C}{\sin A}$, and, therefore,

$$\log c = \log a + \text{t}\log\sin 117^0.20' - \text{t}\log\sin 47^0.15'$$

$$= \log 74.67 + \text{t}\log\sin 62^0.40' - \text{t}\log\sin 47^0.15',$$

$$\begin{array}{rr}
\log 74.67 = & 1.8731462 \\
\text{t}\log\sin 62^0.40' = & \underline{9.9485852} \\
 & 11.8217314 \\
\log\sin 47^0.15' = & \underline{9.8658868} \\
\log 90.333 = & \underline{1.9558446}
\end{array}$$

and, therefore, $c = 90.333$.

(4) The three sides of a triangle are 107.9, 193.4, and 217.12: to find the three angles of the triangle.

It appears from the formulá given in Art. 547, that

Given three sides of a triangle, to find the three angles.

$$\sin A = \frac{2\sqrt{\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)\right\}}}{bc}$$

$$= \frac{2N}{bc} = \frac{2\sqrt{\{s(s-a)\,(s-b)\,(s-c)\}}}{bc}, \text{ if } s = \frac{a+b+c}{2}.$$

Consequently,

$$\text{t}\log\sin A - 10 = \tfrac{1}{2}\,\{\log s + \log(s-a) + \log(s-b) + \log(s-c)\}$$
$$+ \log 2 - \log b - \log c$$

$\log s = \log 259.21 = 2.4136518$
$\log(s-a) = \log 151.31 = 2.1798676$
$\log(s-b) = \log\ \ 65.81 = 1.8182919$
$\log(s-c) = \log\ \ 42.09 = 1.6241789$

2)8.0359902

4.0179951
$\log 2 = \ldots\ldots\ldots$ .3010300
adding 10 14.3190251

$\left.\begin{array}{l}\log b = \log 193.4\ = \ldots\ldots\ldots\ldots\ 2.2864565\\ \log c = \log 217.12 = \ldots\ldots\ldots\ldots\ 2.3366998\end{array}\right\}$

$\log b + \log c = 4.6231563$

$\text{t}\log\sin A = \text{t}\log\sin 29^{0}.46' = 9.6958688$
or $A = 29^{0}.46'$.

Again, $\sin B = \dfrac{2N}{ac}$, and, therefore, as before,

$\log 2 + \log N + 10 = 14.3190251$

$\left.\begin{array}{l}\log a = \log 107.\ 9 \ldots\ldots\ldots\ldots\ 2.0330214\\ \log c = \log 217.12 \ldots\ldots\ldots\ldots\ 2.3366998\end{array}\right\}$

$\log a + \log c = 4.3697212$

$\text{t}\log\sin B = \text{t}\log\sin 62^{0}.51' = 9.9493039$ or $B = 62^{0}.51'$.

The third angle C, may be determined by the formula $\sin C = \dfrac{2N}{ab}$, in the same manner as the two others: but more immediately by subtracting $A + B$ from 180^{0}, which gives $C = 87^{0}.26'$.

Expressions adapted or not adapted to logarithmic computation.

695. The three angles of the triangle in this problem might be determined likewise from the fundamental formulæ, (Art. 527.)

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \cos B = \frac{a^2+c^2-b^2}{2ac},$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}:$$

but inasmuch as the squares of a, b, c, must be determined separately (logarithmically or otherwise) before we can determine the numerical values of the numerators of those fractions, and consequently before we can proceed with the logarithmic computation of the fractions themselves, such formulæ are said *not* to present themselves in a state which is equally adapted to logarithmic computation with those which have been made use of above.

Formulæ generally are said to be adapted to logarithmic computation, when they consist of products, quotients, roots or powers of easily calculated terms: it is in this sense that the formula $\frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{bc}$ is said to be adapted to logarithmic computation: but if the component members of this or any other formula required a mixed application of logarithmic and numerical computation, in order to determine their values, they might be said to be *not* adapted to logarithmic computation: the phrase, however, and its usage, is very vague and indefinite, inasmuch as it does not determine absolutely the conditions of *greatest convenience:* in other words, a formula which is not adapted to logarithmic computation, according to the technical meaning of the term, may admit, in many cases, of more rapid computation, by mixed or even by merely arithmetical means, than one which is so: the selection, therefore, of one or of the other, when both are within our reach, must be determined by the judgement and experience, and sometimes by the *taste*, of the computer.

Formula for the logarithmic computation of $a+b$.

696. A formula, like $a+b$, consisting of two terms, one or both of which are powers or roots, or composed of factors in any way whatever, so as to admit of or require logarithmic computation, may be itself likewise reduced to logarithmic computation in the following manner: since

$$a+b=a\left(1+\frac{b}{a}\right),$$

if we make $\tan^2\theta=\frac{b}{a}$, we get

$$a+b=a\,(1+\tan^2\theta)=a\sec^2\theta \text{ or } \frac{a}{\cos^2\theta}:$$

the process of computation will then be as follows:

$$\text{t}\log\tan\theta=\tfrac{1}{2}(20+\log b-\log a),$$

$$\log(a+b)=\log a+2\,\text{t}\log\sec\theta-20,$$

$$\text{or}=\log a+20-2\,\text{t}\log\cos\theta.$$

Example.

697. An example of the application of this formula may be derived from the solution of the following problem.

If the two sides of a triangle be 729 and 340.5, and the included angle 76°.24′, to find the third side without the intervention of the two angles.

Since

$$\begin{aligned} c^2 &= a^2+b^2-2ab\cos C \qquad \text{(Art. 527.)}\\ &= a^2-2ab+b^2+2ab-2ab\cos C\\ &= (a-b)^2+2ab\,(1-\cos C)\\ &= (a-b)^2\left\{1+\frac{4ab}{(a-b)^2}\cdot\sin^2\frac{C}{2}\right\}: \qquad \text{(Art. 482.)} \end{aligned}$$

make $\tan^2\theta=\frac{4ab}{(a-b)^2}\cdot\sin^2\frac{C}{2}$, and, therefore,

$$\begin{aligned} \text{t}\log\tan\theta &= \tfrac{1}{2}(\log a+\log b)\\ &\quad +\log 2+\text{t}\log\sin\frac{C}{2}-\log(a-b). \end{aligned}$$

$$\begin{array}{rr}
\log a = \log 729 \;= & 2.8627275 \\
\log b = \log 340.5 = & 2.5321171 \\
\hline
 & 2)5.3948446 \\
\hline
 & 2.6974223 \\
\log 2 & .3010300 \\
\text{t} \log \sin \frac{C}{2} = \text{t} \log \sin \; 38^{0}.12' = & 9.7912754 \\
\hline
 & 12.7807277 \\
\log (a-b) = \log 388.5 = & 2.5893910 \\
\hline
\text{t} \log \tan 57^{0}.46' = & 10.2003367 \\
\hline
\end{array}$$

Also $\log c = \log (a-b) + \text{t} \log \sec \theta - 10$

$$\begin{array}{rr}
\log (a-b) = & 2.5893910 \\
\text{t} \log \sec 57^{0}.46' - 10 = & 0.2729727 \\
\hline
\log 728.4 = & 2.8623637
\end{array}$$

and, therefore, $c = 728.4$

Formula for the logarithmic computation of $a-b$.

698. A formula, such as $a-b$, where a is greater than b, may be reduced to the form $a \cos^2 \theta$, if $\sin^2 \theta = \frac{b}{a}$: for $(a-b) = a\left(1 - \frac{b}{a}\right) = a\,(1 - \sin^2 \theta) = a \cos^2 \theta$: consequently, if computed logarithmically, we get

$$\text{t} \log \sin \theta = \tfrac{1}{2} \{20 + \log b - \log a\},$$

and

$$\log (a-b) = \log a + \text{t} \log \cos \theta - 10^{*}.$$

---

* The formula in the problem just solved may be very easily adapted to the case under consideration: for

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$= a^2 + 2ab + b^2 - 2ab - 2ab \cos C$$

=

Second formula for the logarithmic computation of $a-b$.

699. The same formula $a-b$ is likewise reducible to the form $a\,(\sin^2\theta-\sin^2\theta')=a\sin(\theta+\theta')\sin(\theta-\theta')$†, which is also adapted to logarithmic computation: such a formula would be extremely convenient, when one of the terms a or b of $a-b$, presented itself under the form $a\sin^2\theta$: of this kind is the formula

$$\sin^2\left(\frac{a+b}{2}\right)-\sin a\sin b\cos^2\frac{C}{2},$$

which becomes

$$\sin\left(\frac{a+b+\theta}{2}\right)\sin\left(\frac{a+b-\theta}{2}\right),$$

if we make

$$\sin^2\theta=\sin a\sin b\cos^2\frac{C}{2}.$$

Formulæ for the logarithmic computation of a series of terms connected by the signs + and −

700. More generally, any series of terms, such as

$$a-b+c-d+e-\&\text{c.}$$

connected by the signs + and −, may be calculated

$$=(a+b)^2\left\{1-\frac{2ab\,(1+\cos C)}{(a+b)^2}\right\}$$

$$=(a+b)^2\left\{1-\frac{4ab\cos^2\frac{C}{2}}{(a+b)^2}\right\}:$$

if we make therefore $\dfrac{4ab\cos^2\frac{C}{2}}{(a+b)^2}=\sin^2\theta$, we get

$$c^2=(a+b)^2\cos^2\theta,$$

and therefore $\log c=\log(a+b)+\text{t}\log\cos\theta-10.$

† For $\sin^2\theta-\sin^2\theta'=(\sin\theta+\sin\theta')(\sin\theta-\sin\theta')$

$$=2\sin\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right)\times2\cos\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right).$$

$$=2\sin\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta+\theta'}{2}\right)\times2\sin\left(\frac{\theta-\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right)$$

$$=\sin(\theta+\theta')\sin(\theta-\theta').$$

logarithmically by the aid of expedients similar to those above-noticed: for the sum of two, three, four, &c. terms may be put under the successive forms,

$$a-b=a\left(1-\frac{b}{a}\right)=a\cos^2\theta: \text{ where } \sin^2\theta=\frac{b}{a};$$

$$a-b+c=(a-b)\left(1+\frac{c}{a-b}\right)=a\sec^2\theta',$$

$$\text{where } \tan^2\theta'=\frac{c}{a-b};$$

$$a-b+c-d=(a-b+c)\left\{1-\frac{d}{a-b+c}\right\}$$

$$=a\cos^2\theta'', \text{ where } \sin^2\theta''=\frac{d}{a-b+c};$$

and so on, in the same manner, as far as we may choose to proceed: and inasmuch as the process in each case gives the logarithmic value of the sum of the terms already determined, no more openings of the tables would be required than such as are necessary for the logarithmic computation of the additional term of the series which is successively introduced.

It is by means of expedients such as those described above, that all formulæ, whose symbols or component terms have specific arithmetical values, may be brought under the dominion of strictly logarithmic computation: it is neither our object, however, nor would our limits allow us, if it was, to notice in detail the different artifices which are made use of for the purpose of adapting formulæ to the most rapid logarithmic computation, which must vary according to the circumstances of their application: it is in such artifices that the skill of an analyst who constructs formulæ with a view to their practical application, will be chiefly exercised.

701. Having explained the nature and use of logarithmic tables, it remains to notice some of the series by which logarithms are expressed, and from which their arithmetical values may be calculated.

Series for the calculation of the value of A_1 in the equation $e^{A_1}=a$.

We have investigated in the last Chapter, the series for a^x and e^x, and also the equations $e^{A_1}=a$, and $e^{A_1 x}=a^x$, by which the logarithms corresponding to those different bases are connected with each other: it remains to investigate a series by which the arithmetical value of A_1 may be calculated for any assigned value of a, and particularly for the case of tabular logarithms, when $a=10$.

The series in Art. 663.

$$A_1=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.\dots(\alpha),$$

is not convergent, when a exceeds 2, and under such a form possesses no arithmetical value: it may be easily modified, however, so as to be convergent and rapidly so, in all cases whatsoever.

Since $e^{A_1}=a$, it follows that $\log a=A_1$, and consequently,

$$\log a=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.\dots(\beta),$$

or the series for A_1 expresses likewise the Napierian logarithm of a number or quantity a in terms of the number or quantity itself.

Also since $(a^{\frac{1}{m}})^m=a$, and, therefore,

$$\log (a^{\frac{1}{m}})^m=m\log (a^{\frac{1}{m}})=\log a=A_1,$$

it will follow from the form of the series (β) that

$$\log (a^{\frac{1}{m}})=(a^{\frac{1}{m}}-1)-\frac{(a^{\frac{1}{m}}-1)^2}{2}+\frac{(a^{\frac{1}{m}}-1)^3}{3}-\&c.\dots\dots(\gamma),$$

and, therefore,

$$A_1=m\left\{(a^{\frac{1}{m}}-1)-\frac{(a^{\frac{1}{m}}-1)^2}{2}\right.$$

$$\left.+\frac{(a^{\frac{1}{m}}-1)^3}{3}-\frac{(a^{\frac{1}{m}}-1)^4}{4}+\&c.\right\}\dots\dots\dots\dots(\delta).$$

If we suppose a greater than *unity*, it is always possible, by assuming m sufficiently large, to make $a^{\frac{1}{m}}$ differ from 1 by a quantity as small as we choose: thus if $a=10$ and $m=2^{54}$, it has been shewn that

$$10^{\frac{1}{2^{54}}} = 1.00000000000000012781914932003235:$$

and since

$$m=2^{54}=18014398509481984,$$

we get

$$m\,(a^{\frac{1}{m}}-1)=2.3025851\ldots.,$$

which is likewise the accurate value, as far as it goes; of A_1 or of log 10: for the second term of the series (γ) which is $\dfrac{(10^{\frac{1}{2^{54}}}-1)^2}{2}$, has no significant digit in the first 15 places of decimals.

Transition from Napierian to any other logarithms.

702. The preceding calculation gives us the Napierian logarithm of 10, and it is very obvious that a similar process will enable us to calculate the Napierian logarithm of any other number, by means of a series which will possess any required degree of convergency: but the Napierian logarithm being calculated or known, it remains to determine in what manner we may pass from it to the corresponding logarithm of the same number or quantity in any other system, and particularly in that, whose logarithms are registered in tables.

In the first place, it follows from the equation $e^{A_1 x}=a^x$, that the logarithms of any assigned base a will be found by dividing the corresponding Napierian logarithms by A_1 or by multiplying them by $\dfrac{1}{A_1}$ or $\dfrac{1}{\log a}$.

Modulus.

703. It is this invariable multiplier, for the same system, which is called the *modulus* of that system, and

which is always equal to the reciprocal of the Napierian logarithm of its base: consequently, in the case of tabular logarithms, we shall find the modulus or

$$m = \frac{1}{\log 10} = \frac{1}{2.3025851} = .434294481.$$

Objections to the preceding mode of calculating logarithms.

704. The preceding method of calculating tables of logarithms is theoretically perfect: but it is obvious that the operations which it involves (of multiplication, division, and extraction of roots,) are extremely tedious and embarrassing, and such as must necessarily prevent their practical application to any considerable extent: it is for such reasons, that it is usual, in the calculation of the logarithms of *successive* numbers, to resort to artifices of analysis, which furnish methods founded upon the differences of successive logarithms, or to series involving rational terms only, such as those which follow.

Logarithmic series.

705. Since

$$\log a = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c. \ldots (1),$$

we have

$$\log (1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \&c. \ldots\ldots\ldots\ldots (2):$$

if we replace a by $-a$ in this series, we get

$$\log (1-a) = -a - \frac{a^2}{2} - \frac{a^3}{3} - \frac{a^4}{4} - \&c. \ldots\ldots\ldots\ldots (3).$$

If we subtract the last of these series (3) from the second (2), we find

$$\log \left(\frac{1+a}{1-a}\right) = 2\left\{a + \frac{a^3}{3} + \frac{a^5}{5} + \&c.\right\} \ldots\ldots\ldots\ldots (4),$$

$$\text{since } \log (1+a) - \log (1-a) = \log \left(\frac{1+a}{1-a}\right).$$

Series for $\log \frac{m}{n}$.

706. If we make $\frac{1+a}{1-a}=\frac{m}{n}$, we get $a=\frac{m-n}{m+n}$, and, therefore,

$$\log \frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\&c.\right\}\,..(5).$$

If we make $m=n+1$, we shall have $\frac{m-n}{m+n}=\frac{1}{2n+1}$,

and, therefore,

$$\log \frac{n+1}{n}=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\&c.\right\}\ldots\ldots\ldots\ldots\ldots(6),$$

or

$$\log(n+1)=\log n+2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\&c.\right\}\ldots\ldots\ldots\ldots\ldots(7).$$

This is a formula, adapted to the calculation of successive numbers, which involves a series which is rapidly convergent in all cases, and more particularly so when the numbers are large.

When m and n are composed of simple factors.

If m and n are resolvible into *simple* factors, then $\log \frac{m}{n}$ will be equal to the sum of the logarithms of the factors of m diminished by the sum of the logarithms of the factors of n: if we assume p to express the logarithm of the largest factor, and $-q$ to express the algebraical sum of all the others, connected with their proper signs, then we shall have

$$p=q+2\left\{\left(\frac{m-n}{m+n}\right)+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3\right.$$

$$\left.+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\&c.\right\}\ldots\ldots\ldots\ldots\ldots(8),$$

or the logarithm of the number denoted by the largest factor will be expressed in terms of the logarithms of inferior numbers which are already determined, and the terms of an indefinite series which are more or less convergent: if we could assume m and n so that $m-n$ should be equal to 1, or even to an invariable number, then the convergency of the terms of this series would increase rapidly with the increase of the number whose logarithm was required to be determined.

Examples.

Thus if $m=x^2$ and $n=x^2-1=(x+1)(x-1)$, then $p=\log(x+1)$, $q=2\log x-\log(x-1)$, $\dfrac{m-n}{m+n}=\dfrac{1}{2x^2-1}$, and, therefore,

$$\log(x+1)=2\log x-\log(x-1)$$

$$+2\left\{\frac{1}{2x^2-1}+\frac{1}{3}\cdot\frac{1}{(2x^2-1)^3}+\frac{1}{5}\cdot\frac{1}{(2x^2-1)^5}+\&c.\right\};$$

the determination of the logarithm of any number is thus made to be dependent upon that of the two numbers immediately preceding it.

Again, if $m=(x-1)^2(x+2)$, $n=(x+1)^2(x-2)$ $p=\log(x+2)$, $q=2\log(x+1)+\log(x-2)-2\log(x-1)$, and $\dfrac{m-n}{m+n}=\dfrac{2}{x^3-3x}$, then we shall get

$$\log(x+2)=2\log(x+1)+\log(x-2)-2\log(x-1)$$

$$+2\left\{\frac{2}{x^3-3x}+\frac{1}{3}\cdot\left(\frac{2}{x^3-3x}\right)^3+\frac{1}{5}\cdot\left(\frac{2}{x^3-3x}\right)^5+\&c.\right\},$$

a formula, which gives the logarithm of a number, in terms dependent upon the logarithms of three out of four immediately preceding numbers.

If $x=100$, the second term of this series $\frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3$ would have no significant digit in the first 16 places: it is its first term only, therefore, which it is necessary, in the case of considerable numbers, to notice in the calculation of their logarithms by the formula just given.

If we assume

$$m=x^2(x-7)^2(x+7)^2=x^6-98x^4+2401x^2$$

$$n=(x-3)(x+3)(x-5)(x+5)(x-8)(x+8)=x^6$$
$$-98x^4+2401x^2-14400,$$

we shall find

$$\log(x+8)=2\log(x+7)+2\log x+2\log(x-7)$$
$$-\log(x+5)-\log(x+3)-\log(x-3)$$
$$-\log(x-5)-\log(x-8)$$
$$-2\left\{\frac{7200}{x^6-98x^4+2401x^2-7200}+\&\text{c.}\right\}$$

If $x=100$, the first term of this series has no significant digit in the first 7 places of decimals: if $x=1000$, it has no significant digit in the first 13 places of decimals: in such cases, therefore, it may be altogether neglected, and the calculation of the logarithm of a number would be effected by means of the logarithms of 8 preceding numbers alone.

707. Tables of logarithms are constructed for numbers of five places of figures only, but the logarithms of numbers consisting of six or even a greater number of places of figures are sometimes required: for this pur-

Column of differences.

pose, a column of *differences*, to each value of which a small auxiliary column of *proportional* parts corresponds, accompanies each page of tabular logarithms of numbers to five places, the principle of whose construction is as follows.

First difference.

In the first place,

$$\text{t}\log(n+1) - \text{t}\log n = \text{t}\log\left(\frac{n+1}{n}\right)$$

$$= \text{t}\log\left(1+\frac{1}{n}\right) = m\left(\frac{1}{n} - \frac{1}{2n^2} + \&\text{c.}\right) = \frac{m}{n}\text{ nearly,}$$

where m is the modulus or .434294481: let this difference of two successive logarithms be called Δ: in a similar manner we shall find

$$\text{t}\log(n+2) - \text{t}\log(n+1) = \frac{m}{n+1}\text{ nearly:}$$

Second difference.

let this be called Δ': consequently

$$\Delta - \Delta' = m\left\{\frac{1}{n} - \frac{1}{n+1.}\right\} = \frac{m.}{n\,(n+1)}\text{ nearly,}$$

a quantity which has no significant digit in the first 9 places of decimals, when n is a number of five places of figures: consequently, the first differences of a considerable number of successive logarithms will remain the same*: their successive values are registered in a column which is usually headed by the letter *D*.

Use of differences in constructing tables of logarithms.

* This will shew how easily tables of logarithms of large numbers may be constructed by differences only: for since

$$\log(n+1) - \log n = \Delta = m\left\{\frac{1}{n} - \frac{1}{2n^2} + \&\text{c.}\right\},$$

and the second difference

$$\Delta - \Delta' = \log(n+2) - 2\log(n+1) + \log n$$

$$= \frac{m}{n\,(n+1)} = \frac{m}{n^2}\text{ very nearly,}$$

Mode of finding the logarithm of a number of six places.

708. Again,

$$\text{t} \log N = \text{t} \log (10n + x) = 1 + \text{t} \log n + \frac{mx}{10n} \text{ nearly,}$$

$$= 1 + \text{t} \log n + \frac{\Delta x}{10}, \text{ since } \Delta = \frac{m}{n}:$$

or in other words, the difference Δ of the two successive logarithms of $n+1$ and n must be multiplied by x, the additional digit and divided by 10: the result obtained, when added to the logarithm of n increased by 1, will give the logarithm required: this operation is performed to the nearest whole number for all the nine values of x, in the small table of *proportional parts* which accompanies every different value of the difference.

Example.

Thus, let it be required to find from the tables, the logarithm of 338188,

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it will follow that

$$\begin{aligned} \log(n+1) &= \log n + \Delta - \tfrac{1}{2}(\Delta - \Delta') \text{ nearly} \\ \log(n+2) &= \log(n+1) + \Delta - \tfrac{1}{2}(\Delta - \Delta') \ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \log(n+x) &= \log(n+x-1) + \Delta - \tfrac{1}{2}(\Delta - \Delta') \ldots\ldots: \end{aligned}$$

if we add these results together, and strike out the quantities which are common to both sides, we shall get

$$\log(n+x) = \log n + x\,\Delta - \frac{x}{2}(\Delta - \Delta'):$$

as long, therefore, as $\frac{x}{2}(\Delta - \Delta')$ is so small as not to affect the last digit of Δ, the first difference remains the same, and the logarithms are computed accordingly: but when the first digit of $\frac{x}{2}(\Delta - \Delta')$ influences the last digit of Δ, then Δ must be diminished by 1, and a new series of values of Δ must be recommenced: in the construction of tables of logarithms by this method, which is of all others the most expeditious, and the only one which is made use of in practice, it will be merely necessary to compute accurately, by independent methods, the logarithms of three successive numbers, at considerable intervals, in order to avoid the influence which terms of the series for $\log\left(1 + \frac{x}{n}\right)$ beyond the second, may exercise, when x becomes a large number.

$$\begin{array}{l}
\log\ 33819 = 4.5291608 \\
\log\ 33818 = 4.5291479 \\
\hline
\qquad \Delta = \qquad 129 \\
\log 338180 = 5.5291479 \\
\qquad \dfrac{8\Delta}{10} = \qquad 103 \\
\hline
\log 338188 = 5.5291582
\end{array}$$

Prop. Parts.		
D		
129	1	13
	2	26
	3	39
	4	52
	5	65
	6	77
	7	90
	8	103
	9	116

Logarithm of a number of seven places of figures.

709. If the logarithm of a number of 7 places of figures be required, whose two last digits are x and y, we must add to the logarithm of the first five places increased by 2, from the table of proportional parts, $\frac{x\Delta}{10}$ for the first digit, and $\frac{y\Delta}{100}$ for the second: thus suppose it was required to find the logarithm of 3381886.

$$\begin{array}{l}
\text{Log } 3381800 = 6.5291479 \\
\qquad \dfrac{8\Delta}{10} = \qquad 103 \\
\qquad \dfrac{6\Delta}{100} = \qquad\quad 8 \\
\hline
\log 3381886 = 6.5291590
\end{array}$$

To find the number corresponding to a logarithm which is not in the tables.

710. The problem which is the converse of the one last considered, would be "to find the number corresponding to a logarithm not in the tables:" for that purpose, let δ be the difference between the given logarithm and the next inferior logarithm in the tables, and Δ the corresponding difference of the logarithms of the two successive nearest whole numbers; then we should have

$$\log (n + x) - \log n = \delta = x\Delta \text{ nearly,}$$

$$\log 1 = \log e^{2r\pi\sqrt{-1}} = 2r\pi\sqrt{-1},$$

and

$$\log -1 = \log e^{(2r+1)\pi\sqrt{-1}} = (2r+1)\pi\sqrt{-1}:$$

therefore also

$$\log a = 2r\pi\sqrt{-1} + \acute{\log}\, a,$$

$$\log -a = (2r+1)\pi\sqrt{-1} + \acute{\log}\, a.$$

Symbolical logarithms of $\epsilon^{\theta}a$.

715. In a similar manner, since $\epsilon^{\theta} = \cos\theta + \sqrt{-1}\sin\theta = \epsilon^{2r\pi+\theta} = e^{(2r\pi+\theta)\sqrt{-1}}$, we get

$$\log \epsilon^{\theta} = \log(\cos\theta + \sqrt{-1}\sin\theta) = (2r\pi+\theta)\sqrt{-1},$$

where r is a whole number or zero: therefore also

$$\log \epsilon^{\theta}a = \log(\cos\theta + \sqrt{-1}\sin\theta)\, a$$

$$= (2r\pi+\theta)\sqrt{-1} + \acute{\log}\, a.$$

Symbolical logarithms of a^m.

716. Also, since

$$1^m = \cos 2rm\pi + \sqrt{-1}\sin 2rm\pi = \epsilon^{2mr\pi} = e^{2mr\pi\sqrt{-1}},$$

we get

$$\log a^m = 2mr\pi\sqrt{-1} + m\,\acute{\log}\, a.$$

If $r = 0$, $2mr\pi\sqrt{-1} = 0$, and therefore one value of $\log 1^m$ is always zero.

When m is a whole number.

717. If m be a whole number, the values of $2mr\pi\sqrt{-1}$ are included amongst those of $2r\pi\sqrt{-1}$, but not conversely: or, in other words, the values of $\log 1^m$ are included in those of $\log 1$, but not conversely.

When $m = \frac{1}{2}$.

718. If $m = \frac{1}{2}$, then $2mr\pi\sqrt{-1}$ becomes $r\pi\sqrt{-1}$, and, therefore,

$$\log \sqrt{a} = r\pi\sqrt{-1} + \tfrac{1}{2}\,\acute{\log}\, a.$$

719. Again, since

Symbolical logarithms of $(-a)^m$.

$$(-1)^m = \cos(2r+1)\,m\pi + \sqrt{-1}\,\sin(2r+1)\,m\pi,$$

we get

$$\begin{aligned}\log(-a)^m &= \log(-1)^m a^m\\ &= (2r+1)\,m\pi\sqrt{-1} + m\,\text{lóg}\,a,\end{aligned}$$

720. If $m=2$, we get

When $m=2$.

$$\begin{aligned}\log(-a)^2 &= (2r+1)\,2\pi\sqrt{-1} + 2\,\text{lóg}\,a\\ &= (4r+2)\,\pi\sqrt{-1} + 2\,\text{lóg}\,a.\end{aligned}$$

721. Inasmuch as

$$\log a^2 = 2r\pi\sqrt{-1} + 2\,\text{lóg}\,a,$$

it follows that the values of $\log(-a)^2$ are always included amongst those of $\log a^2$, but not conversely.

The logarithms of a^2 are not all of them the same as those of $(-a)^2$.

722. Since $-1^m = (-1)\,1^m$

$$\begin{aligned}&= \{\cos(2r+1)\pi + \sqrt{-1}\,\sin(2r+1)\pi\}\times\\ &\qquad\{\cos 2mr'\pi + \sqrt{-1}\,\sin 2mr'\pi\}\\ &= \cos(2r+2mr'+1)\pi + \sqrt{-1}\,\sin(2r+2mr'+1)\pi,\end{aligned}$$

Symbolical logarithms of $-a^m$.

we get

$$\log - a^m = (2r+2mr'+1)\,\pi\sqrt{-1} + m\,\text{lóg}\,a.$$

723. Let $m=\frac{1}{2}$, then if $r=0$, $r'=-1$, we get

When $m=\frac{1}{2}$.

$$\log - \sqrt{a} = \tfrac{1}{2}\,\text{lóg}\,a,$$

which is also the arithmetical value of $\log\sqrt{a}$. (Art. 717.)

724. More generally, if $m=\dfrac{p}{2n}$, where p is an odd number prime to n, and if $r'=-n$, and $r=\dfrac{p-1}{2}$, then

When $m=\frac{p}{2n}$.

$$2r+2mr'+1=0,$$

and the corresponding value of $\log -a^m$ becomes the arithmetical value of $\log a^m$: such arithmetical values of the logarithms of negative quantities are confined to those powers of a symbol preceded by a negative sign, whose indices are fractions with even denominators.

Symbolical logarithms of $a \pm b\sqrt{-1}$.

725. Since

$$a+b\sqrt{-1} = \epsilon^{\theta}\sqrt{(a^2+b^2)} = \epsilon^{\theta}\rho \text{ where } \rho = \sqrt{(a^2+b^2)},$$

and

$$a-b\sqrt{-1} = \epsilon^{-\theta}\sqrt{(a^2+b^2)} = \epsilon^{-\theta}\rho,$$

where $\theta = \cos^{-1}\dfrac{a}{\sqrt{(a^2+b^2)}} = \sin^{-1}\dfrac{b}{\sqrt{(a^2+b^2)}}$*, or any arc whose cosine is $\dfrac{a}{\sqrt{(a^2+b^2)}}$ or sine $\dfrac{b}{\sqrt{(a^2+b^2)}}$, we therefore have

$$\log(a+b\sqrt{-1}) = (2r\pi+\theta)\sqrt{-1} + \text{lóg}\,\rho,$$

$$\log(a-b\sqrt{-1}) = (2r\pi-\theta)\sqrt{-1} + \text{lóg}\,\rho.$$

Symbolical logarithms of $(a+b\sqrt{-1})^{\alpha+\beta\sqrt{-1}}$.

726. If we take the more general formula

$$(a+b\sqrt{-1})^{(\alpha+\beta\sqrt{-1})},$$

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* This notation, which is extremely convenient, is a very natural extension of the use of negative indices: if we consider cos and $\cos^{-1}$ as simple symbols like a and a^{-1}, or $\sqrt{-1}$ and $(\sqrt{-1})^{-1}$, which indicate a peculiar and definable modification of, or quantity dependent upon, the quantity or symbol which follows them, then the equation $\theta = \cos^{-1}x$, will necessarily lead to the equation $\cos\theta = x$, in the same manner that $\theta = (\sqrt{-1})^{-1}x$ necessarily leads to $x = \sqrt{-1}\,\theta$: in other words, we consider $\cos^{-1}$ and $\dfrac{1}{\cos}$ as equivalent symbolical representations, and the operation of multiplication by one or the other of them, as imposing upon the other factor the peculiar modification of value which it is assumed to indicate: the same observations apply to other and similar symbolical representations, such as $\sin^{-1}$, $\tan^{-1}$, $\sec^{-1}$, &c. and generally to f^{-1}, when assumed to represent the inverse operation or modification of value which may be denoted by f, when prefixed to a symbol.

its logarithm

$$=(\alpha+\beta\sqrt{-1})\log(a+b\sqrt{-1})$$

$$=(\alpha+\beta\sqrt{-1})\left\{\cos^{-1}\frac{a}{\sqrt{(a^2+b^2)}}\sqrt{-1}+\text{lóg}\sqrt{(a^2+b^2)}\right\}$$

$$=(\alpha+\beta\sqrt{-1})\left\{(2r\pi+\theta)\sqrt{-1}+\text{lóg}\,\rho\right\},$$

(replacing $\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}$ by $2r\pi+\theta$, and $\sqrt{a^2+b^2}$ by ρ)

$$=\left\{(2r\pi+\theta)\,\alpha+\beta\,\text{lóg}\,\rho\right\}\sqrt{-1}+\alpha\,\text{lóg}\,\rho-(2r\pi+\theta)\,\beta.$$

The examination of this formula would shew that it would possess arithmetical values, under the circumstances which have been noticed above, and under no other.

727. If we suppose $b=0$, then $\theta=0$, $\rho=a$, and

When $b=0$.

$$\log a^{\alpha+\beta\sqrt{-1}}=(2\alpha r\pi+\beta\,\text{lóg}\,a)\sqrt{-1}$$
$$+\alpha\,\text{lóg}\,a-2\beta r\pi.$$

728. If we suppose $a=0$, then $\text{cós}^{-1}\,0=(2r+1)\dfrac{\pi}{2}$, and, therefore,

When $a=0$.

$$\log(b\sqrt{-1})^{\alpha+\beta\sqrt{-1}}$$
$$=\left\{(2r+1)\,\alpha\frac{\pi}{2}+\beta\,\text{lóg}\,b\right\}\sqrt{-1},$$
$$+\alpha\,\text{lóg}\,b-(2r+1)\,\beta\frac{\pi}{2}.$$

729. If we suppose $a=0$, and $\alpha=0$, then we should find

When $a=0$ and $\alpha=0$.

$$\log(b\sqrt{-1})^{\beta\sqrt{-1}}=\beta\,\text{lóg}\,b\sqrt{-1}-(2r+1)\,\beta\frac{\pi}{2}.$$

When $a=0, \alpha=0$, $b=\beta=1$.

730. If we further suppose $b=\beta=1$, we shall get

$$\log (\sqrt{-1})^{\sqrt{-1}} = \sqrt{-1} \log \sqrt{-1} = -(2r+1)\frac{\pi}{2},$$

and, therefore,

$$\pi = -\frac{2\sqrt{-1} \log \sqrt{-1}}{2r+1} = \frac{1}{2r+1} \cdot \frac{2 \log \sqrt{-1}}{\sqrt{-1}},$$

a very remarkable result, which is essentially symbolical.

Symbolical logarithms to any base different from e.

731. Napierian logarithms, which are denoted by log. are the only logarithms which occur in general analytical enquiries, and the only logarithms therefore, which can properly present themselves in those cases in which their symbolical values are required: but the preceding formulæ will admit of very easy adaptation to any other base, whose modulus is M: for if ϵ^θ be the sign of affection of a quantity a, we get

$$\log' \epsilon^\theta a = \log' \epsilon^\theta + \log' a;$$

$$\text{but } \log' \epsilon^\theta = \log' e^{\theta\sqrt{-1}} = (2r\pi + \theta)\sqrt{-1} \log' e$$

$$= M(2r\pi + \theta)\sqrt{-1}, \text{ since } \log' e = M:$$

therefore,

$$\log' \epsilon^\theta a = M(2r\pi + \theta)\sqrt{-1} + \log' a.$$

732. The preceding formulæ for symbolical logarithms have been obtained, without the aid of any enlarged definition of a logarithm, by considering all quantities as the continued products of the sign of affection by which they are preceded, of the signs of affection which are involved in them, whether from roots, or powers, or otherwise, and of their arithmetical values: we have thus been enabled to exhibit the symbolical distinction between $\log a^2$, and $\log (-a)^2$, and to refute the argument which has sometimes been

founded upon the asserted identity of their logarithms*: we have thus, likewise, been enabled to determine that there is a symbolical value of log $-a^{\frac{1}{2}}$, and likewise of log $-a^{\frac{p}{2n}}$, which is identical with the corresponding arithmetical logarithm, a conclusion which might naturally be expected to follow from the definition of a logarithm.

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* Since $a^2 = (-a)^2$, it was concluded that their logarithms were also equal, and therefore $2 \log a = 2 \log - a$ or $\log a = \log - a$: this was one of the principal arguments advanced to prove that the logarithms of a and $-a$ were the same.

◆

CHAP. XV.

ON SIMPLE, QUADRATIC, AND OTHER EQUATIONS, WHICH INVOLVE ONE UNKNOWN QUANTITY.

Identical and other equations.

733. THE term *Equation* is generally applied to any expressions including zero, which are connected by the sign $=$, in whatever manner that sign may be interpreted: but an essential distinction, as we have shewn on a former occasion, (Art. 128, 129, 130.), exists between those equations whose members are reducible to *identity* with each other, and those which are not so: it is with the latter class of equations only that we are concerned, when we speak of the Theory and Solution of Equations*.

Changes in the form of the same equation.

734. If we assume A and B to represent the two members of an equation in its primitive or unreduced form, or

$$A = B,$$

then the equation will continue to exist, if A and B are both of them increased or diminished, or multiplied or divided by the same quantity, or by any two quantities which are reducible to identity with each other: and inasmuch as such processes constitute the chief part of the labour of reducing equations to the form which is preparatory to their solution, or to the consideration of their theory, we shall notice them and their consequences in detail.

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* We owe some apology to our readers for not introducing the subject of this Chapter at a much earlier part of this volume, inasmuch as the processes which it teaches have been more or less required in all the five last Chapters. We were unwilling, however, to interrupt, by so doing, the course of our investigations concerning the discovery and transformation of equivalent forms, and we felt less scruple in thus deferring the consideration of this very important subject, inasmuch as the applications of it which were required, were, with very few exceptions, so very simple and obvious, as to occasion no difficulty to those students who would be capable of reading, and otherwise understanding the Chapters in question.

By the addition or subtraction of the same quantity to, or from, both its members.

735. (*a*) If $A = B$, then

$$A + C = B + C,$$

$$\text{and } A - C = B - C.$$

It is in consequence of this proposition, that quantities may be transferred from one member of an equation to the other by merely changing their signs from + to −, or from − to +: thus if the primitive equation be

$$A - C = B,$$

by adding C to both its members, we get

$$A - C + C = B + C,$$

$$\text{or } A = B + C.$$

In a similar manner, if the primitive equation be

$$A + C = B,$$

then by subtracting C from both its members, we get

$$A + C - C = B - C,$$

$$\text{or } A = B - C.$$

Transposition of terms from one member to the other.

This is a most important principle in the reduction of equations to new and more commodious forms, inasmuch as it enables us to transfer any or all the terms of one of the members of an equation to the other, and conversely: and it follows as an immediate consequence of it, that every equation may admit of such a modification of its form that all its significant terms may be made to form one of its members, and zero the other: for if

$$A = B,$$

and if we subtract B from both its members,

$$A - B = B - B = 0,$$

Examples of transposition.

736. The following are examples:

(1) Let $7 - 3x = 5 - 2x$: and let it be required to transfer $2x$. Add $2x$ to both sides, and we get

$$7 - 3x + 2x = 5 - 2x + 2x,$$

$$\text{or } 7 - x = 5.$$

(2) Let $a - bx = c - dx$: it is required to transfer the terms involving x to one side, and those which do not involve x to the other.

Add dx to, and subtract a from, both sides, and we get

$$a - a - bx + dx = c - a,$$

$$\text{or } -(b - d)\,x = c - a,$$

or, changing the signs on both sides,

$$(b - d)\,x = a - c.$$

(3) Let $ax^2 - bx = c$: it is required to transfer all the significant terms to one side.

Subtract c from both sides, and we get

$$ax^2 - bx - c = c - c = 0.$$

(4) Let $ax^3 - bx^2 + cx - d = ax^3 - \beta x^2 + \gamma x - \delta$.
It is required to transfer all the significant terms to one side.

Subtract $ax^3 - \beta x^2 + \gamma x - \delta$ from both sides, and we get

$$ax^3 - ax^3 - bx^2 + \beta x^2 + cx - \gamma x - d + \delta = 0,$$

or

$$(a - a)\,x^3 - (b - \beta)\,x^2 + (c - \gamma)\,x - (d - \delta) = 0.$$

By multiplication or division.

737. (β) If $A = B$, then

$$AC = BC,$$

$$\text{and } \frac{A}{C} = \frac{B}{C}.$$

If one or both of the members of an equation involve fractional terms, they may be exterminated, in conformity with the preceding proposition, by multiplying those members either successively by the several denominators of the fractions involved, or at once by their least or lowest common multiple: the following are examples.

Examples of clearing equations of fractions.

738. (1) Let $\frac{x}{2} + 3 = 8 - x$: it is required to clear this equation from fractions.

Multiply both sides by the only denominator 2, and we get

$$x + 6 = 16 - 2x.$$

(2) Let $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13$: it is required to clear this equation from fractional terms.

Multiply both sides of the equation by 12, the least common multiple of the denominators, and we get

$$6x + 4x + 3x = 156.$$

(3) Let $\frac{a}{x} = \frac{b}{b + x}$.

Multiply both sides of this equation, first by x, and secondly by $b + x$, and we get

$$a = \frac{bx}{b + x},$$

$$a(b + x) = bx \text{ or } ab + ax = bx.$$

(4) Let $\frac{10}{x} - \frac{3}{x + 2} = \frac{10}{x + 1}$.

Multiply both sides of the equation successively by x, $x + 2$ and $x + 1$, and we shall get

$$10 - \frac{3x}{x + 2} = \frac{10x}{x + 1},$$

$$10x + 20 - 3x = \frac{10x^2 + 20x}{x + 1}.$$

$$10x^2 + 10x + 20x + 20 - 3x^2 - 3x = 10x^2 + 20x.$$

(5) Let $\frac{7}{x-2} - \frac{10}{4x-8} = \frac{32}{7x^2 - 14x}$.

Multiply both sides of the equation by

$$28x^2 - 56x \text{ or } 4 \times 7 \times x \times (x-2),$$

which is the lowest common multiple of the denominators, and we get

$$7 \times 28x - 10 \times 7x = 32 \times 4,$$

$$\text{or } 196x - 70x = 128.$$

(6) Let $\frac{20x}{7-x} + \frac{140 - 20x}{x} = 580$.

Divide both sides of the equation by 20, and we get

$$\frac{x}{7-x} + \frac{7-x}{x} = 29.$$

Multiply both sides of this equation by $x(7-x)$, and we get

$$x^2 + (7-x)^2 = 29x(7-x),$$

$$\text{or } x^2 + 49 - 14x + x^2 = 203x - 29x^2.$$

Mode of ascertaining whether equations are identical or not identical.

739. When equations under a rational form are proposed for solution, or for any examination which may be connected with the discussion of their properties, we commence by clearing them of products and fractional terms, in the manner taught in the preceding examples, and we then proceed to transfer all the significant terms to one side, and to arrange them according to the powers of some one letter: if, after such reductions, the significant terms disappear by opposition of signs, the equation is *identical*, and all its general symbols, as far as the equation is concerned, are equally indeterminate: but if those terms, under such circumstances, do not disappear, then there is one symbol which must be dependent upon the other symbols or quantities which the equation involves: for if not, the symbol in question must be equally indeterminate and arbitrary with all the others, and the equation itself must be identical, inasmuch as it would, under no other circumstances, be independent of the particular value of any one of its symbols, when all the rest were assigned.

740. In all equations, therefore, which are not identical, there is one symbol which is dependent upon all the others, and which is said to be determined by them, when their values are assigned or assignable: but as the determination of this symbol is consequent upon the solution of the equation, it is said to be *unknown*, antecedently to that solution, and of course it may be considered as unknown, whether such solution be practicable or not: it is according to the powers of this unknown quantity usually designated by x, that the terms of the reduced equation are arranged, and it is by the indices of the highest powers of this unknown quantity that the different classes and degrees of equations are determined: thus, if the simple or first power only of x or of the unknown quantity be involved in it, the equation is termed a *Simple Equation*; if the second power of the unknown quantity be involved in it, with or without its simple power, it is termed a *quadratic* equation, or an equation of the *second* degree: if the third power of the unknown quantity be involved in it, with or without the inferior powers, it is termed a *cubic* equation, or an equation of the *third* degree: if the fourth power of x be involved in it, with or without the inferior powers, it is termed a *biquadratic* equation, or an equation of the *fourth* degree: and so on, as far as an equation of the n^{th} degree, involving the n^{th} power of the unknown quantity (where n is a whole number) with or without the inferior powers.

Classification of equations according to their degrees.

741. It would follow from the preceding definitions of equations of different degrees, that they are severally reducible to the following general forms:

General forms of simple, quadratic and other equations.

(α) Simple equations are reducible to the form

$$ax+b=0,$$

or dividing by a, and replacing $\frac{b}{a}$ by $-a$,

$$x-a=0.$$

(β) Quadratic equations are reducible to the form

$$ax^2+bx+c=0,$$

or, dividing by a (the coefficient of x^2) and replacing $\frac{b}{a}$ and $\frac{c}{a}$ by p and q respectively, to the form

$$x^2+px+q=0.$$

(γ) Cubic equations are reducible to the form

$$x^3+px^2+qx+r=0,$$

where the coefficient of the highest power of x is 1.

(δ) Biquadratic equations are reducible to the form

$$x^4+px^3+qx^2+rx+s=0,$$

where the coefficient of the highest power of x is 1, as in the former cases.

(ϵ) Generally, an equation of n dimensions, is reducible to the form

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\dots\dots\dots p_n=0,$$

where the coefficient of the highest power of x is 1, and where the subscript numbers determine the order of succession of the other terms of the equation.

Inasmuch as the degree of an equation is determined by the highest power of the unknown quantity involved in it, it will follow, that $x^2+q=0$ is equally a quadratic equation with $x^2+px+q=0$: $x^3+r=0$ is equally a cubic equation with $x^3+qx+r=0$ or $x^3+px^2+qx+r=0$: and $x^n+p_n=0$ is equally an equation of n dimensions with the general form of the equation which is given above, where any of the coefficients between p_1 and p_{n-1} inclusive may become equal to zero.

Meaning of a solution of an equation.

742. An equation is said to be solved, when the value or values of the unknown quantity are determined, which being substituted for it in the original equation, would make its two members identical with each other, or if zero

was one member, would make the other identically equal to zero likewise: the general theory of equations would shew that the number of values (or *roots*, as they are commonly called) would be equal to the degree of the equation: but without venturing upon the discussion of this question, which is one of very considerable difficulty, we shall confine our attention in this and the following Chapters to Simple Equations, which obviously admit but of one value of x, and to such other equations as admit of being easily transformed into *binomial* equations, whose theory has already been completely investigated, and their solutions determined. (Art. 456). **Roots of equations.**

743. Resuming therefore the consideration of simple equations, it will be very easy to form a rule for their solution: for the preparatory reductions which have been exemplified above, would bring the equation to the form **Solution of simple equations.**

$$ax+b=0,$$

which assumes, upon transferring b to the other side of the equation, the form

$$ax=-b,$$

and becomes, therefore, by dividing both sides of the equation by a or by the coefficient of the unknown quantity,

$$x=-\frac{b}{a},$$

which is the solution required.

744. The process above-mentioned may be at once translated into the following rule. **Rule.**

"Clear the equation of fractions (1): transfer the terms involving the unknown quantity to one side, and those which do not involve it to the other (2): collect the separate terms, in that member of the equation which involves the unknown quantity, into one (3): divide both members of the equation by the coefficient of the unknown quantity (4), which gives the solution required."

745. In the following examples, reference will be made to the different steps of this process, by means of the numbers (1), (2), (3), and (4).

1. Let $9 - x = 3x - 7$.

By (2), $9 + 7 = 3x + x$:

or by (3), $4x = 16$.

By (4), $x = 4$.

2. Let $\frac{x}{4} + \frac{5x}{6} = \frac{x}{3} + 9$.

$$(1) \qquad 3x + 10x = 4x + 108,$$

$$(2) \qquad 3x + 10x - 4x = 108,$$

$$(3) \qquad 9x = 108,$$

$$(4) \qquad x = 12.$$

3. Let $\frac{ax}{a-b} + \frac{bx}{b-c} = \frac{d}{a-c}$.

$$(1) \quad (a-c)(b-c)ax + (a-b)(a-c)bx = (a-b)(b-c)d,$$

$$(3) \quad \{(a-c)(b-c)a + (a-b)(a-c)b\}x = (a-b)(b-c)d.$$

$$(4) \quad x = \frac{(a-b)(b-c)d}{(a-c)(b-c)a + (a-b)(a-c)b}.$$

4. Let $\frac{cx}{c-dx} - \frac{c+bcx}{a-bx} = c - \frac{d-cx}{c-dx}$.

$$(1) \quad acx - bcx^2 - c^2 - bc^2x + cdx + bcdx^2$$

$$= ac^2 - acdx - bc^2x + bcdx^2 - ad + acx + bdx - bcx^2$$

$$(2) \quad cdx + acdx - bdx = c^2 + ac^2 - ad,$$

omitting all those terms which disappear by opposition of signs.

$$(3) \quad (cd + acd - bd)x = c^2 + ac^2 - ad.$$

$$(4) \qquad x = \frac{(1+a)c^2 - ad}{\{(1+a)c - b\}d}.$$

5. $(7 + x)(8 - x) - \frac{7x}{3} = 17x + 1 - x^2$.

We commence in this and all similar cases, with performing the operations which are indicated, if they lead to a result which terminates: we thus get

$$56 + x - x^2 - \frac{7x}{3} = 17x + 1 - x^2.$$

(1) $168 + 3x - 3x^2 - 7x = 51x + 3 - 3x^2.$

(2) $51x + 7x - 3x = 168 - 3,$

(3) $55x = 165.$

(4) $x = 3.$

6. $3 + \sqrt{x} = 10.$

(2) $\sqrt{x} = 10 - 3 = 7,$

and, therefore,

$$x = 49.$$

The process of solution, in the first instance, determines the value of $\sqrt{x}$: it requires an additional step to determine from thence the value of x.

Generally, if the equation, when reduced, is a simple equation with respect to the n^{th} root of x, where n is a whole number, it is a simple equation also, with respect to x: for it is a matter of indifference which of the n values of $x^{\frac{1}{n}}$, the value determined may be assumed to represent, inasmuch as they will all of them equally lead to the same value of x: the case is altogether different, when it is sought to determine the value of x, from the assigned value of x^n, where n is a whole number.

Equations involving $x^{\frac{1}{n}}$ only, are simple equations with respect to x.

7. $3\sqrt[3]{x} + 4 = 24 - 2\sqrt[3]{x}.$

(2) $3\sqrt[3]{x} + 2\sqrt[3]{x} = 20.$

(3) $5\sqrt[3]{x} = 20.$

(4) $\sqrt[3]{x} = 4,$

and, therefore,

$$x = 64.$$

8. $\sqrt{x} + \sqrt{(a + x)} = \sqrt{b}.$

This equation may be freed from the expression $\sqrt{(a + x)}$, as follows:

$$\sqrt{(a + x)} = \sqrt{b} - \sqrt{x}.$$

Squaring both sides, we get

$$a + x = b - 2\sqrt{bx} + x.$$

$$(2)\quad 2\sqrt{bx} = b - a,$$

$$4bx = (b-a)^2 = (a-b)^2$$

$$(4)\qquad x = \frac{(a-b)^2}{4b}.$$

If the equation had been

$$\sqrt{x} + \sqrt{(a-x)} = \sqrt{b},$$

the same process of reduction would have led to the equation

$$x^2 - ax + \frac{(a-b)^2}{4} = 0,$$

a quadratic equation which will be noticed hereafter.

The same remarks apply to the equations

$$\sqrt{a+x} + \sqrt{x} = \frac{b}{\sqrt{(a+x)}},$$

$$\text{and } \sqrt{a-x} + \sqrt{x} = \frac{b}{\sqrt{(a-x)}},$$

the first of which leads to a simple, and the second to a quadratic, equation.

9. $\sqrt[m]{(a+x)} = \sqrt[2m]{(x^2 + 7ax + b^2)}$.

Both members of this equation being raised to the $2m^{\text{th}}$ power, will give us

$$(a+x)^2 = x^2 + 7ax + b^2$$

$$a^2 + 2ax + x^2 = x^2 + 7ax + b^2.$$

$$(3)\quad 5ax = a^2 - b^2,$$

$$(4)\qquad x = \frac{a^2 - b^2}{5a}.$$

10. $\sqrt{\{a^2 + x\sqrt{(b^2 + x^2)}\}} = a + x$.

Squaring both sides, we get

$$a^2 + x\sqrt{(b^2 + x^2)} = a^2 + 2ax + x^2,$$

$$\text{or } x\sqrt{(b^2 + x^2)} = 2ax + x^2,$$

or, dividing by x,

$$\sqrt{(b^2+x^2)} = 2a + x.$$

If we square both sides again, we get

$$b^2 + x^2 = 4a^2 + 4ax + x^2.$$

$$(2) \quad 4ax = b^2 - 4a^2.$$

$$(4) \quad x = \frac{b^2 - 4a^2}{4a}.$$

The original equation is satisfied by making $x = 0$, which is therefore one of its roots.

11. $a^{mx} b^{nx} = c$.

This equation is reducible by means of logarithms to the form

$$mx \log a + nx \log b = \log c.$$

$$(3) \quad (m \log a + n \log b)\, x = \log c.$$

$$(4) \quad x = \frac{\log c}{m \log a + n \log b} = \frac{\log c}{\log a^m b^n}.$$

This result is true, whatever be the base or modulus of the logarithms expressed by log: it is obvious that arithmetical values only of the logarithms are considered.

12. $3^{2x} \times 5^{3x-4} = 7^{x-1} \times 11^{2-x}$,

$$2x \log 3 + (3x - 4) \log 5 = (x - 1) \log 7 + (2 - x) \log 11$$

$$(2) \quad 2x \log 3 + 3x \log 5 - x \log 7 + x \log 11$$

$$= 4 \log 5 - \log 7 + 2 \log 11.$$

$$(4) \quad x = \frac{4 \log 5 - \log 7 + 2 \log 11}{2 \log 3 + 3 \log 5 - \log 7 + \log 11}$$

$$= 1.242076 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

746. Binomial equations are those which, in their reduced form, consist of two terms only, one of which is a power of the unknown quantity, and the other does not involve it: such equations may be referred therefore to the general forms **Binomial equations.**

$$x^n - a = 0,$$

$$\text{or } x^n + a = 0.$$

Their solutions.

747. The general solution of the first of these equations, as we have already shewn, (Art. 497. and 499.) is

$$x=(1)^{\frac{1}{n}}\rho$$

$$=\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\rho,$$

and of the second

$$x=(-1)^{\frac{1}{n}}\rho$$

$$=\left\{\cos\frac{(2r+1)\pi}{n}+\sqrt{-1}\sin\frac{(2r+1)\pi}{n}\right\}\rho,$$

where ρ is the arithmetical value of $a^{\frac{1}{n}}$.

The number of their roots.

748. If n be a fraction, whose numerator is 1, there is only one value of $(1)^{\frac{1}{n}}$ or of $(-1)^{\frac{1}{n}}$, and the equation, under such circumstances, is essentially a simple equation: but if n be a whole number or a fraction (in its lowest terms) whose numerator is p, then the equation is one of the n^{th} degree in one case, and of the p^{th} degree in the other: and in all cases the number of values of x will be the same as the number which expresses the degree of the equation.

Examples of binomial equations.

749. The following are examples of binomial equations exceeding the first degree.

(1) $x^2=36$,

$$x=(1)^{\frac{1}{2}}6=\pm 6: \text{ for } (1)^{\frac{1}{2}}=+1, \text{ or } -1.$$

(2) $$\frac{1}{x+\sqrt{(2-x^2)}}+\frac{1}{x-\sqrt{(2-x^2)}}=ax,$$

$$x-\sqrt{(2-x^2)}+x+\sqrt{(2-x^2)}=ax(2x^2-2),$$

$$2x=ax(2x^2-2),$$

$$1=a(x^2-1)=ax^2-a,$$

$$ax^2=1+a,$$

$$x^2 = \frac{1+a}{a},$$

$$x = \pm\sqrt{\left(\frac{1+a}{a}\right)}.$$

(3) $$\frac{x^2+ax+bc}{x^2+bx+ad} = \frac{a}{b}.$$

(1) $$bx^2+abx+b^2c = ax^2+abx+a^2d.$$

(2) (3) $$(a-b)\,x^2 = b^2c-a^2d.$$

(4) $$x^2 = \frac{b^2c-a^2d}{a-b},$$

and, therefore,

$$x = \pm\sqrt{\left\{\frac{b^2c-a^2d}{a-b}\right\}}.$$

(4) $$\frac{28\,(x-18)}{x} = \frac{63x}{4\,(x-18)}.$$

Dividing by 7, we get

$$\frac{4\,(x-18)}{x} = \frac{9x}{4\,(x-18)}.$$

(1) $$16\,(x-18)^2 = 9x^2.$$

$$\frac{(x-18)^2}{x^2} = \frac{9}{16}.$$

$$\frac{x-18}{x} = \pm\frac{3}{4},$$

$$4x-72 = \pm 3x,$$

and, therefore,

$$x = 72 \text{ or } \frac{72}{7}.$$

A quadratic equation solved as a binomial.

750. This may be considered as a proper example of a binomial equation, inasmuch as $\frac{x-18}{x}$ is treated, in the first instance, as a simple symbol: in other words, if we make $\frac{x-18}{x} = u$, the equation becomes

$$u^2 = \frac{9}{16}.$$

$$u = \pm \frac{3}{4};$$

and, replacing u by its value, we get

$$\frac{x-18}{x} = \pm \frac{3}{4},$$

forming in this manner, two simple equations, from each of which a value of x may be obtained.

If the original equation had been reduced, by the ordinary process, without resorting to the aid of any artifice for its simplification, we should have obtained

$$7x^2 - 576x + 5184 = 0,$$

$$\text{or } x^2 - \frac{576}{7}x + \frac{5184}{7} = 0,$$

which is a quadratic equation under its general form: it would thus appear that such equations may sometimes at least be solved by means of a binomial equation: and we shall now proceed to shew in what manner all quadratic equations whatever may be thus transformed into binomial equations, and solved accordingly.

General reduction of quadratic equations to binomials.

751. The general form of quadratic equations as we have already shewn Art. 740, is

$$x^2 + px + q = 0,$$

where p and q are any *known* quantities whatever: if we make $x + \frac{p}{2} = u$, we get

$$\left(x + \frac{p}{2}\right)^2 = x^2 + px + \frac{p^2}{4} = u^2;$$

therefore

$$x^2 + px = u^2 - \frac{p^2}{4},$$

$$\text{and } x^2 + px + q = u^2 - \frac{p^2}{4} + q = 0,$$

and the quadratic equation is thus reduced to the binomial equation

$$u^2 - \frac{p^2}{4} + q = 0,$$

$$\text{or } u^2 = \frac{p^2}{4} - q.$$

752. If we solve this equation, we get

Their solutions.

$$u = \pm \sqrt{\left\{\frac{p^2}{4} - q\right\}}:$$

and if we replace u by its value originally assumed, we obtain

$$x + \frac{p}{2} = \pm \sqrt{\left\{\frac{p^2}{4} - q\right\}},$$

$$\text{and } x = -\frac{p}{2} \pm \sqrt{\left\{\frac{p^2}{4} - q\right\}},$$

which are the roots of the quadratic equation

$$x^2 + px + q = 0.$$

753. If we denote these roots by α and β, it will appear that

Formation of the coefficient of the second term and of the last term in terms of the roots.

$$\alpha + \beta = -\frac{p}{2} + \sqrt{\left\{\frac{p^2}{4} - q\right\}} - \frac{p}{2} - \sqrt{\left\{\frac{p^2}{4} - q\right\}} = -p,$$

$$\alpha\beta = \left\{-\frac{p}{2} + \sqrt{\left(\frac{p^2}{4} - q\right)}\right\}\left\{-\frac{p}{2} - \sqrt{\left(\frac{p^2}{4} - q\right)}\right\} = q:$$

consequently, the coefficient of the second term of the quadratic equation is equal to the sum of its roots with their signs changed, and the last term is equal to their product.

Factors of the quadratic equation.

754. Again,

$$(x-\alpha)\,(x-\beta)=\left\{x+\frac{p}{2}-\sqrt{\left(\frac{p^2}{4}-q\right)}\right\}\times$$

$$\left\{x+\frac{p}{2}-\sqrt{\left(\frac{p^2}{4}-q\right)}\right\}=x^2+px+q:$$

consequently the binomial x^2+px+q may be considered as formed by the multiplication of the two simple binomial factors $x-\alpha$ and $x-\beta$, where α and β are the roots of the equation $x^2+px+q=0$: in other words, the quadratic equation $x^2+px+q=0$, may be considered as composed of the two simple equations $x-\alpha=0$ and $x-\beta=0$, and as expressing or involving simultaneously the two values of x by which those simple equations are separately satisfied.

Connection between the signs and form of the roots and the form of the quadratic equation.

755. A very little examination would shew likewise the nature of the roots corresponding to different forms of the equation

$$x^2+px+q=0.$$

If q be positive, and also $\frac{p^2}{4}-q$ positive, then α and β are both negative or both positive, according as p is positive or negative. If q be positive and $\frac{p^2}{4}-q$ negative, then α and β are of the form $-a+b\sqrt{-1}$, $-a-b\sqrt{-1}$, or $a+b\sqrt{-1}$, $a-b\sqrt{-1}$, according as p is positive or negative.

If q be positive, and $\frac{p^2}{4}-q=0$, then α and β are equal to each other, and to $\frac{-p}{2}$. If q be negative, then

α and β have different signs, the greater of which (arithmetically speaking) is positive or negative, according as p is negative or positive. If q be negative, the roots of the quadratic equation can never assume the form

$$a \pm b\sqrt{-1} \text{ or } -a \pm b\sqrt{-1}.$$

Rule for the solution of a quadratic equation.

756. The steps of the process usually adopted in the solution of the equation

$$x^2 + px + q = 0,$$

and the rule into which they are translated, are as follows:

(1) $x^2 + px = -q.$

(2) $x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} - q.$

(3) $x + \frac{p}{2} = \pm\sqrt{\left(\frac{p^2}{4} - q\right)}.$

(4) $x = -\frac{p}{2} \pm\sqrt{\left(\frac{p^2}{4} - q\right)}.$

Rule. "Clear the original equation from those fractions, products or radicals, which involve x; transfer the terms which involve x to one side of the equation, and the terms which do not involve it to the other (1); divide both sides of the equation by the coefficient of x^2 if it differs from 1: add to both sides of the equation thus prepared the square of half the coefficient of x (2): extract the square roots of the resulting members (3), and the values of x are determined from the resulting simple equations (4)."

Examples.

757. The following are examples.

1. $x^2 - 7x + 12 = 0.$

(1) $x^2 - 7x = -12.$

(2) $x^2 - 7x + \frac{49}{4} = \frac{49}{4} - 12 = \frac{1}{4}.$

(3) $x - \frac{7}{2} = \pm\frac{1}{2}.$

(4) $x = 4$ or $3.$

The roots of the equation

$$x^2 + 7x + 12 = 0$$

are -4 and -3, differing from the former in their signs of affection only.

2. $x^2 + 10x + 29 = 0$.

(1) $x^2 + 10x = -29$.

(2) $x^2 + 10x + 25 = 25 - 29 = -4$.

(3) $x + 5 = \pm\sqrt{-4} = \pm 2\sqrt{-1}$.

(4) $x = -5 \pm 2\sqrt{-1}$.

The algebraical sum and product of $-5 + 2\sqrt{-1}$ and $-5 - 2\sqrt{-1}$ are -10 and 29 respectively. Art. 752.

3. $x^2 + x - 90 = 0$.

(1) $x^2 + x = 90$.

(2) $x^2 + x + \frac{1}{4} = \frac{1}{4} + 90 = \frac{361}{4}$.

(3) $x + \frac{1}{2} = \pm\frac{19}{2}$.

(4) $x = 9$ or -10.

The algebraical sum and product of 9 and -10 are -1 and -90 respectively.

The roots of the equation

$$x^2 - x - 90 = 0$$

would be found to be -9 and 10, differing from those of

$$x^2 + x - 90 = 0,$$

in their signs of affection only.

4. $\frac{x + 20}{x} = \frac{x - 5}{20 - x} + \frac{5}{2}$.

$$800 - 2x^2 = 2x^2 - 10x + 100x - 5x^2.$$

(1) $x^2 - 90x = -800$.

(2) $x^2 - 90x + 2025 = 1225$.

(3) $x - 45 = \pm 35$.

(4) $x = 80$ or 10.

5. $$\frac{x^2+1}{a^2+3ab+b^2}=\frac{2x}{a^2+ab+b^2}.$$

(1) $$x^2-\frac{2(a^2+3ab+b^2)}{a^2+ab+b^2}x=-1.$$

(2) $$x^2-\frac{2(a^2+3ab+b^2)}{a^2+ab+b^2}x+\frac{(a^2+3ab+b^2)^2}{(a^2+ab+b^2)^2}$$
$$=\frac{4ab(a^2+2ab+b^2)}{(a^2+ab+b^2)^2}.$$

(3) $$x-\frac{a^2+3ab+b^2}{a^2+ab+b^2}=\pm\frac{2\sqrt{ab}\,(a+b)}{a^2+ab+b^2}.$$

(4) $$x=\frac{a+\sqrt{ab}+b}{a-\sqrt{ab}+b}\text{ or }\frac{a-\sqrt{ab}+b}{a+\sqrt{ab}+b}.$$

6. $\sqrt{x}+\sqrt{a-x}=\sqrt{b}.$

Square both sides of the equation, and we get

$$x+2\sqrt{ax-x^2}+a-x=b,$$
$$\text{or } 2\sqrt{ax-x^2}=b-a.$$

Consequently

$$ax-x^2=\frac{(a-b)^2}{4}.$$

(1) $$x^2-ax=-\frac{(a-b)^2}{4}.$$

(2) $$x^2-ax+\frac{a^2}{4}=\frac{2ab-b^2}{4}.$$

(3) $$x-\frac{a}{2}=\pm\frac{\sqrt{(2ab-b^2)}}{2}.$$

(4) $$x=\frac{a}{2}\pm\frac{\sqrt{(2ab-b^2)}}{2}.$$

If $x=\frac{a}{2}+\frac{\sqrt{(2ab-b^2)}}{2}$, then also $a-x=\frac{a}{2}-\frac{\sqrt{(2ab-b^2)}}{2}$, and conversely: it follows, therefore, from the substitution of either one or the other of these values in the original equation, that

$$\sqrt{\left\{\frac{a}{2}+\frac{\sqrt{2ab-b^2}}{2}\right\}}+\sqrt{\left\{\frac{a}{2}-\frac{\sqrt{2ab-b^2}}{2}\right\}}=\sqrt{b}.$$

7. $\sqrt[3]{(x^3-a^3)}=x-b$.

Therefore, taking the cube of both sides, we get,

$$x^3-a^3=x^3-3bx^2+3b^2x-b^3,$$

$$3bx^2-3b^2x=a^3-b^3.$$

$$(1)\quad x^2-bx=\frac{a^3-b^3}{3b}.$$

$$(2)\quad x^2-bx+\frac{b^2}{4}=\frac{4a^3-b^3}{12b}.$$

$$(3)\quad x-\frac{b}{2}=\pm\sqrt{\left\{\frac{4a^3-b^3}{12b}\right\}}.$$

$$(4)\quad x=\frac{b}{2}\pm\sqrt{\left\{\frac{4a^3-b^3}{12b}\right\}}.$$

Equations reducible to the form of quadratic equations.

758. If the equation is not reducible, or not *conveniently* reducible, to the form

$$x^2+px+q=0,$$

but presents itself, or may be reduced so as to present itself, under the form

$$u^2+Pu+Q=0;$$

where u is an expression different from x, but involving it with or without known quantities, then its complete solution may be obtained, if we can solve the equation or equations.

$$u+\frac{P}{2}\pm\sqrt{\left\{\frac{P^2}{4}-Q\right\}}=0.$$

For whatever values of x satisfy the equation or equations

$$u+\frac{P}{2}\pm\sqrt{\left\{\frac{P^2}{4}-Q\right\}}=0,$$

must satisfy the equation

$$u^2 + Pu + Q = 0,$$

which may be considered as the primitive equation.

759. The following are examples:

(1) $x^3 - 6x^{\frac{3}{2}} = 16$.

In this case, if $u = x^{\frac{3}{2}}$, the equation becomes

$$u^2 - 6u = 16,$$

$$u^2 - 6u + 9 = 25,$$

$$u - 3 = \pm 5,$$

$$u = 8 \text{ or } - 2.$$

If we take the first value, we have

$$x^{\frac{3}{2}} = 8,$$

$$x^3 = 64,$$

$$x = (1)^{\frac{1}{3}} . 4,$$

where 4 is the arithmetical cube root of 64.

$$\text{But } (1)^{\frac{1}{3}} = 1, \text{ or } \frac{-1 + \sqrt{-3}}{2} \text{ or } \frac{-1 - \sqrt{-3}}{2}, \text{ (Art. 498)}:$$

$$\text{therefore } x = 4 \text{ or } -2 + 2\sqrt{-3} \text{ or } -2 - 2\sqrt{-3}.$$

If we take the second value of u, we get

$$x^{\frac{3}{2}} = -2,$$

$$x^3 = 4,$$

and therefore

$$x = (4)^{\frac{1}{3}}, \left(\frac{-1 + \sqrt{-3}}{2}\right) \times (4)^{\frac{1}{3}}, \left(\frac{-1 - \sqrt{-3}}{2}\right) \times (4)^{\frac{1}{3}}.$$

The three first values of x correspond to the binomial equation $x^3 - 64 = 0$: the three last values correspond to the binomial equation $x^3 - 4 = 0$: but all the six values equally correspond to the equation

$$(x^3 - 64)(x^3 - 4) = x^6 - 68x^3 + 256 = 0,$$

which is likewise the product of

$$x^3 - 6x^{\frac{3}{2}} - 16 = 0, \text{ and } x^3 + 6x^{\frac{3}{2}} - 16 = 0,$$

the first of which is the primitive equation.

It is obvious, therefore, that unless we suppose $6x^{\frac{3}{2}}$ to possess the double sign + and − 1, which is proper to it, or the equation

$$x^3 - 6x^{\frac{3}{2}} - 16 = 0,$$

to involve necessarily the equation

$$x^3 + 6x^{\frac{3}{2}} - 16 = 0,$$

that we must reject the second set of values, as altogether foreign to the original equation proposed.

The same observations apply to an extensive class of equations, which involve *implicitly* or *explicitly* fractional powers of the unknown quantity: in such cases, we generally pass, by the process of solution, from the least fractional power of the unknown quantity or of the expression which involves it, to a higher power of the same quantity, whether integral or not, and subsequently, upon the reduction and solution of the resulting equation, we return to a lower power again: under such circumstances, we are not at liberty to assign a greater number of values to the power which we thus arrive at, when this process is concluded, than it possessed, or was supposed to possess, at the commencement of it.

Proper roots and roots of solution.

The roots of equations, which are introduced in this manner, may be termed *roots of solution*, to distinguish them from the *proper* roots, which satisfy the equation in its primitive form: the introduction of them may be generally avoided, by confining ourselves to the arithmetical roots of those powers of the new unknown quantity which is, or may be, employed to designate a fractional power of the primitive unknown quantity or a radical expression involving it, and which is supposed to possess an arithmetical value only.

$$(2) \quad ax = b + \sqrt{cx}.$$

If we make $\sqrt{cx} = u$, we shall find $x = \frac{u^2}{c}$, and the equation will become

$$u^2 - \frac{cu}{a} = \frac{bc}{a},$$

$$u^2 - \frac{cu}{a} + \frac{c^2}{4a^2} = \frac{4abc + c^2}{4a^2},$$

$$u - \frac{c}{2a} = \pm \frac{\sqrt{(4abc + c^2)}}{2a}$$

$$u = \frac{c \pm \sqrt{(4abc + c^2)}}{2a}.$$

Consequently,

$$x = \frac{u^2}{c} = \frac{1}{c}\left\{\frac{c \pm \sqrt{(4abc + c^2)}}{2a}\right\}^2$$

$$= \frac{2ab + c \pm \sqrt{(4abc + c^2)}}{2a}.$$

It is the first of these values *only* which properly belongs to the primitive equation, the second being a *root* of *solution*: in other words, u, and therefore $u - \frac{c}{2a}$ can possess no more values than $\sqrt{cx}$, and if we are confined to an arithmetical value or square root of cx, we must be confined likewise to an arithmetical value of the square root of $\frac{4abc + c^2}{4a^2}$.

If we had begun by *rationalizing* the primitive equation, we should have got

$$a^2x^2 - (2ab + c)x + b^2 = 0,$$

whose roots are the values of x which are given above, and which correspond respectively, but not indifferently, to its two factors $ax - \sqrt{cx} - b = 0$ and $ax + \sqrt{cx} - b = 0$, the first of which is the primitive equation.

The index of the highest power of the unknown quantity involved in this equation is *unity*, and in all such equations, there is only one proper root or value of it: for under such circumstances, we are confined in all other terms, which may involve its fractional powers, to their arithmetical values only, inasmuch as we should otherwise alter the absolute form of the primitive equation which is presented for solution: all such equations, therefore, may be considered as *simple* equations, if their order is to be determined by the number of their roots: but if roots of solution are allowed to be considered as roots of the equation, equally with the proper roots, then the degree of the equation must be determined by the highest power of the unknown quantity which occurs in the equation when *rationalized* with respect to it, or when entirely freed from its fractional powers.

$$(3)\quad 7\sqrt{(3x-6)} = 3\sqrt{(7x+1)} + 3.$$

Make $\sqrt{(3x-6)} = u$, and therefore $x = \dfrac{u^2+6}{3}$: we thus get

$$7u = 3\sqrt{\left(\frac{7u^2+45}{3}\right)} + 3,$$

$$(7u-3)^2 = 21u^2 + 135,$$

$$u^2 - \frac{3u}{2} = \frac{9}{2}$$

$$u^2 - \frac{3u}{2} + \frac{9}{16} = \frac{81}{16},$$

$$u - \frac{3}{4} = \pm\frac{9}{4},$$

$$u = 3 \text{ or } -\frac{3}{2}.$$

If we take the first value of u, we get

$$\sqrt{(3x-6)} = 3,\quad 3x-6 = 9,\quad \text{and } x = 5,$$

which is the *proper* root of the primitive equation.

If we take the second value of u, we get

$$\sqrt{(3x-6)} = -\frac{3}{2},\quad 3x-6 = \frac{9}{4},\quad \text{and } x = \frac{11}{4},$$

which is a *root of solution* only, and satisfies the equation

$$3\sqrt{(7x+1)} - 7\sqrt{(3x-6)} + 3 = 0,$$

which differs from the primitive equation, in the signs of its radicals.

If we had begun by freeing the primitive equation from radicals, without the aid of any substitution, we should have obtained, after making the proper reductions, the equation

$$x^2 - \frac{31x}{4} + \frac{55}{4} = 0,$$

whose *proper* roots are 5 and $\frac{11}{4}$: under such circumstances, the *proper* root of the primitive equation, would be obtained, by confining ourselves to the arithmetical square roots of $\left(x - \frac{31}{8}\right)^2$ and of $\frac{81}{64}$ which is equal to it, in the process of solution of the equation.

$$(4) \quad 3\sqrt{(112 - 8x)} = 19 + \sqrt{(3x + 7)}.$$

The *proper* root of this equation is 6: the *root of solution* is $\frac{7398}{625}$, which satisfies the equation

$$3\sqrt{(112 - 8x)} = \sqrt{(3x + 7)} - 19.$$

Both these roots are *proper roots* of the *rationalized* equation

$$x^2 - \frac{11148}{625}x + \frac{44388}{625} = 0.$$

Similar observations apply to the equations

$$\sqrt{(2x + 7)} + \sqrt{(3x - 18)} = \sqrt{(7x + 1)},$$

and

$$7\sqrt{\left(\frac{3x}{2} - 5\right)} - \sqrt{\left(\frac{x}{5} + 45\right)} - \frac{7}{4}\sqrt{(10x + 56)} = 0,$$

whose proper roots are 9 and 20, and whose roots of solution are $-\frac{18}{5}$ and $\frac{14568980}{2874649}$ respectively.

$$(5) \quad x^{\frac{6}{5}} + 6x^{\frac{3}{5}} = 891.$$

Make $u = x^{\frac{3}{5}}$, and we get

$$u^2 + 6u = 891,$$

$$u^2 + 6u + 9 = 900,$$

$$u + 3 = \pm 30,$$

$$u = 27 \text{ or } -33.$$

Consequently,

$$x^{\frac{3}{5}} = 27 \text{ or } -33.$$

$$x^{\frac{1}{5}} = (1)^{\frac{1}{3}} \times 3 \text{ or } (-1)^{\frac{1}{3}} \times (33)^{\frac{1}{3}},$$

$$x = (1)^{\frac{5}{3}} \times 243 \text{ or } (-1)^{\frac{5}{3}} \times (33)^{\frac{5}{3}},$$

$$= (1)^{\frac{1}{3}} \times 243 \text{ or } (-1)^{\frac{1}{3}} \times (33)^{\frac{5}{3}},$$

for $(1)^{\frac{5}{3}} = (1)^{\frac{1}{3}}$, and $(-1)^{\frac{5}{3}} = (-1)^{\frac{1}{3}}$, (Art. 455).

The number of proper roots of an equation not affected by a denominator of the index of the highest power of the unknown quantity.

The number of *proper* solutions of an equation is not affected by a *common* denominator of the index of the unknown quantity, nor by any denominator of the index of its highest power: for if the highest power of this unknown quantity was $x^{\frac{p}{n}}$, the process of solution could give us the value of $x^{\frac{1}{n}}$, and no new value would be introduced in the transition from the value of $x^{\frac{1}{n}}$ to that of x: thus the equation

$$v^6 + 6v^3 = 891$$

would have the same number of proper solutions with the equation

$$x^{\frac{6}{5}} + 6x^{\frac{3}{5}} = 891,$$

and the solutions themselves would only differ in the second series of values being the fifth powers of those in the first.

If, however, we should suppose the equation

$$x^{\frac{6}{5}} + 6x^{\frac{3}{5}} = 891$$

to admit equally of all the forms which were proper to the different values of the quantities $x^{\frac{6}{5}}$ and $x^{\frac{3}{5}}$, which are 25 in number, we should have 6 roots corresponding to each of them, and the whole number of roots of solution and proper roots would be 150, of which 6 only would belong to the second class: such an example would sufficiently shew the importance and necessity, in a practical point of view, of such a limitation of the roots which are the proper objects of research, when equations of this kind are proposed for solution.

From similar considerations, it will likewise follow, that the reduction of the indices of the unknown quantity in an equation to a common denominator, will convert roots of solution into proper roots, unless the change from one form to the other is accompanied by a limitation of the roots which are to be extracted, in consequence of such a change, to their arithmetical values only: thus the *proper* root of the equation

$$x + \sqrt{x} = 6 \ldots\ldots\ldots\ldots (1)$$

is 4, and 9 is the root of solution: but 4 and 9 are equally proper roots of the equation

$$x^{\frac{2}{2}} + x^{\frac{1}{2}} = 6 \ldots\ldots\ldots\ldots (2),$$

unless we are equally restricted to the arithmetical value of $x^{\frac{1}{2}}$ in the two equations (1) and (2).

In a similar manner, there are only *three proper* roots of the equation

$$x^3 - 6x^{\frac{3}{2}} - 16 = 0 \ldots\ldots\ldots\ldots (1),$$

but there are *six proper* roots of the equation

$$x^{\frac{6}{2}} - 6x^{\frac{3}{2}} - 16 = 0 \ldots\ldots\ldots\ldots (2),$$

which include the roots of solution of the former equation (1).

(6) $x^2 + 15 + 4\sqrt{x^2 + 15} = 96.$

Make $\sqrt{(x^2 + 15)} = u$, and, therefore,

$$u^2 + 4u = 96.$$
$$u^2 + 4u + 4 = 100.$$
$$u + 2 = \pm 10.$$
$$u = 8 \text{ or } - 12.$$

Therefore

$$\sqrt{(x^2 + 15)} = 8 \text{ or } - 12,$$
$$x^2 + 15 = 64 \text{ or } 144,$$
$$x^2 = 49 \text{ or } 129,$$
$$x = \pm 7 \text{ or } \pm \sqrt{129}.$$

The two first values of x are the proper roots of the equation: the two last are merely roots of solution.

(7) $6x - x^2 + 3\sqrt{(x^2 - 6x + 16)} = 12.$

Make $\sqrt{(x^2 - 6x + 16)} = u$, and therefore $x^2 - 6x + 16 = u^2$, and $6x - x^2 = 16 - u^2$: consequently

$$16 - u^2 + 3u = 12,$$

$$u^2 - 3u = 4,$$

$$u^2 - 3u + \frac{9}{4} = \frac{25}{4},$$

$$u - \frac{3}{2} = \pm \frac{5}{2},$$

$$u = 4 \text{ or } -1,$$

the first of which values only is the proper value of u or of $\sqrt{x^2 - 6x + 16}$.

Again, since

$$\sqrt{(x^2 - 6x + 16)} = 4,$$

we get

$$x^2 - 6x + 16 = 16,$$

and therefore

$$x^2 - 6x = 0,$$

the roots of which equation are 0 and 6, which are the *proper* roots of the original equation.

The values of x deduced from the second value of u are roots of solution only.

$$(8) \quad \frac{x}{x^2 + x + 5} + \frac{5}{\sqrt{x^2 + x + 5}} = \frac{116}{25x}.$$

Multiply both sides of the equation by x, and we get

$$\frac{x^2}{x^2 + x + 5} + \frac{5x}{\sqrt{x^2 + x + 5}} = \frac{116}{25}.$$

Make $\frac{x}{\sqrt{(x^2 + x + 5)}} = u$, and we get

$$u^2 + 5u = \frac{116}{25}.$$

$$u^2 + 5u + \frac{25}{4} = \frac{1089}{100},$$

$$u + \frac{5}{2} = \pm \frac{33}{10},$$

$$u = \frac{4}{5} \text{ or } \frac{-29}{5}.$$

If we take the first value of u, we get

$$\frac{x}{\sqrt{(x^2+x+5)}} = \frac{4}{5},$$

$$25x^2 = 16x^2 + 16x + 80,$$

$$x^2 - \frac{16x}{9} = \frac{80}{9}.$$

$$x^2 - \frac{16x}{9} + \frac{64}{81} = \frac{784}{81},$$

$$x - \frac{8}{9} = \pm\frac{28}{9},$$

$$x = 4 \text{ or } -\frac{20}{9}.$$

These are the proper roots of the original equation: the roots derived from the second value of u are roots of solution only.

It is extremely easy to construct equations presenting forms apparently the most dissimilar from each other, but involving radical expressions or fractional powers of the unknown quantity whose indices are in the proportion of 2 to 1, which may be solved by methods similar to those above given: and the observations which we have made will enable us in all such cases to separate the proper roots from the roots of solution: all such equations may be considered as individual members of the corresponding rationalized equations.

760. It remains now to consider some cases of equations, which present themselves under a rational form, and involve powers of the unknown quantity or of expressions involving it, whose indices are also in the proportion of 2 to 1.

(1) $x^4 + 10x^2 = 2891.$

Make $x^2 = u$, and we get

$$u^2 + 10u = 2891,$$

$$u^2 + 10u + 25 = 2916,$$

$$u + 5 = \pm 54.$$

$$u = x^2 = 49 \text{ or } -59,$$

and therefore

$$x = \pm 7 \text{ or } (-1)^{\frac{1}{3}}\,(59)^{\frac{1}{3}}.$$

All these values are equally proper roots of the original equation.

$$(2)\quad \{(x+3)^2 + x + 3\}^2 - 7\,(x+3)^2 = 711 + 7x.$$

Therefore $\{(x+3)^2 + x + 3\}^2 - 7\,\{(x+3)^2 + x + 3\} = 690;$

make $(x+3)^2 + (x+3) = u$, and we get

$$u^2 - 7u = 690,$$

$$u^2 - 7u + \frac{49}{4} = \frac{2809}{4},$$

$$u - \frac{7}{2} = \pm\frac{53}{2}.$$

$$u = 30 \text{ or } -23.$$

If we take the first value of u, we get

$$(x+3)^2 + (x+3) = 30.$$

which becomes, when $x + 3$ is replaced by v,

$$v^2 + v = 30,$$

$$v^2 + v + \frac{1}{4} = \frac{121}{4},$$

$$v + \frac{1}{2} = \pm\frac{11}{2}.$$

$$v = x + 3 = 5 \text{ or } -6,$$

and therefore $x = 2$ or -9.

If we take the second value of u, we shall find, in a similar manner,

$$x = \frac{-1 \pm \sqrt{-91}}{2},$$

which are equally proper roots of the original equation with those already determined.

The primitive equation, when reduced in the ordinary manner, and its terms arranged according to the powers of the unknown quantity, becomes the biquadratic equation

$$x^4 + 14x^3 + 66x^2 + 119x - 630 = 0.$$

(3) $x^4 - 2x^3 - 2x^2 + 3x - 108 = 0$.

This biquadratic equation is reducible to the form

$$(x^2 - x)^2 - 3(x^2 - x) = 108,$$

and therefore may be solved by the methods under consideration: make $x^2 - x = u$, and we get

$$u^2 - 3u = 108,$$

$$u^2 - 3u + \frac{9}{4} = \frac{441}{4},$$

$$u - \frac{3}{2} = \pm \frac{21}{2},$$

$$u = 12 \text{ or } - 9.$$

If we take the first value of u, we get

$$x^2 - x = 12$$

$$x^2 - x + \frac{1}{4} = \frac{49}{4},$$

$$x - \frac{1}{2} = \pm \frac{7}{2}.$$

$$x = 4 \text{ or } - 3.$$

If we take the second value of u, we get

$$x^2 - x = - 9,$$

$$x^2 - x + \frac{1}{4} = - \frac{35}{4}.$$

$$x - \tfrac{1}{2} = \pm \frac{\sqrt{-35}}{2},$$

$$x = \frac{1 \pm \sqrt{-35}}{2}.$$

We thus get the four proper roots of the original equation.

(4) $x^4 + \dfrac{3x^3}{2} - 24x - 256 = 0$.

This is an example of a biquadratic equation, which is the difference of two squares, and which may be resolved as follows:

$$x^4 + \frac{3x^3}{2} = 24x + 256,$$

$$x^4 + \frac{3x^3}{2} + \frac{9x^2}{16} = \frac{9x^2}{16} + 24x + 256,$$

$$\left(x^2 + \frac{3x}{4}\right)^2 = \left(\frac{3x}{4} + 16\right)^2,$$

$$x^2 + \frac{3x}{4} = \pm\left(\frac{3x}{4} + 16\right).$$

If we take the positive sign, we get

$$x^2 + \frac{3x}{4} = \frac{3x}{4} + 16,$$

$$x^2 = 16,$$

$$x = \pm 4.$$

If we take the negative sign, we get

$$x^2 + \frac{3x}{4} = -\frac{3x}{4} - 16,$$

$$x^2 + \frac{3x}{2} = -16,$$

$$x^2 + \frac{3x}{2} + \frac{9}{16} = -\frac{247}{16},$$

$$x + \frac{3}{4} = \pm\frac{\sqrt{-247}}{4},$$

$$x = \frac{-3 \pm \sqrt{-247}}{4}.$$

(4) $x^3 - 3x = 2$.

This is a cubic equation, which may be put under a biquadrati form by multiplying all its terms by x: we should thus get

$$x^4 - 3x^2 = 2x,$$

$$x^4 - 2x^2 = x^2 + 2x,$$

$$x^4 - 2x^2 + 1 = x^2 + 2x + 1,$$

$$(x^2 - 1)^2 = (x + 1)^2,$$

$$x^2 - 1 = \pm(x + 1).$$

If we take the positive sign, we get

$$x^2 - x = 2,$$

$$x^2 - x + \frac{1}{4} = \frac{9}{4},$$

$$x - \frac{1}{2} = \pm\frac{3}{2},$$

$$x = 2 \text{ or } -1.$$

If we take the negative sign, we get

$$x^2 + x = 0,$$

and therefore $x = 0$ or -1.

We thus get four values of x, of which 0 is a root of solution, and two of the other and proper roots are equal to each other, and to -1.

The various expedients which have been noticed in the preceding examples, and many others of a similar kind which experience in such solutions will suggest, enable us to solve cubic and biquadratic equations under particular circumstances: but inasmuch as such solutions are dependent either upon the peculiar form under which the original equation presents itself or to which it is immediately reducible, or upon a peculiar relation of the coefficients of the equation, when completely reduced, they contribute in no respect to guide us to the methods which are necessary for their general solution: it is to the theory and investigation of such methods, that we shall now proceed to direct our attention.

Transformation of a cubic equation so as to want its second term.

761. A complete cubic equation with all its terms, appearing under the form

$$x^3 + p_1 x^2 + p_2 x + p_3 = 0,$$

may be transformed into an equation of the same degree wanting its second term, by a method similar to that which was employed for the transformation of a complete trinomial into a binomial quadratic equation: for if we make $x + \frac{p_1}{3} = u$, and therefore $x = u - \frac{p_1}{3}$, we shall get

$$\begin{array}{lll} x^3 = & u^3 - p_1 u^2 + \frac{p_1^2}{3} u - \frac{p_1^3}{27}, \\ +p_1 x^2 = & \quad +p_1 u^2 - \frac{2p_1^2}{3} u + \frac{p_1^3}{9}, \\ +p_2 x & \qquad\qquad +p_2 u - \frac{p_1 p_2}{3}, \\ +p_3 & \qquad\qquad\qquad +p_3, \end{array}$$

$$x^3 + p_1 x^2 + p_2 x^2 + p_3 = u^3 - \left(\frac{p_1^2}{3} - p_2\right) u + \frac{2p_1^3}{27} - \frac{p_1 p_2}{3} + p_3$$

$$= u^3 - qu + r = 0,$$

if we make

$$q = \frac{p_1^2}{3} - p_2, \text{ and } r = \frac{2p_1^3}{27} - \frac{p_1 p_2}{3} + p_3:$$

in considering, therefore, the general solution of a cubic equation, we should be thus authorized in confining our attention to the equation

$$u^3 - qu + r = 0,$$

$$\text{or } x^3 - qx + r = 0,$$

the most simple form to which a cubic equation is reducible by the preceding process.

Formula of solution of the equation $x^3 - qx + r = 0$.

762. The value or values of x or of the roots of this equation, can be no simple term like a or $\sqrt[3]{a}$, but of such a compound form that we may have $x^3 = qx - r$: such a compound term is $\sqrt[3]{\alpha} + \sqrt[3]{\beta}$: for if we make $x = \sqrt[3]{\alpha} + \sqrt[3]{\beta}$, we get

$$x^3 = \alpha + \beta + 3\sqrt[3]{\alpha\beta}\,(\sqrt[3]{\alpha} + \sqrt[3]{\beta}),$$

$$= 3\sqrt[3]{\alpha\beta}\,.\,x + \alpha + \beta,$$

a result which will be identical with

$$x^3 = qx - r \text{ ....(1),}$$

if we make $3\sqrt[3]{\alpha\beta} = q$, and $\alpha + \beta = -r$: but if $3\sqrt[3]{\alpha\beta} = q$, and therefore $\alpha\beta = \frac{q^3}{27}$ and $\alpha + \beta = -r$, then α and β are the roots of the quadratic equation

$$u^2 + ru + \frac{q^3}{27} = 0 \text{ .... (2), (Art. 753).}$$

and may be determined from it: if we solve this equation, therefore, we shall get

$$u = -\frac{r}{2} \pm \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)} = \alpha \text{ and } \beta,$$

and consequently, $x = \sqrt[3]{\alpha} + \sqrt[3]{\beta}$

$$= \left\{-\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}^{\frac{1}{3}} + \left\{-\frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^2}{27}\right)}\right\}^{\frac{1}{3}}.$$

Separation of the proper roots of the cubic equation from the roots of solution.

763. It thus appears that the roots of the cubic equation

$$x^3 - qx + r = 0 \text{ .... (1)}$$

are dependent upon the roots of the *reducing* quadratic equation

$$u^2 + ru + \frac{q^3}{27} = 0 \text{ .... (2):}$$

it remains to determine the proper roots of the cubic equation as distinguished from those values of x which are introduced by the process of solution.

If we assume ρ and ρ' to represent the arithmetical values of $\sqrt[3]{\alpha}$ and $\sqrt[3]{\beta}$, when the sign $\sqrt{-1}$ does not present itself in α and β, or in other words, when $\frac{r^2}{4}$ is

When $\frac{r^2}{4}$ is greater than $\frac{q^3}{27}$.

greater than $\frac{q^3}{27}$, then we shall have (prefixing to a and β the ambiguous sign $\pm$),

$$x = (\pm 1)^{\frac{1}{3}}\rho + (\pm 1)^{\frac{1}{3}}\rho'.$$

The three values of $(\pm 1)^{\frac{1}{3}}$, are ± 1, $\pm\left(\frac{-1+\sqrt{-3}}{2}\right)$, $\pm\left(\frac{-1-\sqrt{-3}}{2}\right)$, which differ from each other in the signs + or − only: if we agree, therefore, to denote $\pm\rho$ by ρ_1, and $\pm\rho'$ by ρ_1', the preceding expression will become

$$x = (1)^{\frac{1}{3}}\rho_1 + (1)^{\frac{1}{3}}\rho_1'.$$

If we should combine the three values of $(1)^{\frac{1}{3}}\rho_1$ with the three values of $(1)^{\frac{1}{3}}\rho_1'$, we should get *nine* different combinations, and therefore *nine* different values of x, which may all of them be considered as roots of solution of the primitive equation: but if we refer to the process of solution, we find $3\sqrt[3]{a\beta}$ assumed to be equal to q: if, therefore the sign $\sqrt{-1}$ does not exist in q, it can have no existence in $\sqrt[3]{a\beta}$: or, in other words, we shall be confined to those combinations of $(1)^{\frac{1}{3}}\rho_1$ and $(1)^{\frac{1}{3}}\rho_1$, which produce a result independent of $\sqrt{-1}$: we thus get the three *proper* values of x, or roots of the equation $x^3 - qx + r = 0$, which are

$$x = \rho_1 + \rho_1'$$

$$x = \left(\frac{-1+\sqrt{-3}}{2}\right)\rho_1 + \left(\frac{-1-\sqrt{-3}}{2}\right)\rho_1'$$

$$= -\frac{(\rho_1+\rho_1')}{2} + \frac{(\rho_1-\rho_1')\sqrt{-3}}{2},$$

$$x = \left(\frac{-1-\sqrt{-3}}{2}\right)\rho_1 + \left(\frac{-1+\sqrt{-3}}{2}\right)\rho_1'$$

$$= -\frac{(\rho_1+\rho_1')}{2} - \frac{(\rho_1-\rho_1')\sqrt{-1}}{2}.$$

When $\frac{r^2}{4}$ is less than $\frac{q^3}{27}$.

It thus appears, that the three roots of the cubic equation are of the form $2a$, $-a+b\sqrt{-1}$, $-a-b\sqrt{-1}$, when the roots of the reducing quadratic equation α and β are independent of the sign $\sqrt{-1}$: but if the values of α and β be of the form $c+d\sqrt{-1}$ and $c-d\sqrt{-1}$, and therefore $\frac{r^2}{4}$ less than $\frac{q^3}{27}$, we shall have, if we make

$$\rho=\sqrt[6]{(c^2+d^2)}=\sqrt{\frac{q}{3}},$$

$$\text{and } \theta=\cos^{-1}\frac{c}{\sqrt{c^2+d^2}}=\cos^{-1}\frac{-\frac{r}{2}}{\sqrt{\frac{q^3}{27}}}, \text{ (Art. 725).}$$

$$x=\left(\cos\frac{2m\pi+\theta}{3}+\sqrt{-1}\sin\frac{2m\pi+\theta}{3}\right)\rho$$

$$+\left(\cos\frac{2m'\pi+\theta}{3}-\sqrt{-1}\sin\frac{2m'\pi+\theta}{3}\right)\rho. \text{ (Art. 624).}$$

In this expression, the values of m and m' must be the same, inasmuch as $\sqrt[3]{\alpha\beta}$ must be independent of $\sqrt{-1}$: and also the values of m may be confined to 0, 1, 2, inasmuch as the same values of $\cos\frac{2m\pi+\theta}{3}\pm\sqrt{-1}\sin\frac{2m\pi+\theta}{3}$ recur for other values of m in the same series: we are thus confined to the three values of x, which are

$$\text{when } m=0,\ x=2\rho\cos\frac{\theta}{3}=2\sqrt{\frac{q}{3}}\cos\frac{\theta}{3}.$$

$$\text{when } m=1,\ x=2\rho\cos\frac{2\pi+\theta}{3}=2\sqrt{\frac{q}{3}}\cos\frac{2\pi+\theta}{3},$$

$$\text{when } m=2,\ x=2\rho\cos\frac{4\pi+\theta}{3}=2\sqrt{\frac{q}{3}}\cos\frac{2\pi-\theta}{3}.$$

The roots of the cubic equation are independent of $\sqrt{-1}$ when the roots of the reducing quadratic involve it.

It appears, therefore, that the three roots of the cubic equation are independent of the sign $\sqrt{-1}$, when the roots α and β of the reducing quadratic equation involve it: and inasmuch as $\cos\frac{2\pi+\theta}{3}+\cos\frac{2\pi-\theta}{3}$ $=2\cos\frac{2\pi}{3}\cos\frac{\theta}{3}=-\cos\frac{\theta}{3}$, it also follows that one root is equal to the sum of the other two, with their signs changed from + to − 1 or conversely.

Examples.

764. The following are examples:

(1) Let $x^3-3x-2=0$.

In this case $\frac{r}{2}=-1$, $\frac{q}{3}=1$, and therefore

$\alpha=1$, $\beta=1$, and

$$(1)\quad x=1+1=2.$$

$$(2)\quad x=\frac{-1+\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}=-1.$$

$$(3)\quad x=\frac{-1-\sqrt{-3}}{2}+\frac{-1+\sqrt{-3}}{2}=-1.$$

(2) Let $x^3-\frac{15x}{2}+\frac{581}{2}=0$,

$$\alpha=\frac{-581+\sqrt{(337311)}}{4},$$

$$\beta=\frac{-581-\sqrt{(337311)}}{4},$$

$$\rho_1=\frac{-7+\sqrt{39}}{2},$$

$$\rho_1'=\frac{-7-\sqrt{39}}{2}.$$

Therefore

$$x=\rho_1+\rho_1'=-7,$$

$$x=-\frac{(\rho_1+\rho_1')}{2}+\frac{(\rho_1-\rho_1')}{2}\sqrt{-3}=\frac{-7}{2}+\tfrac{1}{2}\sqrt{-117},$$

$$x=-\frac{(\rho_1+\rho_1')}{2}-\frac{(\rho_1-\rho_1')}{2}\sqrt{-3}=\frac{-7}{2}-\tfrac{1}{2}\sqrt{-117}.$$

Extraction of the cube roots of binomial surds.

In this example, we have put down the values of ρ_1 and ρ_1' under a finite form, as determined by a tentative process of the following kind.

If it be suspected from the nature of the case or otherwise that a numerical expression of the form

$$a + \sqrt{b},$$

whose cube root is required, as in the case of a and β in the preceding example, is the complete cube of an expression such as

$$\frac{x + y}{\sqrt[6]{R}},$$

where x and y are one or both of them quadratic surds, whose sum is not reducible to the form ax, where a is a whole number or rational fraction, then we shall also have

$$\sqrt[3]{(a - \sqrt{b})} = \frac{x - y}{\sqrt[3]{R}}:$$

for if $a + \sqrt{b}$ result from the cube of $\frac{x + y}{\sqrt[6]{R}}$, then $a - \sqrt{b}$ must result likewise from the cube of $\frac{x - y}{\sqrt[6]{R}}$: it will follow, therefore, that

$$\sqrt[3]{(a^2 - b)} = \frac{x^2 - y^2}{\sqrt[3]{R}},$$

$$\text{or } \sqrt[3]{\{(a^2 - b)R\}} = x^2 - y^2 = c,$$

an expression, in which R may be always so assumed, that $(a^2 - b)R$ may be a perfect cube* or $x^2 - y^2$ a whole number.

Again, since

$$\sqrt[3]{\{(a + \sqrt{b})^2 R\}} = x^2 + 2xy + y^2,$$

$$\sqrt[3]{\{(a - \sqrt{b})^2 R\}} = x^2 - 2xy + y^2,$$

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* If $a^2 - b$ be not a perfect cube or not resolvible into factors which are repeated, then the least value of R is $(a^2 - b)^2$; thus if

$$a^2 - b = 64, \quad R = 1 \text{ and } x^2 - y^2 = 4:$$

$$\text{if } a^2 - b = 54 = 2 \times 27 = 2 \times 3^3, \text{ then } R = 4 \text{ and } x^2 - y^2 = 6:$$

$$\text{but if } a^2 - b = 58 = 2 \times 29, \text{ then } R = 58^2 \text{ and } x^2 - y^2 = 58.$$

we get

$$\sqrt[3]{\{(a+\sqrt{b})^2 R\}} + \sqrt[3]{\{(a-\sqrt{b})^2 R\}} = 2x^2 + 2y^2 :$$

and inasmuch as it would follow from the hypothesis which we have made, that x^2 and y^2 are whole numbers, the value of $2x^2 + 2y^2$ would be equal to the sum of the two *nearest* integral values ι, ι' of $\sqrt[3]{\{(a+\sqrt{b})^2 R\}}$ and $\sqrt[3]{\{(a-\sqrt{b})^2 R\}}$, one of them being taken in excess, and the other in defect: for their sum could not be a whole number unless one of these cube roots differed as much from one of the numbers ι and ι' in excess, as from the other in defect: we should thus find

$$\left.\begin{array}{l} x^2 + y^2 = \dfrac{\iota + \iota'}{2} \\ x^2 - y^2 = c \end{array}\right\}.$$

Adding these expressions together, we get

$$2x^2 = \frac{\iota + \iota' + c}{2} \text{ and } x = \frac{\sqrt{(\iota + \iota' + c)}}{2},$$

and subtracting them from each other, we get

$$2y^2 = \frac{\iota + \iota' - c}{2} \text{ and } y = \frac{\sqrt{(\iota + \iota' - c)}}{2}.$$

If, *upon trial*, the cube of $\dfrac{x+y}{\sqrt[6]{R}}$ which is thus determined, be found to be equal to $a + \sqrt{b}$, the problem is solved, and one of the roots of the equation is a whole number, or a finite rational fraction; if not, there is no such root of the equation, and its approximate value must be determined by the actual extraction of the roots which are involved in it, for which purpose logarithms may be applied, if required, through the medium of the formula considered in Art. 695.

(3) Let $x^3 + 8x - 9 = 0$,

$$\alpha = \frac{9}{2} + \sqrt{\frac{4235}{108}} = \frac{81 + \sqrt{(12705)}}{18},$$

$$\beta = \frac{9}{2} - \sqrt{\frac{4235}{108}} = \frac{81 - \sqrt{(12705)}}{18},$$

$$\rho_1 = \frac{3 + \sqrt{105}}{6}, \ \rho_1' = \frac{3 - \sqrt{105}}{6}.$$

Therefore

$$x = \rho_1 + \rho_1' = 1,$$

$$x = \frac{-(\rho_1 + \rho_1')}{2} + \frac{(\rho_1 - \rho_1')}{2}\sqrt{-3} = \frac{-1 + \sqrt{-35}}{2},$$

$$x = \frac{-(\rho_1 + \rho_1')}{2} - \frac{(\rho_1 - \rho_1')}{2}\sqrt{-3} = \frac{-1 - \sqrt{-35}}{2}.$$

(4) Let $x^3 - 9x^2 + 26x - 24 = 0$.

If we make $x - 3 = u$ or $x = u + 3$, this equation is transformed into

$$u^3 - u = 0,$$

and therefore $u = 0$, or 1 or -1: the roots of the primitive equation are, therefore, 2, 3, 4.

In this case, the transformation leads to the solution of the equation, without the aid of any formulæ of solution.

(5) Let $x^3 - 18x^2 + 87x - 70 = 0$.

If we make $x - 6 = u$, or $x = u + 6$, we get the transformed equation

$$u^3 - 21u + 20 = 0.$$

From this equation, we find

$$\alpha = -10 + \sqrt{-243} = a + b\sqrt{-1},$$

$$\beta = -10 - \sqrt{-243} = a - b\sqrt{-1}.$$

Make $\cos\theta = \frac{a}{\sqrt{(a^2 + b^2)}} = \frac{-10}{\sqrt{343}} = \frac{-10}{7^{\frac{3}{2}}}$:

and therefore

$$\cos(\pi - \theta) = \frac{10}{7^{\frac{3}{2}}},$$

$$\text{t}\log\cos(\pi - \theta) = 10 + \log 10 - \frac{3}{2}\log 7$$

$$= 11 - \frac{3}{2}\log 7 = 11 - \frac{3}{2}(.8450980)$$

$$= 9.7323530 = \text{t}\log\cos 57^\circ.\ 19'.\ 12'',$$

and therefore $\theta = 180 - 57^\circ.\ 19'.\ 12'' = 120^\circ.\ 40'.\ 48''$.

Also, since $u = 2\sqrt{\frac{q}{3}} \times \cos\frac{\theta}{3}$ or $2\sqrt{\frac{q}{3}} \times \cos\frac{2\pi+\theta}{3}$ or $2\sqrt{\frac{q}{3}} \times \cos\frac{2\pi-\theta}{3}$, we have, for the first value

$$\begin{array}{rl}
\log 2 = & .3010300 \\
\frac{1}{2}\log 7 = & .4225490 \\
\hline
 & .7235790 \\
\text{t} \log \cos 40^\circ.\ 53'.\ 36'' = & .9.8784812 \\
\hline
 & 10.6020602 \\
 & 10.\ \ldots\ldots\ldots \\
\hline
\log u_1 = \log 4 = & .6020602
\end{array}$$

and $u_1 = 4$.

Again, since $\frac{2\pi+\theta}{3} = 160^\circ.\ 53'.\ 36'' = 180^\circ - 19^\circ.\ 6'.\ 24''$, we have

$$\begin{array}{rl}
\text{t} \log \cos 19^\circ.\ 6'.\ 24'' = & 9.9753910 \\
\log 2\sqrt{7} = & .7235790 \\
\hline
\log - u_2 = \log 5 = & .6989700,\ \text{omitting } 10,
\end{array}$$

and therefore $u_2 = -5$.

Lastly, since $\frac{2\pi-\theta}{3} = 79^\circ.\ 6'.\ 24''$, we have

$$\begin{array}{rl}
\text{t} \log \cos 79^\circ.\ 6'.\ 24'' = & 9.2764210 \\
\log 2\sqrt{7} = & .7235790 \\
\hline
\log u_3 = \log 1 = & .0000000,\ \text{omitting } 10,
\end{array}$$

and therefore $u_3 = 1$.

Consequently the values of x, or the roots of the original equation are 10, 1 and 7.

Composition of the coefficients of a cubic equation.

765. It appears from the preceding investigations, that every cubic equation, whether wanting its second term or not, will have three roots, and no more: and it will be very easy, with the aid of this conclusion, to determine the composition of the coefficients of the general cubic equation

$$x^3 + p_1 x^2 + p_2 x + p_3 = 0,$$

in terms of its roots: for if these roots be a, b, c, then $x-a$, $x-b$, $x-c$, are its factors*, and therefore

$$x^3+p_1x^2+p_2x+p_3=(x-a)\,(x^2+Px+Q)$$

$$=(x-a)\,(x-b)\,(x-c)$$

$$=x^3-(a+b+c)\,x^2+(ab+ac+bc)\,x-abc:$$

or

$$p_1=-(a+b+c),$$

$$p_2=\quad ab+ac+bc,$$

$$p_3=-abc.$$

If $p=0$, the sum of the roots is zero.

766. It follows, therefore, that if $p_1=0$, the sum of the roots of the equation is equal to zero, a conclusion which likewise follows from the formulæ which express the roots themselves under the same circumstances.

General form of biquadratic equations.

767. The general form of a biquadratic equation

$$x^4+p_1x^3+p_2x^2+p_3x+p_4=0 \;\ldots\ldots\; (a)$$

may be transformed into an equation of the same degree, without its second term, if we make $x+\frac{p_1}{4}=u$: in con-

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If a be a root of an equation, $x-a$ is a factor.

* If a be a root of the equation

$$x^3+p_1x^2+p_2x+p_3=0,$$

$$\text{then } a^3+p_1a^2+p_2a+p_3=0,$$

and therefore

$$x^3+p_1x^2+p_2x+p_3=x^3+p_1x^2+p_2x+p_3-(a^3+p_1a^2+p_2a+p_3)$$

$$=x^3-a^3+p_1(x^2-a^2)+p_2(x-a):$$

consequently

$$\frac{x^3+p_1x^2+p^2x+p_3}{x-a}=x^2+ax+a^2+p_1(x+a)+p_2,$$

or $x-a$ divides $x^3+p_1x^2+p_2x+p_3$ without a remainder.

The same process of reasoning would equally shew, that if a be a root of an equation of n dimensions, then $x-a$ must be one of its factors.

Transformed to a form wanting the second term.

sidering, therefore, the solution of such equations, we may confine our attention to the form

$$x^4 - qx^2 - rx - s = 0 \ldots\ldots\ldots\ldots (b),$$

which results from such a transformation, whenever the existence of a second term in the primitive equation may render its application necessary: and as we have seen that the roots of a cubic equation were expressible by means of the roots of a quadratic equation, whose coefficients or terms were determinable from it, so likewise we shall find that the roots of the biquadratic equation (*b*) are similarly expressible by means of the roots of a cubic equation whose coefficients or terms are determinable from it.

The roots of a biquadratic equation without its second term expressed by means of those of a cubic equation.

768. Let us suppose, therefore, α, β, γ to be the roots of a cubic equation

$$u^3 + Pu^2 + Qu + R = 0 \ldots\ldots\ldots\ldots (c),$$

which is connected with the biquadratic equation

$$x^4 - qx^2 - rx - s = 0 \ldots\ldots\ldots\ldots\ldots\ldots (b),$$

in such a manner, that

$$x = \sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma}:$$

it remains to shew that such an hypothesis will lead to the determination of P, Q, R in the cubic equation (*c*), and therefore to the knowledge of its roots α, β, γ.

For, in the first place, we get, by squaring the assumed value of x,

$$x^2 = \alpha + \beta + \gamma + 2\sqrt{\alpha\beta} + 2\sqrt{\alpha\gamma} + 2\sqrt{\beta\gamma}$$

$$= P + 2\{\sqrt{\alpha\beta} + \sqrt{\alpha\gamma} + \sqrt{\beta\gamma}\};$$

and therefore

$$x^2 - P = 2\{\sqrt{\alpha\beta} + \sqrt{\alpha\gamma} + \sqrt{\beta\gamma}\}.$$

If we square both sides of this equation, we get

$$x^4 - 2Px^2 + P^2 = 4(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$+ 8\sqrt{\alpha\beta\gamma}\,\{\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma}\}$$

$$= 4Q + 8\sqrt{R}x,$$

since $\alpha\beta + \alpha\gamma + \beta\gamma = Q$, $\alpha\beta\gamma = R$,

and $\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} = x$.

If we now transpose all the significant terms to one side of the equation, we get

$$x^4 - 2Px^2 - 8\sqrt{R}.x + P^2 - 4Q = 0,$$

a biquadratic equation, which will have the same roots with the original equation

$$x^4 - qx^2 - rx - s = 0,$$

$$\text{if } 2P = q,$$

$$8\sqrt{R} = r,$$

$$P^2 - 4Q = -s:$$

$$\text{or if } P = \frac{q}{2},$$

$$Q = \frac{q^2 + 4s}{16},$$

$$R = \frac{r^2}{64}.$$

Consequently the cubic equation, whose roots answer the required conditions, is

$$u^3 - \frac{q}{2}u^2 + \frac{(q^2 + 4s)}{16}u - \frac{r^2}{64} = 0^*.$$

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* If we suppose the roots, if any exist, of the biquadratic equation, to be expressible by means of those of a cubic equation, then it is obvious, in the first

Separation of the proper roots from the roots of solution.

769. If we were at liberty to assume indifferently any combination of the negative and positive values of $\sqrt{a}$,

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The assumption in the text is the only one which fulfils the required conditions.

first instance, that they must form some symmetrical combination of them, inasmuch as when they are expressed by general symbols, whatever condition affects one of them must equally affect the other two: and inasmuch as

$$x^4 = qx^2 + rx + s,$$

it will likewise follow, that whatever be the symmetrical expression of the roots of the *reducing* cubic equation which is equal to x, its fourth power must involve x^2 and x with coefficients and a term independent of x, which may be made to coincide with q, r and s respectively: if we should suppose

$$x = \alpha + \beta + \gamma,$$

where α, β, γ are the roots of the reducing cubic equation

$$u^3 - Pu^2 + Qa - R = 0,$$

then we might replace x by P, and therefore the determination of the values of one and of the other, would depend upon the solution of the same equation. If, again, we should assume

$$x = \sqrt[4]{\alpha} + \sqrt[4]{\beta} + \sqrt[4]{\gamma},$$

we should find

$$x^2 = \sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} + 2\sqrt[4]{\alpha\beta} + 2\sqrt[4]{\alpha\gamma} + 2\sqrt[4]{\beta\gamma},$$

and unless we should be able to express $\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma}$ in terms of P, Q, or R, we should make no progress towards the formation of a biquadratic equation which may be made to coincide with

$$x^4 - qx^2 - rx - s = 0:$$

similar observations would apply to the assumption of the sum of any *other* even, or to the sum of any odd, roots of α, β, γ, to express the value of x: and similar considerations would enable us to shew the impossibility of extending the same or any similar method to establish the dependence of the roots of an equation of the fifth or of a higher degree upon any such symmetrical combination of the roots of an equation of the next inferior degree.

Solutions of biquadratic equations which assume the existence of symmetrical combinations of the roots which are fewer in number than the roots themselves.

Most of the methods which have been proposed for the resolution of a biquadratic equation, have assumed the composition of the equation itself, and have proceeded to the determination of the terms of an equation of inferior degree, whose roots were symmetrical combinations of the roots of the primitive equation: thus, if a, b, c, d, be the roots of the equation

$$x^4 - qx^2 - rx - s = 0,$$

then, since $a + b + c + d = 0$, there are only three *different* values of $(a + b)^2$, $(a + c)^2$, $(a + d)^2$, $(b + c)^2$, $(b + d)^2$ and $(c + d)^2$, since $(a + b)^2 = (c + d)^2$, $(a + c)^2 = (b + c)^2$ and $(a + d)^2 = (b + c)^2$: if we assume, therefore, α, β, γ to represent those three different values, and to be also the roots of the cubic equation

$$u^3 - Pu^2 + Qu - R = 0,$$

Then

$\sqrt{\beta}$, $\sqrt{\gamma}$, we should find *eight* different combinations which would express values of x: but inasmuch as the process

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then any methods which serve to determine the values of $\alpha + \beta + \gamma$, $\alpha\beta + \alpha\gamma + \beta\gamma$ and $\alpha\beta\gamma$ in terms of q, r and s, will determine the values of P, Q and R, and, therefore, the values of α, β and γ: the transition from the values of α, β, γ to those of a, b, c and d may be effected as follows:

$$\begin{aligned} \sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} &= (a + b) + (a + b) + a + c \\ &= 2a + (a + b + c + d) = 2a, \\ \sqrt{\alpha} - \sqrt{\beta} - \sqrt{\gamma} &= 2b - (a + b + c + d) = 2b, \\ \sqrt{\beta} - \sqrt{\alpha} - \sqrt{\gamma} &= 2c - (a + b + c + d) = 2c, \\ \sqrt{\gamma} - \sqrt{\alpha} - \sqrt{\beta} &= 2d - (a + b + c + d) = 2d. \end{aligned}$$

If we compare the expressions thus obtained for a, b, c, d, with those which are given in the method of solution which is followed in the text, it will readily appear that the values of α, β and γ are quadruple of the roots of the reducing cubic which is there deduced, inasmuch as the values of certain combinations of their square roots in one case are double of those of corresponding combinations of them in the other: if, therefore,

$$u^3 - \frac{q}{2}u^2 + \frac{q^2 + 4s}{16}\,u - \frac{r^2}{64} = 0,$$

be the reducing cubic equation in one case, then

$$u^3 - 4.\frac{q}{2}.u + 4^2.\left(\frac{q^2 + 4s}{16}\right).u - 4^3.\frac{r^2}{64} = 0,$$

$$\text{or } u^3 - 2qu + (q^2 + 4s)\,u - r^2 = 0$$

must be the reducing cubic equation to the other: and this is the equation which is obtained by the artifice of transformation, of which Descartes was the author.

Again, in the complete biquadratic equation

$$x^4 - px^3 + qx^2 - rx + s = 0,$$

there are only *three different* values of combinations such as $ab + \frac{s}{ab}$, $ac + \frac{s}{ac}$, $ad + \frac{s}{ad}$, $bc + \frac{s}{bc}$, $bd + \frac{s}{bd}$, $cd + \frac{s}{cd}$; for inasmuch as $s = abcd$, it follows that the first and last, the second and fifth, the third and sixth, are equal to each other: if we assume, therefore, α, β, γ to represent those three different combinations, where α, β, γ are the roots of the equation

$$u^3 - Pu^2 + Qu - R = 0,$$

we shall be enabled to express P, Q and R in terms of the coefficients p, q, r and s of the original biquadratic equation: for

$P =$

of solution makes $8\sqrt{R} = 8\sqrt{\alpha}\sqrt{\beta}\sqrt{\gamma} = -r$, it is necessary that the continued product of those square roots

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$$\begin{aligned}
P = \alpha + \beta + \gamma &= ab + \frac{s}{ab} + ac + \frac{s}{ac} + ad + \frac{s}{ad} \\
&= ab + cd + ac + bd + ad + bc \\
&= q. \\
Q = (ab + bd)(ac + bd) &+ (ab + cd)(ad + bc) \\
&+ (ac + bd)(ad + bc) \\
= (a + b + c + d)&(abc + abd + acd + bcd) - 4abcd \\
= pr - 4s. & \\
R = (ab + cd)&(ac + bd)(ad + bc) \\
= abcd(a^2 + b^2 + c^2 + d^2) &+ a^2b^2c^2d^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right) \\
= s(p^2 - 2q) + s^2&\left(\frac{r^2}{s^2} - \frac{2q}{s}\right) \\
= (p^2 - 4q)s + r^2; &
\end{aligned}$$

and therefore the reducing cubic equation required is

$$u^3 - qu^2 + (pr - 4s)u - (p^2 - 4q)s - r^2 = 0,$$

which becomes, when $p = 0$,

$$u^3 - qu^2 - 4su + 4qs - r^2 = 0.$$

The nature of the value of the root of the reducing cubic equation in this case would shew that it was equally applicable to the solution of a biquadratic equation with or without its second term: and it will be very easy likewise to pass from the values of α, β, γ, or of $ab + \frac{s}{db}$, $ac + \frac{s}{ac}$, $ad + \frac{s}{ad}$, to those of ab, $\frac{s}{ab}$, ac, $\frac{s}{ac}$, ad, $\frac{s}{ad}$, and from them to the values of a, b, c, d, or the roots of the primitive equation: for since

$$\alpha = ab + \frac{s}{ab} = y + \frac{s}{y},$$

if $y = ab$, we have

$$y^2 - \alpha y = -s,$$

$$y = \frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha^2}{4} - s\right)},$$

or if

$$ab = \frac{\alpha}{2} + \sqrt{\left(\frac{\alpha^2}{4} - s\right)} = e',$$

$$\text{then } \frac{s}{ab} = cd = \frac{\alpha}{2} - \sqrt{\left(\frac{\alpha^2}{4} - s\right)} = e_1,$$

In

should always have the same sign, and therefore they must change their signs by pairs: we are thus restricted to the following values of x:

In a similar manner, we should find, if

$$ac = \frac{\beta}{2} + \sqrt{\left(\frac{\beta^2}{4} - s\right)} = e'',$$

$$\text{then } \frac{s}{ac} = bd = \frac{\beta}{2} - \sqrt{\left(\frac{\beta^2}{4} - s\right)} = e_2;$$

$$\text{and, if } ad = \frac{\gamma}{2} + \sqrt{\left(\frac{\gamma^2}{4} - s\right)} = e''',$$

$$\text{then } \frac{s}{ad} = bc = \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma^2}{4} - s\right)} = e_3.$$

Again, since

$$ab = e',$$
$$ac = e'',$$
$$ad = e''',$$
$$abcd = s,$$

if we multiply the three first equations together, we get

$$a^2bcd = e'e''e'''$$

$$\text{or } a^2 . abcd = a^2 s = e'e''e''',$$

$$a^2 = \frac{e'e''e'''}{s},$$

$$\text{and } a = \sqrt{\left(\frac{e'e''e'''}{s}\right)}.$$

In a similar manner we should find

$$b = \sqrt{\left(\frac{e'e_2e_3}{s}\right)},$$

$$c = \sqrt{\left(\frac{e_1e''e_3}{s}\right)},$$

$$d = \sqrt{\left(\frac{e_1e_2e'''}{s}\right)}.$$

We must confine ourselves to the positive values of these square roots, the other values of a, b, c, d being values of solution merely.

Since $u = y + \frac{s}{y}$, if we replace u by this value in the equation

$$u^3 - P u^2 + Q u - R = 0,$$

we

$$(1)\quad x_1=\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma},$$
$$(2)\quad x_2=\sqrt{\alpha}-\sqrt{\beta}-\sqrt{\gamma},$$
$$(3)\quad x_3=\sqrt{\beta}-\sqrt{\alpha}-\sqrt{\gamma},$$
$$(4)\quad x_4=\sqrt{\gamma}-\sqrt{\alpha}-\sqrt{\beta},$$

which are the proper roots of the biquadratic equation.

Composition of a biquadratic equation.

770. We thus determine both the existence and values of the four roots of the biquadratic equation

$$x^4-qx^2-rx-s=0,$$

and it is evident, from the expressions which we have obtained for them, that their sum is equal to zero: and it would likewise appear, that if

$$x^4-qx^2-rx-s=0,$$

was the result of the transformation of the equation

$$x^4+p_1x^3+p_2x^2+p_3x+p_4=0,$$

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we shall get

$$\left(y+\frac{s}{y}\right)^3-P\left(y+\frac{s}{y}\right)^2+Q\left(y+\frac{s}{y}\right)-R=0,$$

or multiplying by y^3, and expanding, we shall get

$$y^6-Py^5+Qy^4-(2Ps+R)\,y^3+Qsy^2-Ps^2y+s^3=0,$$

an equation of six dimensions, where the values of y are ab, ac, ad, bc, bd, cd, or ab, ac, ad and $\frac{s}{ab}$, $\frac{s}{ac}$, $\frac{s}{ad}$: the converse process for the reduction of this equation to a cubic, would direct us to combine the first and last, the second and last but one, the third and last but two, terms together, to divide by y^3, and to replace $y+\frac{s}{y}$, $y^2+\frac{s^2}{y^2}$ and $y^3+\frac{s^3}{y^3}$ by u, u^2-2 and u^3-3su.

The preceding methods of solution not applicable to equations of higher dimensions than the fourth.

The methods of solution of biquadratic equations which are founded upon the existence of symmetrical combinations of their roots, which are fewer in number than the roots themselves, are clearly not generally applicable to equations of higher dimensions: thus an equation of six dimensions wanting its second term would admit of 10 *different* values of the squares of the sum of any three of its roots, and also of 10 different values of the sum of the product of any three of its roots and of the last term divided by it, and the methods of solution founded upon such *transformations* of the primitive equation would lead to *general* equations of 10 dimensions: and such symmetrical combinations of the roots are of all others the most favourable, inasmuch as the transformed equation in all other cases would be at least of 15 dimensions.

so as to want its second term, that the roots of the primitive and complete equation would be $\frac{-p_1}{4} + x_1$, $\frac{-p_1}{4} + x_2$, $\frac{-p_1}{4} + x_3$, $\frac{-p_1}{4} + x_4$: if, again, we agree to denote these four roots by a, b, c, d, then since $x-a$, $x-b$, $x-c$, $x-d$, are severally factors of

$$x^4 + p_1 x^3 + p_2 x^2 + p_2 x + p_4,$$

we must have

$$x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4 = (x-a)\,(x-b)\,(x-c)\,(x-d)$$

$$= x^4 - (a+b+c+d)\,x^3 + (ab+ac+ad+bc+bd+cd)\,x^2$$

$$-(abc+abd+acd+bcd)\,x + abcd,$$

and, therefore,

$$p_1 = -(a+b+c+d),$$

$$p_2 = ab+ac+ad+bc+bd+cd,$$

$$p_3 = -(abc+abd+acd+bcd),$$

$$p_4 = abcd.$$

771. The following are examples of the solution of biquadratic equations: **Examples.**

(1) $x^4 - 25x^2 + 60x - 16 = 0$.

The reducing cubic equation is

$$u^3 - \frac{25}{2}u^2 + \frac{769}{16}u - \frac{225}{4} = 0,$$

whose roots determined as in Ex. 765. Art. 5. are

$$\frac{9}{4},\ 4,\ \frac{25}{4},$$

Consequently

$$x_1 = \sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3},$$

$$= \frac{3}{2} + 2 + \frac{5}{2} = 6,$$

$$x_2 = \frac{3}{2} - 2 - \frac{5}{2} = -3,$$

$$x_3 = 2 - \frac{3}{2} - \frac{5}{2} = -2,$$

$$x_4 = \frac{5}{2} - 2 - \frac{3}{2} = -1.$$

(2) $x^4 - 20x^3 + 148x^2 + 464x + 480 = 0.$

The transformed equation (wanting its second term) is

$$x'^4 - 2x'^2 + 16x' - 15 = 0.$$

The reducing cubic equation is

$$u^3 - u^2 + 4u - 4 = 0,$$

whose roots are 1, $2\sqrt{-1}$ and $-2\sqrt{-1}$.

The values of x' are 1, -3, $1 + 2\sqrt{-1}$ and $1 - 2\sqrt{-1}$.

The values of x are 6, 2, $6 + 2\sqrt{-1}$, $6 - 2\sqrt{-1}$.

Depression of recurring equations of even degrees to other equations of half the number of dimensions.

772. In the absence of general methods of solving equations of degrees superior to the fourth, there are many cases where equations present themselves with particular relations of their coefficients, which admit of reduction to binomial or other equations, whose degrees are within the limits of general solution: of this kind are the equations

$$x^6 + px^5 + qx^4 + rx^3 + qx^2 + px + 1 = 0,$$

and

$$x^8 + px^7 + qx^6 + rx^5 + sx^4 + rx^3 + qx^2 + px + 1 = 0,$$

called *recurring* equations, where the coefficients of the powers of x taken from the beginning and the end are the same: if in the second of these equations, (and the same may be done in all other recurring equations of even degrees) we combine together the terms with the same coefficients, and divide by x^4, we shall get

$$x^4+\frac{1}{x^4}+p\left(x^3+\frac{1}{x^3}\right)+q\left(x^2+\frac{1}{x^2}\right)+r\left(x+\frac{1}{x}\right)+s=0:$$

if we assume $x+\frac{1}{x}=u$, we shall also find

$$x^2+\frac{1}{x^2}=u^2-2,$$

$$x^3+\frac{1}{x^3}=u^3-3u,$$

$$x^4+\frac{1}{x^4}=u^4-4u^2+2;$$

and the substitution of these values will give the biquadratic equation

$$u^4+pu^3+(q-4)u^2+(r-3p)u+s-2q+2=0.$$

Reciprocal equations.

773. We have thus reduced the primitive recurring equation to one of half the number of dimensions, and a similar reduction may be effected in all recurring equations of even degrees: and inasmuch as $x+\frac{1}{x}=u$, there are two values of x for every value of u, which are clearly the reciprocals of each other: or, in other words, if a be a value of x, or a root of the recurring equation, $\frac{1}{a}$ must also be a root, and similarly for all the other roots of the equation*: it is for this reason that such equations are sometimes called *reciprocal* equations.

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* If $x+\frac{1}{x}=a$, then $x=\frac{a}{2}\pm\sqrt{\left\{\frac{a^2}{4}-1\right\}}$ and $\frac{a}{2}+\sqrt{\left\{\frac{a^2}{4}-1\right\}}=\frac{1}{\frac{a}{2}-\sqrt{\left\{\frac{a^2}{4}-1\right\}}}$, and conversely.

Depression of recurring equations of odd degrees.

774. If the recurring equation be of odd dimensions, it can have no middle term, and there can exist no coefficient of the equation without an equal coefficient corresponding to it, which are severally attached to powers of x, of which one is odd, and the other even: under such circumstances + 1 or − 1 must necessarily be a root, and therefore $x \pm 1$ a factor of the equation, inasmuch as the substitution of − 1 or + 1 for x, must make the pairs of terms with equal coefficients, destroy each other, according as they have the same or different signs: if the equation be divided by this factor, the quotient will form a recurring equation of even dimensions, which may therefore be reduced to one of half of the remaining dimensions, as in the case of such equations which we have considered above.

Examples.

775. Thus 1 is a root of the equation

$$x^5 - px^4 + qx^3 + qx^2 + px - 1 = 0,$$

which becomes, when divided by $x - 1$,

$$x^4 - (p-1)x^3 + (q-p+1)x^2 - (p-1)x + 1 = 0,$$

which is a biquadratic recurring equation, and which may be solved therefore by means of a quadratic equation.

Again, − 1 is a root of the equation

$$x^7 + px^6 - qx^5 + rx^4 + rx^3 - qx^2 + px + 1 = 0,$$

which becomes, when divided by $x + 1$,

$$x^6 + (p-1)x^5 - (q+p-1)x^4 + (r+q+p-1)x^3 - (q+p-1)x^2$$
$$+ (p-1)x + 1 = 0,$$

a recurring equation of *six* dimensions, which may be solved, therefore, by means of a cubic equation.

Quasi recurring equations.

776. A very remarkable case of a *quasi* recurring equation, admitting of reductions precisely similar to, and co-extensive with, the recurring equations which we have just been considering, has been noticed and solved in the Note to Art. 768. page 624.

777. Other equations, which are not *quasi* recurring equations, but whose coefficients possess a particular relation to each other, may be solved by formulæ similar to that which occurs in the general solution of a cubic equation. Thus, if

Equations resolvible by formulæ similar to those which are employed for the solution of a cubic equation.

$$x^5 + \frac{s^5}{x^5} = a,$$

$$\text{then } x^{10} - ax^5 = -s^5,$$

$$\text{and } x = \left\{\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - s^5\right)}\right\}^{\frac{1}{5}},$$

and therefore

$$x + \frac{s}{x} = \left\{\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - s^5\right)}\right\}^{\frac{1}{5}} + \frac{s}{\left\{\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - s^5\right)}\right\}^{\frac{1}{5}}}$$

$$= \left\{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - s^5\right)}\right\}^{\frac{1}{5}} + \left\{\frac{a}{2} - \sqrt{\left(\frac{a^2}{4} - s^5\right)}\right\}^{\frac{1}{5}}.$$

But, if we make $u = x + \frac{s}{x}$, we get

$$u^5 = \left(x + \frac{s}{x}\right)^5 = x^5 + \frac{s^5}{x^5} + 5s\left(x^3 + \frac{s^3}{x^3}\right) + 10s^2\left(x + \frac{s}{x}\right)$$

$$= u + 5s\,(u^3 - 3su) + 10s^2u,$$

which becomes, by transferring all the terms to one side,

$$u^5 - 5su^3 + 5s^2u - a = 0,$$

an equation of the fifth degree, whose roots have been determined.

Again, the formula

$$\left\{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - s^6\right)}\right\}^{\frac{1}{6}} + \left\{\frac{a}{2} - \sqrt{\left(\frac{a^2}{4} - s^6\right)}\right\}^{\frac{1}{6}}$$

will express the roots of the equation

$$u^6 - 6su^4 + 9s^2u^2 - 2s^3 - a = 0;$$

and in a similar manner we may proceed to determine generally, the form of the equation whose roots are expressed by the formula

$$\left\{\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - s^n\right)}\right\}^{\frac{1}{n}} + \left\{\frac{a}{2} - \sqrt{\left(\frac{a^2}{4} - s^n\right)}\right\}^{\frac{1}{n}},$$

both when n is an odd and an even number.

CHAP. XVI.

On the Solution of Simultaneous Equations.

Dependent and independent symbols.

778. The object proposed in the solution of those equations which have been considered in the preceding Chapter, has been to express one of the symbols in terms of all the other symbols of the equation, upon whose values therefore whether assigned or assignable, the value of this symbol is dependent, and by means of which it is determined: and the same necessary dependence between one symbol and all the others is presumed to exist in all equations whatsoever, whether the law and nature of that dependence is assignable or not: it is for this reason that such a symbol is termed the *unknown* quantity in the primitive equation, when all the other symbols are assigned or assignable, and when its determination is the object proposed by the solution of the equation; or the *dependent* symbol, when the other symbols are not assigned, but are perfectly arbitrary and independent of each other: in other words, the dependent symbol can only become known and determined, upon the assignation of specific values of the arbitrary and independent symbols, in terms of which it is expressed.

Connection of the character of dependent and independent symbols with the solution of the equation which involves them.

779. Thus the equation

$$ax + by - c = 0,$$

when solved with respect to x, gives us

$$x = \frac{c - by}{a},$$

where x is *dependent* upon y and the symbols a, b and c, which are all of them equally arbitrary and *independent*, so far as the conditions of the equation determine them: but if we suppose (as is most commonly the case), that the first letters of the alphabet a, b, c, denote quantities which are known or determinate, then x is the *dependent*, and y the *independent* symbol: if, however, we should solve the primitive equation (1) with respect to y, we should get

$$y = \frac{c - ax}{b},$$

when y will become the *dependent*, and x the *independent* symbol: the character of *dependence* and *independence*, therefore, as distinguishing one symbol of an equation from the others, which are unknown and indeterminate, is determined by the solution, or *presumed* solution, of the equation with respect to one or other of those symbols*.

Meaning of the term function: explicit and implicit functions.

* The dependence of one symbol upon an independent symbol or symbols, whether accompanied by known and determinate symbols or not, is usually expressed by the term *function;* and the function (for the term is used absolutely) is said to be *explicit* or *implicit*, according as the equation which involves the dependent and independent symbols, is solved with respect to the dependent symbol or not: thus x is an *explicit* function of y, in the equation

$$x = \frac{c - by}{a},$$

and y is an *explicit* function of x, in the equation

$$y = \frac{c - ax}{b};$$

but x is an *implicit* function of y, or y of x, (for those relations, as we have shewn in the text, are convertible), in the equation

$$ax + by - c = 0.$$

Presumed functions.

The term *function* is not only used to denote the dependence of one symbol or quantity upon another or others, when their dependence is completely exhibited in a symbolical equation, whether *implicitly* or *explicitly*, but also when such dependence is *presumed* to exist, from the nature of the case, anteriorly to the investigation of any equation by which it may be expressed: thus if we should wish to express the dependence of the *space* described by a body when acted upon by *determinate* forces according to *determinate* laws, upon the *time* of its motion, we should say that the *space* was a *function* of the *time:*

The exhibition of the dependence of one symbol upon the others in an equation.

780. When more than one unknown or indeterminate symbol presents itself in an equation, the solution of the equation with respect to any one of them, and therefore the exhibition of its *dependence* upon the others must be effected by the general methods which have been taught in the last Chapter, and must be limited by the limitation of those methods: thus the equation

$$x+y+\sqrt{(x+y)}=12$$

may be reduced to the equivalent equation

$$x=9-y,$$

where the actual *dependence* of x upon y is exhibited: in a similar manner the equation

$$\frac{x^2}{y^2}+\frac{2x+y}{\sqrt{y}}=20-\frac{y^2+x}{y}$$

leads to the equivalent equation

$$x=\frac{-y}{2}\pm\frac{9y}{2}-y^{\frac{3}{2}}:$$

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time: and if we should agree to denote the space by s, and the time by t, we should express the same proposition by means of the equation $s=f(t)$, where the letter f prefixed to the independent symbol (t), is used to express the term *function*.

Inverse functions.

We have on a former occasion, (Note to Art. 725.), shewn that if $y=\log x$, then $x=\log^{-1} y$, if $y=\sin x$, then $x=\sin^{-1} y$, if $y=\tan x$, then $x=\tan^{-1} y$, and so on; if instead of *log*, or *sin*, or *tan*, which are used to denote particular modifications, or rather *functions*, of the symbol which succeeds them, we had used the common letter f, this *reciprocation* of the dependent and independent symbol, would have been expressed by the equations $y=f(x)$ and $x=f^{-1}(y)$: such functions are said to be the *inverse* of each other: the existence of this relation is a necessary consequence of the equation by which they are expressed, and its presumed solution with respect to any of its symbols, and the principles of the notation by which it is expressed may be seen in the Note to which we have just referred.

Dependent and independent variables.

Quantities and their symbols may be distinguished from each other as known and unknown, but more commonly when their values are indeterminate, as variable and invariable: it is in this latter character that they are considered in many of the most important applications of mathematics, and the symbols which denote them when involved in an equation, or presumed to be so, assume the relation of dependent and independent variables: it is sufficient for our purpose on the present occasion merely to notice these terms, and their usage.

and the equation

$$x^4 - 2x^2y^2 + y^4 - 2a^2x^2 + 2a^2y^2 = b^4 - a^4,$$

leads to the equation

$$x = (a^2 \pm b^2 + y^2)^{\frac{1}{2}}.$$

Contributes nothing to the determination of their values.

781. It is obvious, however, that this exhibition of the dependence of one symbol upon one or more other indeterminate symbols leaves them necessarily as indeterminate as in the primitive equation: and however important such reductions may be for many of the purposes for which such equations may be applied, it can only be by the aid of other hypotheses or conditions, that their values can be absolutely determined: such conditions, however various when considered in connection with the problems from which they arise, will generally resolve themselves into the *simultaneous* existence of as many independent equations, as there are indeterminate, and in this case, unknown quantities involved in any one of them.

Process by which such equations may be solved or one or more unknown quantities eliminated.

782. Thus, if we have two equations involving x and y, possessing *simultaneous* values in both of them: then it is obvious that the value of x determined, in terms of y, from the first equation, must be the same as that of x determined in terms of y from the second: if we equate these values, we get an equation involving y only: the solution of this equation gives the absolute value or values of y, and leads us therefore necessarily to the absolute value or values of x: the following are examples:

Examples.

$$(1)\quad \left.\begin{aligned} 7x - 9y &= 7 \\ 3x + 10y &= 100 \end{aligned}\right\}.$$

The first equation, solved with respect to x, give us

$$x = \frac{7 + 9y}{7};$$

The second equation, solved with respect to x, likewise gives us

$$x = \frac{100 - 10y}{3}.$$

The values of x and y in the two equations being *assumed* to be identical, we must have

$$\frac{7+9y}{7}=\frac{100-10y}{3},$$

$$21+27y=700-70y,$$

$$97y=679,$$

$$y=7,$$

and therefore $x=\frac{7+9y}{7}=\frac{7+63}{7}=\frac{70}{7}=10.$

$$(2)\quad \left.\begin{array}{l} ax+by=c \\ ax+\beta y=\gamma \end{array}\right\}.$$

From the first equation, we get

$$x=\frac{c-by}{a}.$$

From the second equation, we get

$$x=\frac{\gamma-\beta y}{a}.$$

Equating these values of x, we get

$$\frac{c-by}{a}=\frac{\gamma-\beta y}{a},$$

$$ac-aby=a\gamma-a\beta y,$$

$$(a\beta-ab)\,y=a\gamma-ac,$$

$$y=\frac{a\gamma-ac}{a\beta-ab},$$

and, therefore,

$$x=\frac{c-by}{a}=\frac{c-\left(\frac{a\gamma-ac}{a\beta-ab}\right)b}{a}$$

$$=\frac{b\gamma-\beta c}{ab-a\beta}.$$

$$(3)\quad \left.\begin{array}{l} \frac{x^2}{y^2}-\frac{4x}{y}+\frac{35}{9}=0 \\ x-y=2. \end{array}\right\}.$$

If we solve the first equation with respect to x, we get

$$x = \left(2 + (1)^{\frac{1}{2}} \frac{1}{3}\right) y.$$

If we solve the second equation with respect to x, we get

$$x = 2 + y.$$

If we equate these values of x, we get

$$\left(2 + (1)^{\frac{1}{2}} \frac{1}{3}\right) y = 2 + y:$$

$$\text{and therefore } y = 3 \text{ or } \frac{3}{2}.$$

The corresponding values of x are 5 and $\frac{7}{2}$.

$$(4) \quad \left.\begin{aligned} \frac{x+2}{5} - \frac{y+4}{8} + \frac{z+5}{10} - 1 &= 0 \\ x + y + z - 12 &= 0 \end{aligned}\right\}.$$

Reducing the first equation, and solving it with respect to x, we get

$$x = \frac{5y - 4z + 24}{8}.$$

From the second equation, we get

$$x = 12 - y - z.$$

Equating these values of x, we get

$$\frac{5y - 4z + 24}{8} = 12 - y - z,$$

$$\text{or } 13y + 4z - 72 = 0.$$

We have thus reduced the two primitive equations to one, involving two unknown, and in this case, *indeterminate* quantities: the third unknown quantity has been *eliminated* from them, and by the process necessary for that purpose, we have reduced the number of the primitive equations by 1.

If we had commenced by eliminating y, instead of x, from the primitive equations, we should have obtained the equation

$$13x + 9z - 84 = 0:$$

and if we had eliminated z, instead of x or y, we should have obtained the equation

$$4x - 9y + 24 = 0.$$

Elimination: final equation.

783. The *elimination* of symbols or unknown quantities, whether determinate or indeterminate, from equations in which they are involved, is one of the most important operations in Algebra, and the process which is necessary for that purpose is identical with the process for the solution of such equations, or rather for their reduction to a single final equation: if this final equation involves one unknown quantity only, its value may be determined absolutely by means of it: if it involves two unknown quantities, they are both of them indeterminate, and one of them is independent and perfectly arbitrary: if the final equation involves more than two unknown and indeterminate symbols or quantities, then all but one of them are independent and arbitrary.

The number of any system of equations diminished by the number of unknown quantities eliminated.

784. In the preceding examples, the elimination of one unknown quantity has reduced the number of unknown quantities from two to one: and a very little consideration would shew that if n was the number of the primitive equations involving any number of unknown quantities of which x was one, then the number of *independent* equations which would result from the elimination of x, would be $n-1$: for if each of these equations be solved with respect to x, (and we assume the practicability of such solutions) and if we should thus obtain

$$x=A_1,\quad x=A_2,\quad x=A_3,\ \dots\dots\dots\ x=A_n,$$

where A_1, A_2, A_3, ... A_n are the symbolical values of x derived from the several equations, then we should obtain, by equating the first value of x with each of the others, the following $(n-1)$ equations,

$$A_1-A_2=0,\ A_1-A_3=0,\ A_1-A_4=0,\ \dots\dots\ A_1-A_n=0.$$

All other similar combinations of the quantities A_1, A_2,

A_3, ... A_n with each other, though equally admissible with the preceding, will lead to equations immediately derivable from them, and therefore presenting no new and *independent* conditions for the determination of the unknown symbols which they involve: thus

$$\begin{aligned} A_2 \quad - A_3 &= (A_1 - A_3) - (A_1 - A_2) \quad = 0, \\ A_3 \quad - A_5 &= (A_1 - A_5) - (A_1 - A_3) \quad = 0, \\ A_{n-1} - A_n &= (A_1 - A_n) - (A_1 - A_{n-1}) = 0, \\ \ldots\ldots\ldots\ldots \quad & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots : \end{aligned}$$

in other words, the equation corresponding to any such combination will always arise from subtracting from each other some two of the $(n-1)$ equations, which resulted from the first series of combinations which we thought it proper and convenient to select.

Inasmuch as the elimination of one unknown quantity from any number of (n) equations, diminishes the number of independent equations by 1, it will follow that the successive elimination of $(n-1)$ unknown quantities will diminish the number (n) of equations by $(n-1)$, and will therefore leave a single final equation remaining: if the number of unknown quantities be the same as the number of equations which involve them, the final equation will involve one unknown quantity only, which will admit of determination by any methods which enable us to solve the equation itself: but if the number of unknown quantities exceeds the number of equations by (m), the final equation will involve $(m+1)$ unknown quantities, which are therefore indeterminate in common with all the others, and (m) of them are *independent:* but if, on the contrary, the number of equations exceeds the number of unknown quantities (n) by (m), all the unknown quantities may be determined from any combination of n of those equations, which are $\dfrac{(n+m)\,(n+m-1)\,\ldots\ldots\,(m+1)}{1\,.\,2\,\ldots\ldots\,n}$ in number: the values of the unknown quantities which would thus be obtained would or might be different for different combinations, and therefore m out of the $n+m$ equations are at least superfluous.

785. When we speak of equations as independent of each other, we mean such equations as severally contain conditions for the determination of the unknown quantities which they involve, which are not supplied by the other equations, nor derivable from them: we must exclude therefore all equations which are multiples of any other of the equations proposed, when the multiplier is an assigned quantity, and independent of the unknown quantities considered in the equations: for such a factor can express no new condition for the determination of the quantities or symbols which are involved in it: again, we must exclude equations which are multiples of other equations, when the multiplier involves one or more unknown quantities, if such a multiplier be a factor of another equation: for in such a case, this multiplier, (by which alone the equation in question differs from another equation proposed) will express no condition which is not already involved in the other equations: lastly, we must exclude all such equations as are the sums or differences or products of the other equations, or of assigned multiples of them, inasmuch as all such equations are satisfied, by any values of the unknown quantities which satisfy the several equations which are involved in them, and consequently express no new and independent conditions for their determination.

Equations independent of each other.

786. It is not necessary that every equation should involve all the unknown quantities which are required to be determined: thus one equation may involve one unknown quantity, when its value is determined by means of it, or two unknown quantities or more: two equations may involve two unknown quantities, whose values will then be determined by means of them, or they may involve three unknown quantities or more: three equations may involve three unknown quantities, when their values are determined by means of them, or they may involve four unknown

In a system of equations it is not necessary that every equation should involve all the unknown quantities.

quantities or more: but the number of equations in any series of connected equations must not exceed the number of unknown quantities involved in them, inasmuch as in such a case, some of these equations would be superfluous, or would give different values of those unknown quantities when differently combined with each other.

Suppression of common factors of equations.

787. Any common factor, involving one or more unknown quantities, of two or more of a system of equations, may be detected by the methods employed for finding the highest common divisor, and must be excluded from them as foreign to the determination of the unknown quantities involved in them: for if such a factor involve one unknown quantity, any value of it which makes this factor zero, (and therefore determines it) will verify the equations which involve it, whatever be the values of the other unknown quantities; thus if there were two equations with two unknown quantities possessing such a factor, the other unknown quantity would remain perfectly indeterminate, and similarly in other cases. Again, if such a factor involve two or more unknown quantities, there are an infinite number of values which make this factor zero, and which therefore verify the equations which involve it, without any reference to the values of the unknown quantities which the equation divided by this factor contain, and without contributing therefore in any respect to their determination; equations therefore which involve such common factors cannot be considered as independent of each other as long as such factors exist.

General process of elimination.

788. It remains to consider the particular methods which are requisite for the *elimination* of unknown or indeterminate quantities from a system of equations: we shall confine our attention almost exclusively to a system of two equations with two unknown quantities, and to the following method, which is of all others the most general, though not always the most expeditious.

If there be two equations $E_1 = 0$ and $E_2 = 0$, involving x and y, and possessing simultaneous values of them, then for every *proper* value of y, there must correspond a *proper* value of x: in other words, a *proper* value of y must be of such a kind, that if substituted in both the equations, the resulting equations involving x *only*, will have a *common* value or values of x: consequently, if E_1 and E_2 become X_1 and X_2 upon the substitution of a *proper* value of y, then X_1 and X_2 must have a common factor, which, when made equal to zero, will give the corresponding proper value or values of x: in order, therefore, to find such factors, we institute upon E_1 and E_2 arranged according to powers of x, the process for finding their highest common divisor, and continue it (excluding throughout fractional quotients and remainders) until we obtain a remainder Y, which involves y only: whatever value of y makes $Y=0$ will make the last divisor, which involves x, a common factor of E_1 and E_2, which upon this substitution become X_1 and X_2: if we find therefore all the values of y which make $Y=0$, we shall obtain all the corresponding values of x, and we shall thus be enabled to form all the sets of *proper* roots of the equations $E_1=0$ and $E_2=0$.

Circumstances which may modify the preceding process.

789. The preceding is the statement of the general process which must be followed in such cases, without reference to circumstances which may sometimes modify the results which are obtained, or which may apparently cause us to fail in obtaining them: for in the first place, the process of finding the highest common divisor of E_1 and E_2 may introduce factors into Y, and therefore values of y in $Y=0$, which are foreign to the system of equations: in the second place, two, three or more values of x may correspond to the same value of y, in which case the common factor of X_1 and X_2 will be of the form

$$x^2+cx+b \text{ or } x^3+cx^2+bx+d,$$

and so on, and therefore the last divisor in the first case, the two last divisors in the second, and so on, will become equal to zero, for the corresponding proper value of y:

these cases and their theory will be more particularly noticed amongst those examples which follow, in which they first occur.

Incompatible equations.

790. Again, the system of equations may be themselves inconsistent with the existence of simultaneous values of x and y: this would be indicated by the final remainder or the last divisor becoming a numerical quantity which cannot be made equal to zero; circumstances which would indicate in one case that there was no value of x, which would give a common value of y, and in the second case, that there was no value of y, which would give a common value of x.

Equations which have common factors involving x and y or both.

791. Lastly, the final equation $Y=0$, may become an identical equation, in which case the last divisor, if not itself identically equal to zero, or the last divisor which is not so, must be a common factor of E_1 and E_2. This common factor must be excluded from the two primitive equations and the process instituted again with the quotients which result from its exclusion, which, when made equal severally to zero, become the independent equations which are the proper objects of consideration.

Examples.

792. The following are examples:

$$\begin{array}{ll} (1) & ax + by - c = 0 = E_1, \\ & \alpha x + \beta y - \gamma = 0 = E_2, \end{array}$$

$$\begin{array}{ll} ax + by - c) & \alpha x + \beta y - \gamma \\ & \qquad a \\ \hline & a\alpha x + a\beta y - a\gamma \,(\alpha, \quad \text{(Art. 171.)}, \\ & a\alpha x - \alpha b y - \alpha c \\ \hline & (a\beta - \alpha b)\, y - (a\gamma - \alpha c) = Y. \end{array}$$

If $Y = 0$, we find $y = \dfrac{a\gamma - \alpha c}{a\beta - \alpha b}$, and therefore $ax + by - c$ $= ax + b\left(\dfrac{a\gamma - \alpha c}{a\beta - \alpha b}\right) - c = 0$, which gives us $x = \dfrac{\beta c - b\gamma}{a\beta - \alpha b}$.

If $a\beta - \alpha b = 0$, and if $a\gamma - \alpha c$ be not equal to zero, there is no common value of x in the two equations, which are therefore incompatible with each other.

If $a\beta - ab = 0$, and also $a\gamma - ac = 0$, then the two equations have a common measure which is independent of the particular values of x and y: under such circumstances the equations are *not* independent of each other, and the values of x and y continue indeterminate.

$$(2)\quad \left.\begin{aligned} x + y &= 5 \\ x^2 + y^2 &= 13 \end{aligned}\right\} \text{ or } \begin{aligned} x + y - 5 &= 0 = E_1 \\ x^2 + y^2 - 13 &= 0 = E_2 \end{aligned}$$

$$\begin{array}{r} x + y - 5)\ x^2 + y^2 - 13\ (x - y + 5 \\ x^2 + (y - 5)\,x \qquad\qquad\quad \\ \hline -(y - 5)\,x + y^2 - 13 \qquad\quad \\ -(y - 5)\,x - y^2 + 10y - 25 \\ \hline 2y^2 - 10y + 12 = Y. \end{array}$$

If $Y = 0$, or $2y^2 - 10y + 12 = 0$, we find $y = 3$ or 2: if $y = 3$, we have $x + y - 5 = 0$, and therefore $x = 2$: if $y = 2$, we have $x + y - 5 = 0$, and therefore $x = 3$.

$$(3)\quad x + y = a \text{ or } x + y - a = 0 = E_1,$$

$$x^3 + y^3 = b \text{ or } x^3 + y^3 - b = 0 = E_2,$$

$$\begin{array}{r} x + y - a)\ x^3 + y^3 - b\ (x^2 - (y - a)\,x + (y - a)^2 \\ x^3 + (y - a)\,x^2 \qquad\qquad\qquad\qquad\qquad\qquad \\ \hline -(y - a)\,x^2 \qquad\qquad\qquad\qquad\qquad\quad \\ -(y - a)\,x^2 - (y - a)^2\,x \qquad\qquad\qquad \\ \hline (y - a)^2\,x + y^3 - b \qquad\quad \\ (y - a)^2\,x + (y - a)^3 \qquad \\ \hline 3ay^2 - 3a^2y + a^3 - b = Y. \end{array}$$

If $Y = 0$, we find

$$y = \frac{a}{2} + \sqrt{\left(\frac{4b - a^3}{12a}\right)}, \text{ and } \therefore x = \frac{a}{2} - \sqrt{\left(\frac{4b - a^3}{12a}\right)},$$

$$y = \frac{a}{2} - \sqrt{\left(\frac{4b - a^3}{12a}\right)}, \text{ and } \therefore x = \frac{a}{2} + \sqrt{\left(\frac{4b - a^3}{12a}\right)},$$

The equation E_1 is of three dimensions, but the final equation $Y = 0$ is of two dimensions only: it is obvious, however, that the second equation is reducible to the independent equation of two dimensions

$$x^2 - xy + y^2 - \frac{b}{a} = 0,$$

by dividing $x^3 + y^3$ by $x - y$, and b by a: the same remark is applicable to the dimensions of the final equation deducible from the two equations $x + y = a$, and $x^5 + y^5 = b$, or to that of the two equations $x - y = 0$, $x^4 - y^4 = 0$, and similarly in other cases, where such a division of the equal members of one equation by the corresponding equal members of the other is found to be practicable.

$$\begin{array}{lll} (4) & xy = a, \text{ or } xy - a & = 0 = E_1, \\ & x^3 + y^3 = b, \text{ or } x^3 + y^3 - b & = 0 = E_2. \end{array}$$

$$\begin{array}{l} yx - a)\ x^3 + y^3 - b \\ \quad yx^3 + y^4 - by\ (x^2 \\ \quad yx^2 - ax^2 \\ \hline \qquad ax^2 + y^4 - by \\ \qquad ayx^2 + y^5 - by^2\ (ax \\ \qquad ayx^2 - a^2x \\ \hline \qquad\quad a^2x + y^5 - by^2 \\ \qquad\quad a^2yx + y^6 - by^3\ (a^2 \\ \qquad\quad a^2yx - a^3 \\ \hline \qquad\quad y^6 - by^3 + a^3 = Y = 0. \end{array}$$

If ρ and ρ' express the arithmetical values of

$$\left\{\frac{-a^3}{2} + \sqrt{\left(\frac{a^6}{4} - \frac{b^3}{27}\right)}\right\}^{\frac{1}{3}} \text{ and } \left\{\frac{-a^3}{2} - \sqrt{\left(\frac{a^6}{4} - \frac{b^3}{27}\right)}\right\}^{\frac{1}{3}},$$

and if 1, a, a^2 be the three cube roots of 1, then the corresponding values of y and x are

$$y = \rho,\ a\rho,\ a^2\rho,\ \rho_1,\ a\rho_1,\ a^2\rho_1;$$

$$x = \rho_1,\ a\rho_1,\ a^2\rho_1,\ \rho,\ a\rho,\ a^2\rho.$$

$$(5)\quad \left.\begin{aligned} xy + xy^2 - 12 = 0 = E_1 \\ x + xy^3 - 18 = 0 = E_2 \end{aligned}\right\},$$

$$(1 + y^3)\,x - 18)\ (y + y^2)\,x - 12$$

$$y\,(1 + y^3)\,x - 12\,(1 - y + y^2)\ \ (y$$

$$\underline{y\,(1 + y^3)\,x - 18y}$$

$$-12y^2 + 30y - 12 = Y = 0.$$

Therefore

$$y^2 - \frac{5y}{2} + 1 = 0.$$

If $y = 2$, then $x = 2$.

If $y = \frac{1}{2}$, then $x = 16$.

In this case, the coefficients of the first terms of the divisor and dividend have a common factor $1 + y$, and therefore $y\,(1 + y^3)$ is their lowest common multiple: if this circumstance was not attended to, the final equation would become

$$(y + 1)\left(y^2 - \frac{5y}{2} + 1\right) = 0,$$

and would involve a *foreign* factor $y + 1$, and therefore a value of y to which no value of x would correspond.

$$(6)\quad \left.\begin{aligned} yx^2 - 7x + 2 = 0 \\ (y - 1)\,x^2 - 3x - 2 = 0 \end{aligned}\right\},$$

$$yx^2 - 7x + 2)\ (y - 1)\,x^2 - 3x - 2$$

$$y\,(y - 1)\,x^2 - 3yx - 2y\ \ (y - 1$$

$$\underline{y\,(y - 1)\,x^2 - 7\,(y - 1)\,x + 2\,(y - 1)}$$

$$(4y - 7)\,x - (4y - 2)$$

$$y - 7)\,x - (4y - 2))\ yx^2 - 7x + 2$$

$$(4y - 7)^2 yx^2 - 7\,(4y - 7)^2 x + 2\,(4y - 7)^2$$

$$\underline{(4y - 7)^2 yx^2 - (4y - 2)\,(4y - 7)\,yx}$$

$$(4y - 7)\,(4y^2 - 30y + 49)\,x + 2\,(4y - 7)^2$$

$$\underline{(4y - 7)\,(4y^2 - 30y + 49)\,x - (4y - 2)\,(4y^2 - 30y + 49)}$$

$$16y^3 - 96y^2 + 144y = 0 = Y.$$

This final equation is reducible to the form $16y(y-3)^2=0$: if $y=3$, we have $x=2$: but if $y=0$ which is the third value of y in $Y=0$, we find no corresponding value of x: for under such circumstances E_1 and E_2 become $-7x+2$ and $-x^2-3x-2$, which have no common measure: but E_1 and yE_2 become, under the same circumstances, $-7x+2$ and 0, which are equally zero, when $x=\frac{2}{7}$: it appears therefore that $y=0$ is a root of the final equation, which is introduced by the process of solution, and which is altogether *foreign* to the equations $E_1=0$, and $E_2=0$ which were originally proposed.

Theory of foreign factors in the final equation.

If we consider this process generally, it would appear that $Y=0$, would necessarily contain the values of y, which severally give common factors, not merely of E_1 and E_2, but of E_1 and E_2 one or both of them, or of any of their successive remainders, multiplied by such factors as are necessary to avoid the introduction of fractions: or, in other words, the final equation will be the same, as if such factors had existed as essential parts of the primitive equation or equations: if to the values of y which belong to such factors, there are found corresponding values of x, then such values of y must necessarily present themselves as *foreign* roots of the final equation: but if there is no value of x corresponding to the values of y which belong to such factors, then the final equation cannot comprehend such values of y among its roots, and will be the same therefore as if such factors had never been introduced. It is for this reason, that the factors introduced for the last division, if the divisor involves the first power of x *only*, will never affect the degree of the final equation: for the values of x which belong to the factor whose introduction may be necessary in this case, will make the first term of the divisor equal to zero, and will leave therefore no term in which x exists: thus, in the example which we have been considering, if we make $(4y-7)^2=0$, the last divisor becomes -5, which is a numerical quantity.

No foreign factor introduced into the final equation, if the equations do not exceed the second degree.

For the same reason likewise, it follows, that if the primitive equations do not exceed the second degree, no *foreign* root or factor can appear in the final equation: for if the first term of both of them involves x^2, it must present itself with a numerical coefficient only, and will not require therefore the introduction of any factor involving y: and it is obvious that the second divisor can involve x to the first degree only.

Suppression of factors in the primitive equation or the remainders.

It will frequently happen that a factor involving only y may be suppressed in the first or some subsequent remainder, and by such means the operation which is otherwise necessary, may be greatly shortened: if a value or values of x correspond to this factor, the simple factor itself, or powers of it, would appear in the final equation, if not suppressed, according as the remainder in which it first appeared, was the remainder immediately preceding the final equation or not: for if it appeared in the last remainder but one, its square would be involved in the final equation; if in the last remainder but two, its cube, and so on: if therefore it is required to determine the *complete* final equation, such powers of this factor must be restored to it, at the conclusion of the operation: but if it is merely proposed, as is most commonly the case, to determine the different systems of values of x and y, it will be merely necessary to consider the value or values of y deduced from this factor in connection with those which are deduced from the final equation which results from its suppression: these observations would equally apply, if the factor in question was a factor of one of the primitive equations.

If in the example which has given occasion to these remarks, we had made $(y - 1)\,x - 3x - 2$ the divisor, and $yx^2 - 7x + 2$ the dividend, the final equation at which we should have arrived, would have been $(y - 1)\,(y - 3)^2 = 0$, where $y - 1$, and not y, would be the foreign factor introduced by the operation.

$$(7)\quad x^2 + (2y - 7)\,x + y^2 - 7y - 8 = 0,$$

$$x^2 + (2y - 5)\,x + y^2 - 5y - 6 = 0.$$

If we subtract these equations from each other, we get

$$-\,2x - 2y - 2.$$

Therefore

$$\begin{array}{rl}
x + y + 1) & x^2 + (2y - 5)\,x + y^2 - 5y - 6\ (x \\
 & \underline{x^2 + (\ y + 1)\,x\qquad\qquad\qquad\quad} \\
 & \qquad\quad (y - 6)\,x + y^2 - 5y - 6 \\
 & \underline{\qquad\quad (y - 6)\,x + y^2 - 5y - 6}
\end{array}$$

In this case there is no final equation, inasmuch as $x+y+1$ is a common factor, and therefore x and y are indeterminate; but if we take the equations

$$x+y-6=0,$$
$$x+y-8=0,$$

which result from the suppression of this factor, they are obviously incompatible with each other.

In a similar manner it will be found that the equations

$$yx^3+y^2x^2+(y^2+y)yx+y^4+7y^2=0,$$
$$x^3-y^2x+7x=0,$$

are incompatible with each other.

(8) To find the conditions which must be satisfied, in order that x^2+ax+b may be a factor of the equation

$$x^4-qx-rx-s=0.$$

$$\begin{array}{l}
x^2+ax+b)\ x^4-qx^2-rx-s\ (x^2-ax+a^2-b-q \\
\qquad\qquad\quad x^4+ax^3+bx^2 \\
\hline
\qquad\qquad\quad -ax^3-(b+q)x^2-rx \\
\qquad\qquad\quad -ax^3-a^2x^2-abx \\
\hline
\qquad\qquad\qquad\qquad (a^2-b-q)x^2+(ab-r)x-s \\
\qquad\qquad\qquad\qquad (a^2-b-q)x^2+\{a^3-a(b+q)\}x+a^2b-b^2-qb \\
\hline
\qquad\qquad\qquad\quad -(a^3-2ab-qa+r)x-a^2b+b^2+qb-s.
\end{array}$$

Therefore, in order that x^2+ax+b may be the factor required, we must have

$$a^3-2ab-qa+r=0,$$
$$a^2b-b^2-qb+s=0.$$

If we eliminate b from these equations, we find

$$\begin{array}{l}
2ab-a^3+qa-r)\ b^2-(a^2-q)b-s\ (\\
\qquad\qquad\qquad\qquad 2ab^2-2a(a^2-q)b-2as \\
\qquad\qquad\qquad\qquad 2ab^2-(a^3-qa+r)b \\
\hline
\qquad\qquad\qquad\qquad\quad -(a^3-qa-r)b-2as \\
\qquad\qquad\qquad\qquad 2a(a^3-qa-r)b+4a^2s\ (\\
\qquad\qquad\qquad\qquad 2a(a^3-qa-r)b-(a^3-qa-r)(a^3-qa+r) \\
\hline
\qquad\qquad\qquad\qquad\quad a^6-2qa^4+(q^2+4s)a^2-r^2=0
\end{array}$$

This final equation of six dimensions becomes the reducing cubic in Descartes's solution of a biquadratic equation, which is given in the Note to page 621, if we make $a^2 = u$: the elimination of a from the same two equations, would lead also to a final equation of six dimensions in b, which is the *quasi* recurring equation noticed in the same Note.

793. When three equations, involving three unknown or indeterminate quantities are proposed for solution, we commence by reducing them to two equations with two unknown quantities, and subsequently to a single final equation, by a process similar to the one which has been employed in the preceding examples: thus let it be proposed to find the values of x, y and z in the following equations:

Process for the solution of three equations and three unknown quantities.

$$\left.\begin{aligned} x - y + z - 3 &= 0 \\ xy - z^2 + 10 &= 0 \\ x^2 + y^2 + z^2 - 29 &= 0 \end{aligned}\right\};$$

$$x - y + z - 3)\ yx - z^2 + 10$$

$$yx - y^2 + zy - 3y$$

$$\overline{y^2 - (z - 3)y - z^2 + 10 = Y_1 = 0.}$$

$$x - y + z - 3)\ x^2 + y^2 + z^2 - 29\ ($$

$$x^2 - yx + zx - 3x$$

$$\overline{(y - z + 3)x + y^2 + z^2 - 29}$$

$$(y - z + 3)x - (y - z + 3)^2$$

$$\overline{2y^2 - 2(z - 3)y + 2z^2 - 6z - 20 = 0}$$

$$\text{or } y^2 - (z - 3)y + z^2 - 3z - 10 = Y_2 = 0.$$

If we now proceed to eliminate y from $Y_1 = 0$ and $Y_2 = 0$, we find, by subtracting one equation from the other,

$$y^2 - (z - 3)y + z^2 - 3z - 10 = 0$$

$$y^2 - (z - 3)y - z^2 + 10 = 0$$

$$\overline{2z^2 - 3z - 20 = Z = 0.}$$

If we solve the final equation $Z = 0$, we get $z = 4$, or $-\frac{5}{2}$: the substitution of the first value of z in $Y_1 = 0$ or $Y_2 = 0$, gives $y = 3$ or -2: the substitution of the second value of z, gives $y = \frac{-11}{4} \pm \frac{\sqrt{61}}{4}$: the substitution of the corresponding pairs of the values of z and y in any one of the primitive equations, gives $x = 2$ or -3 or $\frac{11}{4} \pm \frac{\sqrt{61}}{4}$.

The values of x and $-y$ are symmetrically involved in the three equations, and the values of x and y, which are four in number, are therefore the same with different signs; if we had begun therefore by eliminating z from the three equations, we should have arrived at the following final biquadratic equations, according as the final equation was formed in x or in y,

$$x^4 - \frac{9x^3}{2} - \frac{31x^2}{4} + \frac{147}{4}x - \frac{45}{2} = 0,$$

$$y^4 + \frac{9y^3}{2} - \frac{31y^2}{4} - \frac{147}{4}y - \frac{45}{2} = 0.$$

General expression for the unknown quantities in a complete system of three equations.

794. The process followed in the Example just given may serve as a guide to that which might be followed in other cases, when the number of unknown quantities, and of the equations which involve them was greater than three: under such circumstances, however, the results which are obtained, when presented in general terms, are extremely complicated: if, however, the equations be of the first degree, it will be possible to exhibit the values of the unknown quantities under symmetrical forms, and to point out the law of their construction: for this purpose, we will consider in the first instance, the three equations of the first degree

$$a_1x + b_1y + c_1z - k_1 = 0 \ \ldots\ldots \ (1),$$

$$a_2x + b_2y + c_2z - k_2 = 0 \ \ldots\ldots \ (2),$$

$$a_3x + b_3y + c_3z - k_3 = 0 \ \ldots\ldots \ (3).$$

If we replace $c_1z - k_1$ by $-K_1$, $c_2z - k_2$ by $-K_2$, $c_3z - k_3$ by $-K_3$, these equations become

$$a_1x + b_1y - K_1 = 0 \;\ldots\ldots\; (4),$$

$$a_2x + b_2y - K_2 = 0 \;\ldots\ldots\; (5),$$

$$a_3x + b_3y - K_3 = 0 \;\ldots\ldots\; (6).$$

If we now solve the two first of these equations (4) and (5) with respect to x and y, we get (Art. 782. Ex. 2).

$$x = \frac{K_1b_2 - K_2b_1}{a_1b_2 - a_2b_1},\quad y = \frac{a_1K_2 - a_2K_1}{a_1b_2 - a_2b_1},$$

expressions which have the same denominator: if we substitute these values of x and y in equation (6), and replace K_1, K_2, K_3 by the values for which they were assumed, and clear the equation from fractional terms, we shall find

$$(a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1)\,z$$
$$- (a_1b_2k_3 - a_1b_3k_2 - a_2b_1k_3 + a_2b_3k_1 + a_3b_1k_2 - a_3b_2k_1) = 0,$$

and therefore

$$z = \frac{a_1b_2k_3 - a_1b_3k_2 - a_2b_1k_3 + a_2b_3k_1 + a_3b_1k_2 - a_3b_2k_1}{a_2b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_1c_2};$$

and in a similar manner, we shall find

$$y = \frac{a_1k_2c_3 - a_1k_3c_2 - a_2k_1c_3 + a_2k_3c_1 + a_3k_1c_2 - a_3k_2c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1},$$

and

$$x = \frac{k_1b_2c_3 - k_1b_3c_2 - k_2b_1c_3 + k_2b_3c_1 + k_3b_1c_2 - k_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1}.$$

Enunciation of the law of formation of the preceding expressions.

795. It remains to ascertain and enunciate the law of the formation of these expressions for x, y, z, and the principle upon which it may be extended to express the unknown quantities in four or a greater number of such equations.

In the first place, the numerator of each fraction differs from its denominator in having k with its different subscript numbers in the place of the coefficient (with the same subscript numbers) of the unknown quantity whose value it expresses.

In the second place, the number of terms in each numerator and denominator is 6, or is equal to the number of permutations of the subscript numbers 1, 2, 3, (Art. 229).

In the third place, the algebraical sign of any term of the numerator or denominator will be the *same* with, or *different* from, the first term in each, according as it involves an *odd* or an *even* number of quantities which are different from those in the first: thus, the terms $a_1b_2c_3$ and $a_1b_3c_2$, which involve two quantities, b_2, c_3, and b_3, c_2, which are different from each other, have *different* signs, whilst the terms $a_1b_2k_3$ and $a_2b_3k_1$, each involving three quantities which are different from each other, have the *same* sign.

It will follow as a consequence of the last observation, that the corresponding terms in the numerator and denominator, that is, the terms with the same subscript numbers in the same order, will have the same algebraical signs.

Determination of the expressions for the unknown quantities in a system of four or of n equations.

796. We shall now proceed to consider the formation of corresponding expressions for the unknown quantities, when there are four or a greater number of such equations.

Let $x = \frac{N_1}{D}$, $y = \frac{N_2}{D}$, $z = \frac{N_3}{D}$, be assumed to represent the expressions for the three unknown quantities in three equations which are given above; and let the four equations be

$$a_1x + b_1y + c_1z + d_1u - k_1 = 0 \;\ldots\ldots\; (1),$$

$$a_2x + b_2y + c_2z + d_2u - k_2 = 0 \;\ldots\ldots\; (2),$$

$$a_3x + b_3y + c_3z + d_3u - k_3 = 0 \;\ldots\ldots\; (3),$$

$$a_4x + b_4y + c_4z + d_4u - k_4 = 0 \;\ldots\ldots\; (4).$$

If in the expressions N_1, N_2, N_3 we replace k_1, k_2, k_3 by $d_1u - k_1$, $d_2u - k_2$, $d_3u - k_3$ respectively, and if n_1, n_2, n_3 be taken to represent the values of N_1, N_2, N_3, when k_1 is replaced by d_1, k_2 by d_2, k_3 by d_3, then we shall find, from the three first equations (1), (2), (3),

$$x = \frac{n_1u - N_1}{D}, \quad y = \frac{n_2u - N_2}{D}, \quad z = \frac{n_3u - N_3}{D};$$

and substituting these values in the last equation (4), and suppressing fractions, we get

$$(a_4 n_1 + b_4 n_2 + c_4 n_3 + d_4 D)\, u - a_4 N_1 - b_4 N_2 - c_4 N_3 - k_4 D = 0,$$

and therefore

$$u = \frac{a_4 N_1 + b_4 N_2 + c_4 N_3 + k_4 D}{a_4 n_1 + b_4 n_2 + c_4 n_3 + d_4 D}:$$

and if N', N'', N''' with their proper subscript numbers be successively taken to represent the values of the numerators of the expressions for any other three amongst the four unknown quantities in the three equations (1), (2), (3), namely, of

x, y, u, when c is replaced by d and d by c,

x, z, u, when b is replaced by c and c by b,

y, z, u, when a is replaced by b and b by a;

and if the corresponding values of n and D be denoted by

$$n',\, n'',\, n''',\ D',\, D'',\, D''',$$

then we shall find

$$z = \frac{a_4 N_1' + b_4 N_2' + c_4 N_3' + k_4 D'}{a_4 n_1' + b_4 n_2' + c_4 n_3' + d_4 D'},$$

$$y = \frac{a_4 N_1'' + b_4 N_2'' + c_4 N_3'' + k_4 D''}{a_4 n_1'' + b_4 n_2'' + c_4 n_3'' + d_4 D''},$$

$$x = \frac{a_4 N_1''' + b_4 N_2''' + c_4 N_3''' + k_4 D'''}{a_4 n_1''' + b_4 n_2''' + c_4 n_3''' + d_4 D'''}.$$

The number of terms in the numerator and denominator of each of these expressions is four, or is equal to the number of unknown quantities: and if similar expressions were formed for five unknown quantities and five equations, they would severally contain five terms in their numerator and denominator; and similarly for whatever number of equations such expressions were investigated: for the substitution of the expressions $\frac{P_1}{Q}, \frac{P_2}{Q}, \ldots \frac{P_{n-1}}{Q}$ for $(n-1)$ unknown quantities derived from $(n-1)$ equations but *adapted* to a system of n equations and n unknown quantities, in the nth or additional equation, will lead to an expression for the new unknown quantity, whose numerator and denominator consist of n terms; or if x_n denote the last unknown quantity introduced, and l and m (with their proper subscript numbers) be the coefficients of x_{n-1} and x_n, then we shall find

$$x_n = \frac{a_n P_1 + l_n P_2 + c_n P_3 + \dots l_n P_{n-1} + k_n Q}{a_n Q + l_n Q + c_n Q + \dots l_n Q \quad + m_n Q};$$

and it is obvious that corresponding symmetrical expressions may be obtained for all the other unknown quantities x_{n-1}, x_{n-2}, ... x_1 in the inverse order of their introduction.

Enunciation the general law of formation of the expressions for the n unknown quantities in a system of n equations.

797. It appears therefore that the number of factors in each product involved in the numerator and denominator of the expression for x_n will be n: for an additional factor is introduced for every additional unknown quantity or additional equation.

Again, the number of terms, when they are completely exhibited, in the numerator and denominator of the expression for x_n is $1 \times 2 \times 3 \times \dots n$, or is equal to the number of permutations of the subscript numbers $1, 2, 3, \dots n$, (Art. 229.): for the number of terms in the numerator and denominator of the expression for x_n is n times the number of them in the expression for x_{n-1} in a system of $(n-1)$ equations, $n(n-1)$ times the number of them in the expression for x_{n-2} in a system of $(n-2)$ equations, and so on, until we descend to the expression for the unknown quantity in a system of two equations.

Again, the number of positive and negative terms in the numerator and denominator being the same for the expression in the unknown quantity in a system of two and three equations, will continue the same likewise in the corresponding expressions in a system of n equations: for the new numerators of these expressions are formed by multiplying the series of literal numerators (which are the same as far as the letters involved and their signs are concerned) of the $(n-1)$ first unknown quantities and also their common denominator, into a_n, b_n, $c_n \dots k_n$ respectively, and connecting their results with the sign $+$: if therefore the number of positive and negative terms be the same in the numerators and denominators of those expressions for $(n-1)$ unknown quantities and $(n-1)$ equations, it must continue the same therefore when there are n unknown quantities and n equations: and inasmuch as this number was the same, when there were two equations and two unknown quantities, it must continue the same therefore whatever be their number: the same observations apply, with a very trifling modification, to the number of positive and negative terms in the denominators.

Lastly, the same law which was noticed as determining the negative and positive terms in the case of the expressions for the unknown quantities in three equations, will prevail likewise for any number of such equations: for whatever condition determines the sign of the separate terms in the numerators and denominator of the expressions for the unknown quantities, when there are $(n - 1)$ unknown quantities, will determine their signs likewise when there are n unknown quantities: for the series of terms involved in their numerators and denominator are multiplied into new factors

$$a_n, \; b_n, \; c_n, \; \ldots\ldots \; k_n,$$

and therefore the condition for the determination of the signs of the resulting products in each series remain the same as before.

CHAP. XVII.

ON THE SOLUTION OF PROBLEMS.

Objects of enquiry in the solution of problems.

798. THE solution of problems, by means of Algebra, will require the expression in algebraical language of the quantities which they involve, and of the conditions to which they are subject, and their reduction, in conformity with such conditions, to the form of an equation or equations, whether identical or not: for this purpose, it will be requisite to consider, *first*, the *data* or things given, and *secondly*, the *quæsita* or things sought for or required to be determined, and *thirdly*, the conditions and consequent operations by which the connection between them is established, and through the medium of which the problem is required to be solved.

The data.

799. The *data* may be *abstract* or *concrete* numbers, to which specific values are assigned, or quantities, whether numerical or not, whose values are supposed to be given or determinate, but whose values are not assigned: in one case they will be expressed by the specific numbers themselves, and in the other by the earlier letters of the alphabet or alphabets, which are commonly appropriated for this purpose: in neither case however can such numbers or symbols express *generally* the *specific qualities* of the magnitudes they represent: thus 7 pence, 7 shillings, 7 feet, 7 yards, 7 hats, 7 horses, and so on, are all equally denoted by the same abstract number 7; and in a similar manner any *determinate* number (not assigned) of pence, shillings, feet,

yards, hats, horses, would be equally denoted by the same general symbols such as *a* or *b* or *c*: the connection between the number or symbol, and the specific quality of the corresponding magnitude, is merely preserved, for the most part, in order to connect the results which are obtained with the problem in whose solution they present themselves.

The quæsita.

800. The *quæsita*, whether expressible by numbers or not, must be denoted in the first instance, by symbols, which are usually the last letters of the alphabet or alphabets; and it is obvious that they must correspond to the *data*: if the *data* are expressed by specific numbers, the *quæsita* are required to be expressed by specific numbers or numerical quantities likewise: if the *data* are expressed by general symbols, the *quæsita* can only be expressed in general symbols: for it is assumed as the primary condition for the solution of the problem, that the *quæsita* must be dependent upon the *data*, and expressible by means of them; and therefore they are expressible by numbers or numerical quantities, if the *data* are expressed in numbers and in the general symbols which express the *data*, if the *data* are expressed by general symbols.

The conditions.

801. The conditions of the problem establish the connection of the *data* and *quæsita*, and their necessary dependence upon each other, and must be of such a kind as may admit of being expressed by arithmetical or algebraical operations, whatever those operations may be: it is the selection and adaptation of those operations which constitute the *reduction* of the problem to algebraical language, and in which the chief difficulty of its solution generally consists: for the subsequent solution of the equation which results from this *reduction*, is a process altogether independent of the problem itself, the consideration of which is only resumed when the values of the *quæsita* or *unknown quantities* which result from the solution, are required to be interpreted with reference to the problem proposed.

The number of equations is properly the same as the number of *quæsita*.

802. Inasmuch as the *quæsita* of the problem are *unknown* quantities, their determination will require, and therefore the *reduction* of the problem will lead to, as many equations as there are *quæsita*: if it should appear, however, as is very frequently the case, that one of those equations involves one unknown quantity only, admitting of immediate determination from it, it may be at once replaced by this value or values, and considered as a *datum* of the problem: or if one or more of such equations involve two unknown quantities, where one is immediately expressible in terms of the other, such an expression may, if convenient, be substituted for it in the other equation or equations, and we shall thus advance one step towards the final equation of the problem: it is difficult however to give any useful general rules for such simplifications of the solution of the equations which the reduction of problems leads to, and the adoption of them must generally be left to the experience and tact of the student himself.

Translation of the conditions into algebraical language.

803. The principles of the translation of the conditions of a problem into the corresponding algebraical conditions are the converse of those which have been already considered as regulating the interpretation of the algebraical signs of affection or of operation: thus if the sign + placed between two magnitudes a and b of the *same* kind means their arithmetical *sum*, and the sign − their arithmetical *difference*, then conversely, the terms *sum* and *difference*, however variously expressed or indicated, whether by the terms *more* or *fewer*, or *greater by* or *less than*, *increased by* or *diminished by*, *added to* or *subtracted from*, will be symbolized by the signs + and − placed between the symbols which are their representatives: again if ab or $\frac{a}{b}$ means the product or quotient of two magnitudes a and b, in the sense in which such operations are interpreted for specific values of a and b, or their arithmetical product and quotient, when their specific nature is not required to be con-

sidered, then such combinations must replace the terms product and quotient, or any equivalent expressions, in the problem proposed: similar observations may be extended to the symbolization of other operations, such as powers, roots, &c. which the verbal conditions of the problem may include.

Affections of magnitude which are capable of algebraical expression.

804. In most cases, the terms in which the problem is expressed will exclude the consideration of the affections of the magnitudes which are introduced into them, and the *reduction* of the problem may take place without any reference to them: but when magnitudes are considered which differ from each other in such affections merely as may be symbolized by the signs + and − prefixed to them or by the more general sign $\cos\theta + \sqrt{-1}\sin\theta$ or its equivalents, and when the conditions of the problem do not supersede their usage, such signs will present themselves in the reduction of the problem, in common with the signs of operation: thus *property* or *debt*, *gain* or *loss*, *time past* or *future*, *parallel lines* with *different directions* may be denoted by symbols with the signs + and −, previously to their combination by the signs of operation which will be necessary in the reduction of the problem: and if the term *direction* be used in its most general sense as applied to lines or planes, it may be symbolized by the sign $\cos\theta + \sqrt{-1}\sin\theta$, where θ is the angle made by the line or plane in question, with the assumed primitive line or plane: examples will occur in the problems which follow, of the original introduction of such signs, both when necessary and not necessary for their reduction.

Absolute and relative interpretation of the results which are obtained.

805. The results which are obtained from the solution of the equations to which the problems are reduced, may be interpreted *absolutely*, with reference to the specific nature of the quantities which they denote and without any reference to the problems in which they occur, or *relatively*, with reference to both: this distinction is extremely important, inasmuch as the *absolute* interpretations of the results will always include the *relative*, but not conversely:

for the terms of the problem will in most cases, though not in all, exclude all results but one, as not answering the conditions which it expresses, in the sense in which its terms are used in ordinary language; whilst the number of results, which may or may not admit of *absolute* interpretation, will be equal to the number which expresses the degree of the final equation: under such circumstances, there is only one proper solution of the problem, all other solutions being *algebraical solutions* only.

Examples. **806.** We shall now proceed to the solution of a few problems, for the purpose of illustrating the principles employed in the translation of problems into algebraical language, and of interpreting the results which are thus obtained with reference to the problems proposed.

(1) The sum of two numbers is 30, and their difference is 6: what are the numbers?

The *data* are 30 and 6.

The *quæsita* are the two unknown numbers, which we denote by x and y. The first condition gives

$$x + y = 30 \;\ldots\ldots\; (1),$$

and the second

$$x - y = 6 \;\ldots\ldots\; (2),$$

where the signs $+$, $-$ and $=$ are used in their ordinary arithmetical meaning.

The solution of these equations (1) and (2) (by adding and subtracting) gives us $x = 18$ and $y = 12$, which answer the conditions, and satisfy the language, of the problem.

If the problem proposed had been, "to find two numbers whose sum is 6 and whose difference is 30," we should find for the equations resulting from the reduction of the problem, without any reference to its possibility,

$$x + y = 6,$$

$$x - y = 30,$$

which would give us, as the algebraical values of x and y,

$$x = 18, \text{ and } y = -12.$$

There is no such *abstract* number as -12, and there is, therefore, no proper solution of the problem in the sense in which it was proposed: if however we should consider numbers as the symbols of concrete quantities, admitting the interpretation of the relation to each other expressed by $+$ and $-$, then a problem might be adapted to the solution which is thus obtained: thus, "if the joint property of A and B was £6, but A was richer than B by £30," then the *property* of A would be £18. and B's *debt* would be £12: in such a case, the *absolute* interpretation of $+18$ and -12 would become *relative* or *congruent* likewise to the problem proposed.

(2) A's property is m times that of B: but after gaining sums of money denoted by a and b respectively, A's property becomes n times that of B: what were their respective properties at first?

The *data* are a, b, m, n.

The *quæsita* are the respective original properties of A and B denoted by x and y.

The *conditions* translated into algebraical language, furnish the equations

$$x = my,$$

$$x + a = n(y + b).$$

If in the second equation, we replace x by my, we get the single or final equation

$$my + a = n(y + b)$$

and therefore

$$y = \frac{nb - a}{m - n},$$

and also

$$x = \frac{m(nb - a)}{m - n}.$$

The verbal conditions of the problem would require these values of x and y to be positive: if not, the original property both of A and B must be replaced in the problem as proposed or interpreted, by the term *debt*.

If $m = n$, and nb bè not equal to a, then x and y become infinite, and therefore admit of no values sufficiently great to satisfy the conditions of the problem.

If $m = n$, and $nb = a$, then $x = \frac{0}{0}$ and $y = \frac{0}{0}$, and the values of x and y are indeterminate: or the conditions of the problem will be satisfied by any values of x and y, which satisfy the equation

$$x = my.$$

If the *gains* a and b become *losses*, the second equation will become

$$x - a = n(y - b):$$

under such circumstances $-a$ and $-b$ will express the magnitudes of those losses with reference to x and y: if however a and b under a positive form are assumed to express negative quantities, the second equation will retain its original form.

Problem of two couriers.

(3) Two couriers leave two places A and B, distant d miles from each other, and travel a and b miles a day respectively: what is their distance at the end of (t) days, and when will they meet?

The *data* are d, a, b.

The *quæsita* are their distance (x) at the end of t days, and the value of t when $x = 0$.

The space travelled by the first courier in t days $= at$.

The space travelled by the second courier in t days $= bt$.

If they travel in the same direction, the excess of the space travelled over by the first courier above the second in t days is

$$at - bt = (a - b)t:$$

and this expresses the diminution of the distance of the couriers at the end of t days: consequently

$$x = d - (a - b)t;$$

and if $x = 0$, then

$$t = \frac{d}{a - b}:$$

If $a = b$, their original distance from each other remains unaltered, and t is infinite: in other words, there is no value of t however great, which satisfies the conditions of the problem: it may be considered as expressing the extreme limit of the continually increasing values of t corresponding to the continually diminished excess of a above b, or of the rate of travelling of one courier above the other: in such a sense, in this and similar cases, the occurrence of such a value may be considered as one of the symbols of *impossibility*: but if $d=0$ at the same time that $a=b$, then the interpretation of the result $t=\frac{0}{0}$, will indicate that the couriers leave the same place at the same time, and travel in the same direction at the same rate, and that all values of t will equally satisfy the conditions of the problem.

If a be less than b, then t is negative: if the problem be taken literally in the sense in which it is proposed, it will, under such circumstances, be impossible: but if the places A and B express their positions, *in the course of their journey*, corresponding to the *zero* of time, then the negative value of t will express the number of days preceding that *zero* (Art. 94.) when the couriers were together. If a be positive, and b be negative, the couriers are moving in opposite directions to meet each other from A and B respectively: but if a be negative, and b positive, then they will be moving at the zero of time, in opposite directions *from* each other, and t will express the number of days preceding that zero when they were at the same place.

The following numerical values of d, a, b, and therefore of t, will illustrate this problem more fully.

$$\text{If } d = 100,\ a = 15 \text{ and } b = 10:$$

$$\text{then } t = \frac{100}{15-10} = 20:$$

or the couriers will meet at the end of 20 days.

$$\text{If } d = 100,\ a = 15 \text{ and } b = -10:$$

$$\text{then } t = \frac{100}{15+10} = 4:$$

or the couriers travel towards each other, and will meet at the end of 4 days.

$$\text{If } d = 100,\ a = 10 \text{ and } b = 15:$$

$$\text{then } t = \frac{100}{10-15} = -20:$$

or the couriers *were* together 20 days before the zero of time indicated in the problem.

$$\text{If } d = 100,\ a = -10 \text{ and } b = 15:$$

$$\text{then } t = \frac{100}{-10-15} = -4:$$

or the couriers *were* together 4 days before the zero of the problem.

$$\text{If } d = 100\ a = -16 \text{ and } b = -15,$$

$$\text{then } t = \frac{100}{-10+15} = 20,$$

or the couriers will meet 20 days after the zero of the problem: they move also in the same direction, which is opposite to that in which they moved when a was 10, and $b = 15$, and opposite to that likewise in which d is measured.

The same problem stated more generally.

This problem, which has been commonly designated the problem of the couriers, may be stated somewhat more generally as follows.

"Two couriers leave two places distant d miles from each other, and travel each day in *given* directions a and a' miles respectively: to find their distance from each other and their position at the end of t days, and the circumstances under which they may meet."

If θ and θ' represent the directions in which they move with respect to their primitive distance, then $\cos\theta + \sqrt{-1}\sin\theta$ and $\cos\theta' + \sqrt{-1}\sin\theta'$ will be the algebraical signs of those directions, and $at\,(\cos\theta + \sqrt{-1}\sin\theta)$ and $a't\,(\cos\theta' + \sqrt{-1}\sin\theta)$ will represent algebraically the actual spaces over which they travel, (Art. 504.): their distance therefore at the end of t days will be represented in quantity and direction by

$$d + (a'\cos\theta - a\cos\theta)\,t + (a'\sin\theta' - a\sin\theta)\,t\sqrt{-1},$$

$$\text{or } \rho\,(\cos\phi + \sqrt{-1}\sin\phi)$$

where

$$\rho = \sqrt{\{d^2 + 2\,(a'\cos\theta' - a\cos\theta)\,dt + a^2 + a'^2 - 2aa'\cos(\theta' - \theta)\,t^2\}}$$

$$\text{and } \cos\phi = \frac{d + (a'\cos\theta - a\cos\theta)\,t}{\rho}.$$

The conditions of their meeting would be expressed by the equations

$$\left.\begin{array}{r} d + (a' \cos \theta' - a \cos \theta)\, t = 0 \\ a' \sin \theta' - a \sin \theta = 0 \end{array}\right\}.$$

If, therefore, the travellers recede from the direction of the primitive line with equal rapidity, they will meet, or will have met, when

$$t = \frac{d}{a \cos \theta - a' \cos \theta'},$$

according as the value of t is positive or negative: under no other circumstances can they ever meet, or ever have met, with each other.

(4) A labourer is engaged for n days, upon condition of receiving a pence for every day he works, and of paying b pence for every day he is idle: at the end of the time he receives c pence: How many days did he work, and during how many days was he idle?

The *data* are n, a, b, c.

The *quæsita* are the number of days (x) he worked, and the number of days (y) during which he was idle.

The conditions of the problem give us the two equations

$$\left.\begin{array}{r} x + y = n \\ ax - by = c \end{array}\right\}:$$

from the solution of which we find

$$x = \frac{bn + c}{a + b},$$

$$y = \frac{an - c}{a + b}.$$

If $n = 0$, we find $x = \frac{c}{a + b}$, and $y = \frac{-c}{a + b}$, values which are not *relative* or *congruent* to the problem proposed, inasmuch as the supposition is incompatible with the existence of the other conditions and the consequences which are supposed to follow from them.

If $an = c$, $y = 0$ and $x = n$, and he labours during the whole period.

If $bn = -c$, $x = 0$, and $y = n$, and he is idle during the whole period: under such circumstances he *pays* a sum c, and its relation to the same sum *received* is symbolized by a change of sign from $+$ to $-$: without such a change of the conditions, the problem would be *impossible*.

If an be less than c, then y is negative, and the problem is *impossible:* for under such circumstances, he would receive a greater sum than he would have been entitled to, if he had laboured during the whole period.

Geometrical problem.

(5) To divide a line a into two such parts, that the rectangle contained by them may be equal to b^2.

The *data* are a and b^2.

The *quæsita* are the two parts x and y, whose sum is equal to the given line.

The conditions of the problem give us the equations

$$\left.\begin{array}{r} x + y = a \\ xy = b^2 \end{array}\right\}.$$

The final equation in x is

$$x(a - x) = b^2.$$

$$\text{Therefore } ax - x^2 = b^2,$$

$$x^2 - ax = -b^2.$$

$$x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - b^2.$$

$$x - \frac{a}{2} = \pm\sqrt{\left(\frac{a^2}{4} - b^2\right)},$$

$$x = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - b^2\right)},$$

$$y = \frac{a}{2} \mp \sqrt{\left(\frac{a^2}{4} - b^2\right)}.$$

Since x and y are *symmetrically* involved in the two equations, their values are interchangeable: in other words, x as well as y

may represent either of the two parts whose sum is equal to the given line, and therefore the general solution of the problem must furnish equally both these values of x and of y.

As long as $\frac{a^2}{4}$ is greater than b^2, the problem admits of solution in the sense in which it is proposed: but if $\frac{a^2}{4}$ be less than b^2, then the problem becomes impossible, unless its language be interpreted in an algebraical and not in an arithmetical or geometrical sense.

Interpretation of the results when they involve the sign $\sqrt{-1}$.

In this last case, we have

$$x = \frac{a}{2} \pm \sqrt{-1}\sqrt{\left(b^2 - \frac{a^2}{4}\right)},$$

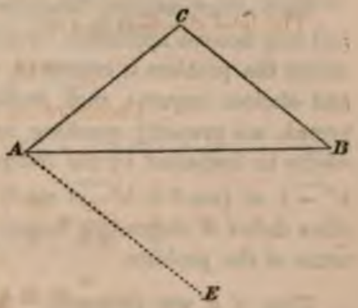
$$y = \frac{a}{2} \mp \sqrt{-1}\sqrt{\left(b^2 - \frac{a^2}{4}\right)};$$

or otherwise

$$x = b\,(\cos\phi \pm \sqrt{-1}\,\sin\phi),$$

$$y = b\,(\cos\phi \mp \sqrt{-1}\,\sin\phi),$$

where $\phi = \cos^{-1}\frac{a}{2b}$, (Art. 725.): consequently if AC be arithmetically equal to b, and inclined to AB or a at an angle ϕ, and if an equal line AE or CB which is equal and algebraically parallel (Art. 559.) to it, make an angle $-\phi$ with the same line AB, then the algebraical sum (Art. 511.) of AC and AE, or of the two sides AC and CB of the isosceles triangle ACD is equal to AB, the third side of the triangle: the conditions of the problem would be thus satisfied in the general sense to which its terms have been adapted.



A similar problem where the data are numbers.

If it had been proposed as a problem to divide the number 10 into two parts, such that their product might be equal to 21, the numbers 7 and 3 would be found to answer the conditions of the question: but if it had been proposed to divide the number 10 into two such parts, that their product should be equal to 26, then the resulting values of the parts $5+\sqrt{-1}$ and $5-\sqrt{-1}$, which satisfy the algebraical equation to which the problem is reduced, would shew that problem itself to be impossible, inasmuch as $\sqrt{-1}$ and the sign $+$ which connects it with 5, would admit of no interpretation in conformity with its terms: if, however, the units in the number 10 were inches, feet or any other equal measures of length, then 10 would be a line, and the results $5+\sqrt{-1}$ and $5-\sqrt{-1}$ would admit of interpretation in conformity with a certain enlarged sense of the terms of the problem, such as we have considered above.

Interpretation of $a+b\sqrt{-1}$ when a and b denote property.

In a similar manner, if a and b denoted property, and $a+b\sqrt{-1}$ denoted property whose absolute magnitude was $\sqrt{(a^2+b^2)}$, but which was so constituted that a portion of it equal to a *only* belongs to me, or is available to my use, then we should arrive at, and be able to interpret, the results $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$, if the problem proposed should require us to find two properties so constituted, that their sum (taken in this enlarged sense) shall be equal to $2a$, and their product numerically equal to a^2+b^2.

Such interpretations however are not *necessary* (Art. 442.), and only become applicable by extending to the ordinary terms in which the problem is expressed, a meaning beyond their simple and obvious import: such problems, indeed, if strictly interpreted, are properly speaking *impossible*, whether such *impossibility* be indicated by the independent sign $-$, or by the sign $\sqrt{-1}$ or $(\cos\theta+\sqrt{-1}\sin\theta)$ and its equivalents, or by any other defect of *congruence* between the result obtained and the terms of the problem.

Congruent and incongruent roots not necessarily connected with the presence or absence of the signs $-$ or $\sqrt{-1}$.

Thus if it was proposed, "to find a number, the double of whose square exceeded three times the number itself by 5", we should get, by assuming x for the unknown number, the equation

$$2x^2-3x-5=0,$$

the solution of which gives us

$$x=\frac{5}{2} \text{ or } -1:$$

but neither of these values is *congruent* to the problem proposed, inasmuch as one of them is a fraction, and the other negative, and therefore the problem is impossible in the sense in which it was proposed: but if by the term *number* in the problem we mean fractions equally with whole numbers, then the first root $\frac{5}{2}$ is *congruent* to the problem, which becomes possible therefore in this enlarged meaning of its terms: but there is no proper interpretation of the second root -1, or modified meaning of the terms of the problem, which can make it *congruent* with respect to it.

Again, the reduction of the following problem will lead to a *congruent* and *incongruent* result, which are both affected with the same sign.

"There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the first digit. But if the digits be inverted, and the resulting number be divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Required the number."

If x be the first digit, and y the second, then $10x + y$ expresses the number, which becomes, when its digits are inverted, $10y + x$: the first condition gives the equation

$$\frac{10x + y}{x + y} = x + 2,$$

$$\text{or } x^2 + (y - 8)x + y = 0,$$

and the second, the equation

$$\frac{10y + x}{x + y + 1} = x + 4,$$

$$\text{or } x^2 + (y + 4)x - 6y + 4 = 0.$$

The *elimination* of x from these equations, gives us the final equation

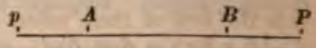
$$133y^2 - 632y + 400 = 0,$$

the roots of which are 4 and $\frac{100}{133}$: and the corresponding values of x are 2 and $\frac{12}{133}$.

The first pair of values 2 and 4 are *congruent* to the problem, and the number required is 24: the second pair of values $\frac{12}{133}$ and $\frac{100}{133}$ are *incongruent* with the terms of the problem, inasmuch as the term *digit* confines the values of the *quæsita* to whole numbers only.

The examples which we have just given would shew that the possibility of a problem, or the *congruence* of the roots of the equation to which it is reduced, has no necessary connection with the absence or presence of the signs $-$ or $\sqrt{-1}$, but is determined by the accordance of the absolute interpretation of the results which are obtained, with the terms of the problem in whatever sense they can be, or may be, understood.

(6) To find a point in a line produced, such that the rectangle contained by the whole line produced and the part produced may be of a given magnitude.



Let $AB\,(a)$ be the given line, and P the point to which it is produced in the same direction with AB: then the conditions of the problem express that the rectangle under $AP\,(a+x)$ and $BP\,(x)$ shall be of a given magnitude (b^2).

Therefore

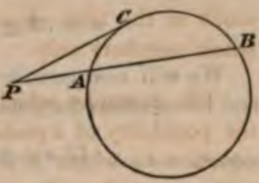
$$(a+x)\,x = b^2,$$

$$x^2 + ax + \frac{a^2}{4} = \frac{a^2}{4} + b^2,$$

$$x = -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b^2\right)}.$$

The values of x in this result have different signs, the positive value $-\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} + b^2\right)}$ denoting BP, drawn in the same direction, and having the same sign therefore with AB or a; and the negative value denoting Bp drawn in a contrary direction to, and having therefore a different sign from, AB: the rectangles of AP and BP, and of Ap and Bp equally answer the conditions of the problem proposed: but one only of these values would be *congruent* to the problem, if the line is required to be produced in the same direction *only* in which it is originally reckoned.

The problem just considered will be found to coincide in its conditions and its solution with the following.



"To draw from a point P without a circle, a line PAB cutting the circle, so that the chord AB intercepted may be of a given length."

If PC be a tangent to the circle, it follows from a well known proposition in Geometry that PC^2 (b^2) is equal to the rectangle under PA and PB: if we make $PA = x$, and therefore $PB = x + a$, we get the equation

$$x(x+a) = b^2,$$

and therefore

$$x = \frac{-a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b^2\right)}.$$

It is the first of these values of x which has the same sign with a, and which is therefore *congruent* to the problem proposed: the second, or negative value of x, would denote BP drawn in a contrary direction to AB, and differing from it therefore in sign: but if the problem was reduced in accordance with this hypothesis, the equation would become

$$x^2 - ax = b^2,$$

whose positive root $\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} + b^2\right)}$ is the value of x which is congruent with the conditions of the problem.

(7) By selling a horse for a £'s I lose as much per cent. as the horse cost me: Required the prime cost of the horse.

The *datum* is a, the price for which the horse was sold.

The *quæsitum* (x) is the prime cost of the horse.

The conditions of the problem express that the loss ($x - a$) upon the sale bears to (x) the prime cost, the same ratio which the prime cost (x) bears to £100: consequently, we get

$$\frac{x-a}{x}=\frac{x}{100},$$

$$100x-100a=x^2,$$

$$x^2-100x+2500=2500-100a,$$

$$x=50\pm\sqrt{(2500-100a)}.$$

Case in which this problem is ambiguous.

We will now consider the connection between this problem and its solutions for different values of a.

If $a=24$, then

$$x=50\pm10=60 \text{ or } 40.$$

In this case, both the values of x are *congruent* to the problem proposed, which is therefore ambiguous, there being two values of the prime cost of the horse which equally answer to its conditions:

$$\begin{array}{ccccccc} & \text{loss} & & \text{prime cost} & & \text{prime cost} & & \pounds \\ 60-24 & : & 60 & :: & 60 & : & 100 \\ 40-24 & : & 40 & :: & 40 & : & 100 \end{array}\Bigg\}[*].$$

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[*] In this case, and in others similar to it, the process to be followed in a system of purely arithmetical algebra would be as follows:

$$\frac{x-a}{x}=\frac{x}{100},$$

$$100x-100a=x^2,$$

$$100x-x^2=100a,$$

and subtracting both members of this equation from 2500, the square of half the coefficient of x, we get

$$x^2-100x+2500=2500-100a,$$

and extracting the square root

$$x-50=\sqrt{(2500-100a)},$$

$$x=50+\sqrt{(2500-100a)}$$

$$=60, \text{ if } a=24.$$

The square root of $x^2-100x+2500$, would be $x-50$ or $50-x$, according as x was greater or less than 50, inasmuch as it may be arranged in the order $x^2-100x+2500$ or $2500-100x+x^2$ indifferently, without altering its

When the problem is not ambiguous.

If $a = 25$, then we get

$$x = 50 \pm 0 = 50.$$

In this case, the problem proposed ceases to be ambiguous, there being only one value of x, or of the prime cost of the horse, by which its conditions can be satisfied.

When the problem is impossible.

If $a = 26$, then we get

$$x = 50 \pm \sqrt{-100} = 50 \pm 10\sqrt{-1}.$$

In this case, the values of x are not *congruent* to the problem proposed, which is therefore *impossible*, there being no prime cost of the horse which can satisfy the conditions required.

The problem differently stated.

If the problem proposed had been the following; "by selling a horse for a £'s, I gained as much per cent. as the horse cost me: required the prime cost of the horse;" where the term *loss* in the former problem is changed into *gain*, we should have got the equation

$$\frac{a - x}{x} = \frac{x}{100}:$$

and therefore its solution would give us

$$x = -50 \pm \sqrt{(2500 + 100a)}.$$

If $a = 13$, we get

$$x = 10 \text{ or } -110.$$

The first of these values *only* is *congruent* to the problem, which admits but of one solution: there is no proper interpretation which can make the second value *congruent* to the problem proposed: but if we should propose a new problem of the following kind: "I sold a horse so as to lose £13. more than his prime cost, and by so doing, I lost as much per cent. as the prime cost

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its value or signification: if we adopted the second arrangement, we should obtain

$$50 - x = 10,$$

$$\text{and therefore } x = 40;$$

we should thus be enabled to obtain the double value of x, without assuming the independent use of the signs + and −, and confining them to their primitive meaning of addition and subtraction.

of the horse: required the prime cost of the horse;" then the reduction of the problem would lead to the equation

$$\frac{13+x}{x}=\frac{x}{100},$$

Interchange of the congruent and incongruent roots.

and therefore $x=110$ or -10. It would thus appear that the *congruent* value of x in this equation, was the *incongruent* value of x in the former equation: the equations themselves however are different from each other, and the transition from one to the other is not such as can be effected by a change of the terms of the problem, so that the *data* and *quæsita* may change their signs: it is the change of the conditions which leads to an equation whose roots are the negative roots of the former, and which produces this interchange, therefore, under such circumstances, of the *congruent* and *incongruent* roots.

A convertible problem.

In some cases, however, the terms of the problem will naturally suggest the change of the conditions which are requisite to produce this change of the signs of the roots: an example of this kind occurs in the following problem.

"A person *bought* some sheep for £72, and found that if he had *bought* 6 *more* for the same money, he would have *paid* £1. *less* for each: How many sheep did he *buy?*"

If we convert *buying* into *selling*, *more* into *fewer*, *paying* into *receiving*, *less* into *more*, relations which are all symbolized by the signs + and −, we shall get the following problem.

"A person *sold* some sheep for £72, and found that if he had sold 6 *fewer* for the same money, he would have *received* £1. *more* for each: How many sheep did he *sell?*"

The equation which results from the reduction of the first problem is

$$x^2+6x-432=0 \;\ldots\ldots\; (1):$$

and that which results from the reduction of the second, is

$$x^2-6x-432=0 \;\ldots\ldots\; (2).$$

The roots of the first equation (1) are 18 and − 24, of which the first *is congruent*, and the second *incongruent:* the roots of the second equation (2) are 24 and − 18, of which the first is *congruent* and the second *incongruent:* it is the number 6 to which the term *more* is attached in the first problem, and which becomes the coefficient of the second term in the equation resulting from its reduction, which suggests the change of the requisite conditions in passing from one problem to the other: for the change of the sign of this term determines the change of the signs of the roots of the equation, and this will result from the change of the term *more* into *fewer,* which changes its relative sign: it will then be necessary to accommodate the other conditions of the problem to the change which has thus been made, and upon which they are in some measure, if not altogether, dependent.

Geometri problem.

(8) There are two square surfaces, one of whose sides exceeds the other by 2 feet, and the sum of their areas is 1 square foot: Required the sides of the squares, and their position.

If x and $x + 2$ represent the sides of the squares, then the conditions of the problem give us the equation

$$x^2 + (x + 2)^2 = 1,$$

$$x^2 + 2x = -\frac{3}{2},$$

$$x = -1 \pm \sqrt{\left(\frac{-1}{2}\right)},$$

and therefore

$$x + 2 = 1 \pm \sqrt{\left(\frac{-1}{2}\right)}.$$

The expressions obtained for x and $x + 2$ are convertible into the equivalent expressions

$$\sqrt{\frac{3}{2}}\,\{\cos(\pi - \phi) \pm \sqrt{-1}\,\sin(\pi - \phi)\},$$

and $\sqrt{\frac{3}{2}}\,(\cos\phi \pm \sqrt{-1}\,\sin\phi)$,

where $\phi = \cos^{-1}\sqrt{\frac{2}{3}} = 35^\circ . 16'$ nearly: and they may be interpreted to represent lines, equal in length to $\sqrt{\frac{3}{2}}$, and making angles of 35°. 16′ and 144°. 44′ with the primitive line (Art. 504.) or axis, and whose algebraical difference is

$$\sqrt{\frac{3}{2}}\,\{\cos\phi - \cos(\pi - \phi)\} = 2\sqrt{\frac{3}{2}}\cos\phi = 2.$$

The squares constructed upon those sides are represented by

$$\frac{1}{2} \pm 2\sqrt{\frac{-1}{2}} \text{ and } \frac{1}{2} \mp 2\sqrt{\frac{-1}{2}},$$

or their equivalents

$$\frac{3}{2}(\cos 2\phi \pm \sqrt{-1}\sin 2\phi) \text{ and } \frac{3}{2}(\cos 2\phi \mp \sqrt{-1}\sin 2\phi),$$

$$\text{where } \phi = \cos^{-1}\sqrt{\frac{2}{3}} \text{ and } 2\phi = \cos^{-1}\frac{1}{3} = 70^0.32':$$

and they may be interpreted to represent two squares, constructed upon sides equal in length to $\sqrt{\frac{3}{2}}$, and inclined at angles of $70^0.32'$ and $-70^0.32'$ to the primitive plane (Art. 611.): the sum of their projections upon this plane (Art. 612.) is equal to

$$2\left(\frac{3}{2}\right)\cos 2\phi = 2\left(\frac{3}{2}\right)\left(\frac{1}{3}\right) = 1,$$

a result which satisfies the conditions of the problem proposed.

(9) Given the sum (s), the first term (a), the common difference (b) of an arithmetical series, to find the number of terms.

The relation between the first term, common difference, number of terms, and sum of an arithmetic series.

An arithmetic series is one whose successive terms have equal differences, and the *equation* which connects the sum (s), the first term (a), the common difference (b), and the number of terms (n) may be investigated as follows.

The r^{th} term of the series is $a + (r-1)\,b$; for the first term is a, the second $a + b$, the third term $a + 2b$, and so on, the multiple of b in any term being equal to the number of terms after the first, which in the r^{th} term is $(r-1)$: the last, or n^{th} term of the series is therefore $a + (n-1)\,b$ (Art. 226. Note).

The sum of the r^{th} and $(n - r + 1)^{th}$ terms, which are equidistant from the first and last terms respectively

$$= a + (r-1)\,b + a + (n-r)\,b = 2a + (n-1)\,b;$$

and is equal therefore to the sum of the first and last terms of the series.

If a second and identical series (s) be combined with the first and the r^{th} term of one be added to the $(n-r+1)^{\text{th}}$ term of the other continually, taking all values of r from 1 to n, we shall get a series of n terms equal to each other and to $2a+(n-1)\,b$, whose sum is equal therefore to $\{2a+(n-1)\,b\}\,n$: consequently

$$2s = \{2a+(n-1)\,b\}\,n,$$

$$\text{or } s = \{2a+(n-1)\,b\}\,\frac{n}{2} \;\ldots\ldots\; (1).$$

Solution of the problem and discussion of the meaning of different values of n.

In the problem proposed, s, a and b are given, and n is required to be determined: consequently if the equation (1) be solved with respect to n, we get

$$n = -\left(\frac{a}{b}-\frac{1}{2}\right) \pm \sqrt{\left\{\left(\frac{a}{b}-\frac{1}{2}\right)^2+\frac{2s}{b}\right\}}.$$

If both the values of n be positive and whole numbers, they are equally *congruent* to the problem, and there are therefore 2 series which satisfy the required conditions. Thus, if $s=100$, $a=28$ and $b=-4$, we find $n=5$ or 10, which are *congruent* values corresponding to the two series

$$28,\ 24,\ 20,\ 16,\ 12,$$

$$\text{and } 28,\ 24,\ 20,\ 16,\ 12,\ 8,\ 4,\ 0,\ -4,\ -8.$$

If one value of n be positive and the other negative, but both of them whole numbers, then the first value is *congruent*, and the other *incongruent*, though adaptable to the sum of a number of terms of the same series reckoned in an inverse order, which is equal to this *incongruent* value with its sign changed.

Thus if $s=100$, $d=12$, and $b=4$,

then we find $n=5$ or -10:

it is the second value which expresses the number of terms of the series

$$28,\ 24,\ 20,\ 16,\ 12,\ 8,\ 4,\ 0,\ -4,\ -8,$$

where the last term of the series

$$12,\ 16,\ 20,\ 24,\ 28,$$

becomes the first term of the other, and the terms are reckoned in an inverse order.

If one only of the values of n be a whole number and positive, and the other be fractional, whether positive or negative, it is the only value which is *congruent* to the problem proposed.

Thus, if $s = 33$, $a = 18$ and $b = -5$, we find $n = 6$ or $\frac{11}{5}$: the first value is alone congruent to the problem; there is no adaptation of the problem which can enable us to give a meaning or interpretation to the second.

If neither of the values of n be a whole number, whether positive or negative, they are altogether *incongruent* to the problem proposed, which is therefore *impossible:* or, in other words, there is no series, which can satisfy the required conditions.

Thus, if $s = 35$, $a = 20$ and $b = -8$, we shall find

$$n = \frac{7}{2} \text{ or } \frac{5}{2}:$$

there is no series therefore such as the problem supposes, the sum of any number of whose terms can be equal to 35.

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ADDITIONS AND CORRECTIONS.

CHAP. I. THE exposition of the grounds upon which many of these definitions or first principles rest, and the justification of the form under which they are exhibited, will be found in the Third Chapter, and we shall therefore add some references to the Articles in that and also in other Chapters, in which the discussion of them is resumed: it ought to be observed, likewise, that Arithmetic throughout is considered as the science of Suggestion (see Preface), and that the operations coincide with those in Arithmetic, which bear the same name, when the quantities which are subjected to them are arithmetical: it has been thought proper to adopt this course, in order to avoid the very abstract form under which those definitions and principles would otherwise present themselves.

Page 1. Art. 1. See Art. 47. and those which immediately follow.

——— Art. 2. See Art. 49. 72, 73, 74.

——— Art. 5. The operations denoted by $+$ and $-$ are *assumed* to be identical with those termed Addition and Subtraction, according to their ordinary meaning in Arithmetic, when the quantities which are subjected to them are of the *same* kind: see Art. 79: under such circumstances, the specific nature of the magnitudes which the symbols denote, will neither limit their application, nor affect their meaning. It is for this reason, that the result of the addition of a symbol a (whatever it may denote) to itself, will be denoted by $a + a$, and will mean the *double* of a: if the same quantity or symbol be added to the result just obtained, the result will be represented by $a + a + a$, and will be equivalent to *thrice a*, or *three times a* as expressed in ordinary language: in a similar manner $a + a + a + a$ will be equivalent to *four* times a, and so on, to whatever extent we choose to repeat this operation: the same series of results are represented by $2a$, $3a$, $4a$, and so on, where 2, 3, 4, &c. denote the number of times that a is repeated in each result: such numbers are called the *coefficients* of a, and the results $2a$, $3a$, $4a$, &c. are equivalent in meaning to the arithmetical products of a, by 2, 3, 4, and so on, whatever be the arithmetical quantity, whether integral or fractional, which is thus combined with a; in thus interpreting therefore the meaning of such numerical coefficients, we assume that the operation termed *multiplication* in Algebra, will coincide in meaning with arithmetical *multiplication*, when one or both of the symbols in each product is arithmetical: see Art. 80.

The term coefficient is mentioned in Art. 14, without being accompanied by a sufficient definition of its meaning: and the use which is made of it in the simplification of the results of the operations of Addition and Subtraction in the second Chapter, requires to be justified by some explanation similar to the one just given.

Page 2. Art. 6. See Art. 74, 75, 76, 77: the independent existence of the signs + and −, is an *assumption* which is necessary in order that the values of the general symbols may be unlimited, and the operations indicated by the same signs + and − universally possible: the identity of those signs which we call signs of affection, with the signs of operation, is likewise a necessary consequence of the symbolical definition of those operations.

Page 3. Art. 7. See Art. 25. 28. 34. 65. 75, 76, 77. 80. and Note: the rule of the *concurrence* of signs contains the symbolical definition of those operations which are *called* Addition and Subtraction, and which are denoted by + and −: in other words, Addition is the *affectation* of symbols (with their proper signs) with the sign + (Art. 25.), and Subtraction is their *affectation* with the sign −, (Art. 25.), in conformity with this rule: the definition of those operations, when considered with reference to each other, is contained in Art. 10, when they are assumed to be the inverse of each other: it is only by means of a *partial interpretation* of those operations that we are enabled to combine *like* terms into one, and thus to simplify the results which we obtain.

The rule for the incorporation of signs contributes to define the operation *called* Multiplication and Division, but not to the same extent, nor in the same manner, that the operations of

Addition and Subtraction are defined by the rule for their concurrence: for these operations are symbolically defined by the mode of denoting them (Art. 8.), and by being considered as the inverse of each other (Art. 10.), and by this rule: it is the partial interpretation of the meaning of those operations, in conformity with those definitions and the results of the *science of suggestion,* which enables us to incorporate the numerical coefficients of the symbols which are subject to them.

Page 4. Art. 9. The principle stated in this Article, is a consequence of the assumed invariableness of the operations of Addition, Subtraction, Multiplication and Division: if the effect or result of such operations was in any way dependent upon the order of their succession with respect to other operations, whether of the same or of a different kind, the nature and extent of the influence of that position must be comprehended in the definitions of the operations themselves: it is the absence of any such conditions in those definitions which leads to the conclusion or principle stated in this Article as a necessary consequence: and it is the accordance of this conclusion with the results of the operations with the same names in the *science of suggestion,* which is essential to the transfer by interpretation of the conclusions of the more general science to the science which is subordinate to it.

Page 5. Art. 12. The general principle of indices which is stated in this Article, will be noticed again in Art. 131, where it will be shewn to be a consequence of the *principle of the permanence of equivalent forms:* the complete developement of its consequences will form the subject of Chapter VI.

The sign = which is used in this Article is defined in Art. 22.

Page 6. Art. 13. It might be more correct to say *a root* rather than *the root;* the phraseology will then be adapted to the multiplicity of roots, the existence of which will be afterwards established: see Chap. XII.

Page 7. Art. 14. We have explained more particularly the meaning of a numerical coefficient in our observations on Art. 5: the same observations will explain likewise the sense in which it is used in Art. 20, when we assert that a numerical coefficient does not effect the dimensions of the quantity denoted by the symbols with which it is connected.

Page 8. Art. 21. Like quantities as defined in this Article are not necessarily quantities of the same nature, inasmuch as their signs of affection may be different from each other.

—— 8. Art. 22. See Art. 117. 120, 121, 122, 127, 128, 129.

—— 11. Art. 25. Ex. 3, 4. 15: the *disappearance* of symbols by *opposition* of signs is a consequence of the defined relation of operations of Addition and Subtraction, as *inverse* operations, (Art. 10).

—— 21. Art. 34. The meaning which is agreed to be attached to the term *coefficient*, and their treatment in the operation of multiplication, as defined by this rule, are stated in our observations on Art. 5.

—— 21. Art. 35. line 1: *for* $a \times - b = ab$ *read* $a \times b = ab$.

—— 22. Art. 35. Ex. 13. line 3: *for* Art. 11. *read* Art. 9.

—— 27. Art. 35. Ex. 2. The principle of symmetrical combinations which is used in this and the following examples, is stated *generally* in Art. 136.

—— 35. Ex. 11. See Art. 243. and Chapter 10.

—— 36. Ex. 12. See Art. 241. for the law of formation of n binomial factors with the same first term.

—— 43. Art. 42. Ex. 5. last line but 1: *for* $ab \times \frac{1}{c}$ *read* $ab \times \frac{1}{b}$.

—— 43. Art. 42. Ex. 9: *for* $\frac{12bc}{8} = \frac{3bc}{2}$ *read* $\frac{12be}{8} = \frac{3be}{2}$.

—— 62. Art. 52. Or more accurately thus: Multiplication is equivalent to the result of the addition of the same number, which is called the *multiplicand*, repeated as often as unity is contained in another number which is called the *multiplier*.

—— 66. Art 65. (1): *for* $a \times (c - d)$ *read* $a \times (c + d)$.

—— 79. Art. 92: *for* A' in the figure *read* A.

Page 80. Art. 92. Note: the consideration of the sign $\cos\theta + \sqrt{-1}\sin\theta$ forms the principal subject of Chap. XII.

—— 90. Art. 110. line 11: velocity in this place means the number of feet described in a *unit* of time: see the third paragraph in this Article.

—— 96. Art. 115. This subject, which is one of the most difficult and embarrassing which occurs in the elements of Algebra, is resumed in Chap. X. Art. 321. and those which follow it, as far as Art. 342.

—— 101. Art. 128. and 129. See Art. 733. and those which follow in Chap. XV.

—— 105. Art. 133. line 3: *for algebraical read equivalent.*

—— 107. Art. 139. Note, last line but 3: *for* 2^n *read* 2^{2^n}.

—— 111. Art. 145. Note. The subject of this note is resumed at great length in Chap. XII. Art. 528—578.

—— 124. Art 170: after the words "which is obviously not a divisor both of A and B," add "and which has no factor which is common to them."

———— Art. 173. Ex. 7. When the same divisor, as in this case, is employed twice, it would have been convenient to have multiplied the original dividend by 4^2 instead of 4: we should thus have avoided the necessity of multiplying the first remainder, and of interrupting by so doing the process of division.

—— 146. Art. 180. line 10: *for* a^{mn} *read* a^{nm}.

—— 162. Art. 184: bottom of page: *for* $754 = \frac{754}{10^3}$ *read* $.754 = \frac{754}{10^3}$.

—— 169. Art. 194. line 9: *for* Art. 193. *read* Art. 192.

—— 179. Art. 205. first line: *for* define *read* determine.

—— 187. Art. 208. Ex. 4. line 4: *for* $-\frac{3}{16}.\frac{a^3}{x}$ *read* $-\frac{3}{16}.\frac{a^3}{x^2}$.

—— 194. Art. 213 Ex. 10. first line: *for* $x^{\frac{5}{2}}$ *read* $x^{\frac{5}{3}}$, *for* $x^{\frac{8}{2}}$ *read* $x^{\frac{8}{3}}$, *for* $x^{\frac{11}{2}}$ *read* $x^{\frac{11}{3}}$.

Page 212. Art. 239. sixth line from the top of the page: *for* fourth *read* third.

—— 215. Art. 245. sixth line: *for* $a_1 + a_2$ *read* $a_2 + a_3$.

—— 217. Art. 247. first line from the top of the page: *for* a_n *read* a_m.

—— 218. Art. 258. line 7: *for* their *read* it.

———————— line 8: *for* them *read* it.

———————— line 11: for a^2 read a_1^2.

———————— line 15: for $n - r + 2$ read $n + r - 2$.

Page 232. Art. 270. Ex. γ in margin: *for* tetrahedial *read* tetrahedral.

———————— in Note: *for* Art. 126. *read* Art. 136.

—— 290. Art. 335. last line but 2: *for* the $\surd 5$ *read* the value of $\surd 5$.

—— 304. Art. 345. Note, line 7 from bottom: *for* a^{1^2} *read* a_1^2, and *for* a^{2^2} *read* a_2^2.

—— 306. Art. 347. line 5: *for* fraction *read* fracture.

—— 330. Art. 393. line 5: *for* $-\frac{a}{b}$ *read* $\frac{a}{b}$.

—— 337. Art. 403: The chapter on continued fractions and their applications to the theory of incommensurable magnitudes, to which a reference is made in this article, has been necessarily omitted in consequence of the great and unexpected extent which other and more important parts of this work were found to occupy.

—— 355. Art. 436. Ex. 8. If however we consider $\surd b$ and $\surd \beta$ as representing the arithmetical values of $\sqrt{-b}$ and $\sqrt{-\beta}$, they will cease to admit of the double sign $+$ and $-$: the same remark applies to $\surd b\beta$.

—— 356. Art. 436. Ex. 20: *for* occur *read* recur.

—— 358. Ex. 41: *for* $(a-b\sqrt{-1})+$ *read* $(a-b\sqrt{-1})\times$.

—— 375. Art. 456: The same conclusion is otherwise obtained Art. 497.

Page 380. Art. 462. Note: the subject of the measure of angles is resumed in Art. 549, 666 and 667.

—— 387. Art. 470. line 10 from bottom:

for $\sin\theta = \sin(-\theta)$ *read* $\sin\theta = -\sin(-\theta)$.

—— 429. Art. 532. It will likewise follow from this definition, that a straight line will be *symmetrical* in all its parts: for any one part of it may be transferred (Art. 539) to any other part, and if any two points of one part be made to coincide with any two points of the other, they must coincide throughout, however far they may be produced: it will thus appear also that whatever property is proved to belong to one point of a straight line, must belong likewise to every other point of the same or of any other straight line.

—— 522. Art. 644. Note, line 3 from bottom: *for* $\frac{B_1^2}{1.2.1.2}$ *read* $\frac{B_1^4}{1.2.1.2}$.

—— 618. Art. 768. line 3: *for* $u^3 + Pu^2 + Qu + R = 0$ *read* $u^3 - Pu^2 + Qu - R = 0$.

—— 620. Art. 768. Note, line 11: *for* Qa *read* Qu.

———— line 3 from bottom:

for $(a + c)^2 = (b + c)^2$ *read* $(a + c)^2 = (b + d)^2$.

—— 621. Art. 768. Note, line 5: *for* $(a + b) + (a + b) + (a + c)$ *read* $(a + b) + (a + c) + (a + d)$.

—— 638. Art. 784. line 16 from top: *for* of (n) *read* (n) of.

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